MACROSCOPIC BEHAVIOR IN THE ABLOWITZ-LADIK EQUATIONS

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Abstract

Modulation theory is used to study the Ablowitz-Ladik equations. Exact multiphase wavetrain solutions are found, and local conservation laws are averaged to obtain a macroscopic description of this lattice problem.

1 Transport in Nonlinear Lattices

Physical systems like organic polymers and waveguide arrays that are necessarily spatially discrete and yet translationally symmetric are *lattices* that may be modeled by an infinite number of coupled ODEs, indexed by integers n, that are invariant under the substitution $n \mapsto n+1$. One interesting question to ask about such systems is: to what degree is it possible for the *microscopic dynamics* (as described by the coupled ODEs) to contribute to collective phenomena on scales much larger than the lattice spacing? Such collective phenomena that can be described by closed sets of effective equations are called *macroscopic dynamics*. The hope is that robust macroscopic dynamics can be understood as a mechanism for coherent energy transport over very long scales. One way to seek macroscopic dynamics is to assume that all dynamical variables vary slowly along the lattice; this is the usual idea of the continuum limit. However, such a limit ignores the possibility that essential effects of the spatial discreteness can contribute to the macroscopic dynamics. We want to explore this possibility as well.

As a mathematical model of a lattice, we consider here the system of ODEs

$$i\partial_t A_n + f(|A_n|^2)(A_{n+1} + A_{n-1}) + F(|A_n|^2)A_n = 0\,. \tag{1}$$

The complex numbers $A_n(t)$ can be considered to be *complex mode amplitudes* of the form $q_n + ip_n$ for a system of oscillators with displacements q_n and momenta p_n. By choosing the real-valued functions $f(\rho)$ and $F(\rho)$ appropriately, it is possible to obtain a variety of effects. Nonconstant terms in the function F give rise to anharmonicity in the individual oscillators, and nonconstant terms in the function f introduce anharmonic coupling between these oscillators. For example, choosing $f(\rho) = 1$ and $F(\rho) = \pm 2\rho - 2$ results in the discrete nonlinear Schrödinger system (DNLS)

$$i\partial_t A_n + (A_{n+1} - 2A_n + A_{n-1}) \pm 2|A_n|^2 A_n = 0\,. \tag{2}$$

On the microscopic level, the DNLS is a description of a lattice of anharmonic oscillators, each linearly coupled to its two nearest neighbors. On the other hand, choosing $F(\rho) = -2$ and $f(\rho) = 1 \pm \rho$ results in the Ablowitz-Ladik system (AL)

$$i\partial_t A_n + (A_{n+1} - 2A_n + A_{n-1}) \pm |A_n|^2 (A_{n+1} + A_{n-1}) = 0\,. \tag{3}$$

In contrast with the DNLS, AL describes on the microscopic level a lattice of harmonic oscillators, each coupled to its two nearest neighbors by nonlinear interactions. In spite of the differences on the microscopic level, these two systems of differential equations can both be scaled to yield NLS in the continuum limit. However, the fact that the two agree in this limit is no reason to believe that there should be any correspondence for general initial data.

The system of equations Eq. (1) is Hamiltonian. The conservation of energy can be written in local form, where the time derivative of the energy density is equal to a pure difference in n of an energy flux. There is also an additional local conservation law that follows from the gauge symmetry group $A_n \mapsto \exp(i\alpha)A_n$ for real α. There are many more local conservation laws for the special case of AL, which will allow us to obtain a very complete picture of macroscopic dynamics for this system.

2 Modulation Equations for Harmonic Plane Waves

As an initial example of the kind of transport mechanisms we have in mind, and to provide a demonstration of the way that spatial discreteness can

contribute to macroscopic phenomena, we briefly describe the behavior of slowly varying harmonic wavetrains in the lattice system Eq. (1). The results of this section first appeared in the work of Hays, Levermore, and Miller[5].

The lattice system Eq. (1) has exact solutions of the simple form

$$A_n(t) = \sqrt{\rho}\exp(i(kn - \omega t))\,, \tag{4}$$

for constants ρ, k, and ω satisfying the dispersion relation

$$\omega + 2f(\rho)\cos k + F(\rho) = 0\,. \tag{5}$$

We now consider approximate solutions, for which ρ and k are taken to be slowly varying, ω being determined locally from the dispersion relation Eq. (5). To obtain dynamical equations for the quantities ρ and k, we consider the solution Ansatz Eq. (4) with ρ, k, and ω depending on n and t, insert the Ansatz into the two local conservation laws for Eq. (1), and pass to the continuum limit, not in the field $A_n(t)$, but in the quantities ρ and k. This procedure yields two quasilinear PDEs for the quantities $\rho(X,T)$ and $k(X,T)$ on the scales $X = \varepsilon n$ and $T = \varepsilon t$ as $\varepsilon \downarrow 0$. These PDEs are called the *modulation equations* for the harmonic plane waves given by Eq. (4). If the characteristic speeds for these PDEs are real and distinct, the PDEs are *strictly hyperbolic*, and the corresponding Cauchy problem at $T = 0$ is well-posed, which is interpreted as indicating modulational stability of the underlying wavetrain. Otherwise, the PDEs are *elliptic*, and the Cauchy problem at $T = 0$ is ill-posed, which is interpreted as indicating modulational instability of the wavetrain. As a result of nonlinearity, the characteristic speeds depend on the values of ρ and k. Thus, it is possible for the modulation equations, posed for globally hyperbolic initial data, to evolve the fields $\rho(X,T)$ and $k(X,T)$ forward in T in such a way that the speeds cease to be real for some X and $T > 0$. We call this phenomenon *dynamic change of type*. It is also possible for $\partial_X\rho$ and $\partial_X k$ to become infinite for some X and $T > 0$, even though the speeds are real. Both of these catastrophes are interpreted as the local breakdown of the modulational Ansatz, Eq. (4).

These macroscopic limits, unlike the usual continuum limit, are able to distinguish qualitatively between different discretizations of the same equation. Figure 2 summarizes the results of the analysis described for the two discretizations of NLS discussed above, and gives the analogous results for harmonic plane wave modulation in NLS for comparison.

One may ask whether the analysis described above can be generalized. Is is possible to replace the exact solutions, Eq. (4), with a more complicated

	DNLS	AL	NLS
Microscopic Equation	$i\partial_t A_n + (A_{n+1} - 2A_n + A_{n-1})$ $\pm 2\|A_n\|^2 A_n = 0$	$i\partial_t A_n + (A_{n+1} - 2A_n + A_{n-1})$ $\pm\|A_n\|^2(A_{n+1} + A_{n-1}) = 0$	$i\partial_t A + \partial_x^2 A \pm 2\|A\|^2 A = 0$
Modulation Equations	$\partial_T \rho = \partial_X[2\rho \sin k]$ $\partial_T k = \partial_X[\mp 2\rho - 2\cos k]$	$\partial_T \rho = (1 \pm \rho)\partial_X[2\rho \sin k]$ $\partial_T k = \partial_X[-2(1 \pm \rho)\cos k]$	$\partial_T \rho = \partial_X[2\rho k]$ $\partial_T k = \partial_X[\mp 2\rho - k^2]$
Stability Criterion	$+$: $\cos k < 0$ $-$: $\cos k > 0$	$+$: No Stable Waves $-$: $\rho < 1$	$+$: No Stable Waves $-$: All Waves Stable
Dynamic Change of Type?	Sometimes	Never	Never

Figure 1: Harmonic plane wave modulation for two discretizations of NLS.

family of exact solutions, and obtain modulation equations describing the slow evolution of the parameters that index these solutions? If one is content to work with the Ablowitz-Ladik equations, then the answer to this question is in the affirmative.

3 Wavetrains in AL

The Ablowitz-Ladik equations, Eq. (3), deserve special treatment in the study of macroscopic limits of Eq. (1) because in this case there exists an inverse scattering transform (IST) that can be used to linearize Eq. (3) to obtain exact solutions and also to identify an inexhaustible number of local conservation laws. Eq. (3) is the reduction of

$$\begin{aligned} -i\partial_t Q(n,t) &- [Q(n+1,t) - 2Q(n,t) + Q(n-1,t)] \\ &+ Q(n,t)R(n,t)[Q(n+1,t) + Q(n-1,t)] = 0\,, \\ -i\partial_t R(n,t) &+ [R(n+1,t) - 2R(n,t) + R(n-1,t)] \\ &- Q(n,t)R(n,t)[R(n+1,t) + R(n-1,t)] = 0\,, \end{aligned} \tag{6}$$

where n has been promoted from a subscript to an argument, when $R = \mp\overline{Q}$ for $t \in \mathbb{R}$. These ODEs were found by Ablowitz and Ladik [1], who were looking for lattice systems that could be solved with an IST. We refer to Eq. (6) as the complex Ablowitz-Ladik equations (CAL).

The goal of the rest of this paper is to describe the use of the machinery of the IST to find classes of exact wavetrain solutions that will play the role

of generalizations of the plane wave Ansatz, Eq. (4), and to provide tools for describing the modulational behavior of these solutions. As we are only interested in producing functions that solve CAL and not in solving an initial value problem, we will use this machinery in a way that differs in appearance and structure, though certainly not in spirit, from its traditional usage in solving initial value problems. The details not mentioned here can be found in the paper of Miller, Ercolani, Krichever, and Levermore [6].

We use the following Lax pair representation of CAL:

$$\mathbf{u}(n+1,t,z) = \mathbf{L}(n,t,z)\mathbf{u}(n,t,z)\,, \tag{7}$$

where

$$\mathbf{L}(n,t,z) = \begin{bmatrix} z & Q(n,t) \\ zR(n,t) & 1 \end{bmatrix}, \tag{8}$$

and

$$-i\partial_t \mathbf{u}(n,t,z) = \mathbf{B}(n,t,z)\mathbf{u}(n,t,z)\,, \tag{9}$$

where

$$\mathbf{B}(n,t,z) = \begin{bmatrix} z-1-Q(n,t)R(n-1,t) & Q(n,t)-z^{-1}Q(n-1,t) \\ zR(n-1,t)-R(n,t) & 1-z^{-1}+R(n,t)Q(n-1,t) \end{bmatrix}. \tag{10}$$

The consistency of these two linear problems is a condition on the potentials Q and R that is equivalent to CAL. This Lax pair is more useful for our purposes than that given for a vector $\mathbf{v}(n,t,\lambda)$ in the paper of Ablowitz and Ladik [1]. The latter Lax pair can be obtained from the one above by the transformation

$$\begin{aligned} v_1(n,t,\lambda) &= \lambda^{-n}u_1(n,t,z=\lambda^2)\,, \\ v_2(n,t,\lambda) &= \lambda^{-n-1}u_2(n,t,z=\lambda^2)\,. \end{aligned} \tag{11}$$

3.1 Constructing Finite Genus Solutions

The motivation for the Riemann surface method to be used below to produce multiphase wavetrain solutions to Eq. (6) stems from the study of the

solutions satisfying periodic boundary conditions in n. If one uses Floquet theory to study Eq. (7) when Q and R are periodic in n with period N, it is possible to deduce that the Floquet multipliers are independent of both n and t. Because the spatial linear system for $\mathbf{u}$ is second order, the Floquet multipliers satisfy a quadratic equation with coefficients that turn out to have the simple structure of polynomials in z. The graph of this algebraic equation is the Floquet multiplier curve, Γ. It is a finite genus Riemann surface that is a two-sheeted branched cover of the z-plane. The vector $\mathbf{u}$ is most naturally considered as a function of n, t, and a point on Γ. For values of z different from the branch points, there are two linearly independent simultaneous solutions $\mathbf{u}$ of the linear problems making up the Lax pair. The two solutions fail to be linearly independent when z is a branch point of Γ. Formulas for $\mathbf{u}$ can be found using the Riemann theta function, which is the central object of analytic function theory on Riemann surfaces. From $\mathbf{u}$, it is easy to recover the potentials Q and R. As it happens in the focusing and defocusing cases, the corresponding formulas for the potentials can be written in such a way that it is clear that they depend periodically on a number of phase variables of the form $kn - \omega t$. Thus the potentials have the form of multiphase waves, but since spatial periodicity has been imposed the wavenumbers satisfy certain quantization conditions.

It is reasonable that to relax the constraint of spatial periodicity in an appropriate way should yield a family of multiphase wavetrain solutions of similar form, but without the quantization conditions on the wavenumbers. The analytic manifold of the spectral parameter should still be a finite genus Riemann surface, but we can no longer interpret this surface as a Floquet multiplier curve. Below, we will show how to construct simultaneous solutions $\mathbf{u}$ of the two linear problems of the Lax pair, Eq. (7) and Eq. (9), by first selecting an arbitrary set of branch points in the z-plane to determine an appropriate hyperelliptic Riemann surface. Because we are interested in general solutions, we must proceed without the Floquet scattering theory of the periodic problem. From $\mathbf{u}$ we will see exactly how to obtain a solution (Q, R) of Eq. (6).

The main idea here is that it is difficult to produce exact solutions to the nonlinear system Eq. (6), and it is easier to produce two linearly independent simultaneous solutions $\mathbf{u}(n, t, z)$ to the two linear problems, Eq. (7) and Eq. (9). The existence of two linearly independent solutions, $\mathbf{u}^{\pm}(n, t, z)$, even for an arbitrary value of the complex parameter z, guarantees the compatibility of the two linear problems, and thus guarantees that the potentials $Q(n, t)$ and $R(n, t)$ solve Eq. (6).

Let us proceed to find some solutions $\mathbf{u}$ of the Lax pair. In particular,

we must make sure that $\mathbf{u}$ simultaneously satisfies Eq. (7) and Eq. (9) in the asymptotic limits $z \downarrow 0$ and $z \uparrow \infty$. When z is small, there are two possible dominant balances in the two problems:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \exp(i(1-z^{-1})t) \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}(1) \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = z^n \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}(z) \end{pmatrix}. \tag{13}$$

In order to distinguish these two kinds of expansions, we refer to the former as 0^+ and the latter as 0^-. When z is large, there are also two possible dominant balances:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = z^n \exp(i(z-1)t) \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}(1) \end{pmatrix}, \tag{14}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}(z) \end{pmatrix}. \tag{15}$$

Once again, to distinguish these two kinds of expansions, we refer to the former as ∞^+ and the latter as ∞^-.

These facts, along with the analytic dependence of $\mathbf{u}$ on z suggest building the two linearly independent solutions $\mathbf{u}^{\pm}(n,t,z)$ as projections of a single vector function $\mathbf{u}(n,t,P)$, where P denotes a point on a two-sheeted branched cover, Γ, of the z-sphere. The manifold Γ consists of two copies of the Riemann sphere, ramified at an even number of fixed but arbitrary points $z_1, \ldots, z_{2g+2}$. Each copy of the sphere may be identified with the complex z-plane through stereographic projection. This is how we recover from $\mathbf{u}(n,t,P)$ two different vector functions $\mathbf{u}(n,t,z)$, both solving Eq. (7) and Eq. (9) for all complex z. As long as Γ does not branch at $z=0$ or $z=\infty$, there are two points of Γ over each of these z values. We may thus insist that the function $\mathbf{u}(P)$ have behavior determined by the expansions $0^{\pm}$ at the two points above $z=0$ and likewise that $\mathbf{u}(P)$ have behavior determined by the expansions $\infty^{\pm}$ at the two points above $z=\infty$. We will label these points on Γ as $0^{\pm}$ and $\infty^{\pm}$ according to the kind of expansion $\mathbf{u}$ has at these points. There is an ambiguity stemming from the way one chooses points $0^{\pm}$ above $z=0$ with respect to the choice of points $\infty^{\pm}$ over $z=\infty$ that results in two essentially different kinds of asymptotic behavior for the function $\mathbf{u}(P)$ for a given choice of branch points z_k. This is a feature of the construction of finite genus solutions to the Ablowitz-Ladik equations that distinguishes it from constructions in which the Lax pair becomes singular for only one value of the parameter z.

So let Γ be such a Riemann surface, branched at the n and t independent distinct z-values $z_1, \ldots, z_{2g+2}$, having the interpretation of constants of motion or action variables. Next, assume that $u_1(P)$ and $u_2(P)$ both have their poles in $\Gamma \setminus \{0^\pm, \infty^\pm\}$ confined to the n and t independent points $P_1, \ldots, P_g$, having the interpretation of initial ($n = 0$, $t = 0$) values of angle variables. Finally, normalize the functions $u_1(P)$ and $u_2(P)$ so that

$$u_1 = z^n \exp(i(z-1)t)\Big\{1 + \mathcal{O}(z^{-1})\Big\}, \tag{16}$$

near ∞^+, and

$$u_2 = \exp(i(1-z^{-1})t)\Big\{1 + \mathcal{O}(z)\Big\}, \tag{17}$$

near 0^+. Then we have the following (for most sets of g points $P_1, \ldots, P_g$).

Lemma 1 *The function $\mathbf{u}(n,t,P)$ with the above properties exists and is constructed uniquely from the data $z_1, \ldots, z_{2g+2}$, $P_1, \ldots, P_g$, and the choice of expansions over $z = 0$ and $z = \infty$. Furthermore, the function $\mathbf{u}(n,t,P)$ solves both the linear problems Eq. (7) and Eq. (9) globally in n, t, and P as long as the potentials $Q(n,t)$ and $R(n,t)$ are such that*

$$u_1 = \exp(i(1-z^{-1})t)\Big\{Q(n-1,t) + \mathcal{O}(z)\Big\},$$

near 0^+ and

$$u_2 = \exp(i(z-1)t)\Big\{R(n-1,t) + \mathcal{O}(z^{-1})\Big\},$$

near ∞^+.

The compatibility of the two linear problems Eq. (7) and Eq. (9) for these potentials then gives the following theorem.

Theorem 1 *The functions $Q(n,t)$ and $R(n,t)$ derived from the expansions for $\mathbf{u}$ solve the complex Ablowitz-Ladik equations Eq. (6).*

These solutions of CAL have the form of multiphase wavetrains:

$$Q(n,t) = A\exp(i\theta_0)\frac{\Theta((\theta_1, \ldots, \theta_g)^T)}{\Theta((\theta_1+\beta_1, \ldots, \theta_g+\beta_g)^T)}, \tag{18}$$

where $\theta_j = k_j n - \omega_j t - \delta_j$ are phase variables and Θ is the Riemann theta function defined by

$$\Theta(\mathbf{w}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp\left[\frac{1}{2}\mathbf{m}^T\mathbf{B}\mathbf{m} + \mathbf{m}^T\mathbf{w}\right], \tag{19}$$

and $\mathbf{B}$ is a symmetric matrix with negative definite real part obtained by integrating holomorphic differentials around closed loops on Γ. This indicates that, while the function $\mathbf{u}$ and the corresponding potentials Q and R are unique given the data, the representations of these in terms of theta functions are not well defined, depending on a choice of *homology cycles* on the surface Γ. This is a gauge symmetry that may be used to our advantage in simplifying final formulas.

If the genus g is less than 3, the procedure we have outlined above is equivalent to substitution of an Ansatz of the form given by Eq. (18) into Eq. (6) and solving for relations among the various parameters. If $g \geq 3$, it is no longer possible to find solutions without employing the complex geometry of the Riemann surface Γ; the dispersion relations become transcendental.

The solutions we have described all obey the constraint $Q(-1,0) = 1$. It is easy to free up this constraint by using the following symmetry group of CAL: if (Q, R) is a solution of CAL, then for each complex number ξ, $(\xi Q, \xi^{-1} R)$ is also a solution of CAL.

It is also possible to relax the constraint that the branch points be distinct, although this requires more complicated tools[6]. These degenerate solutions can correspond to homoclinic orbits and solitons as described by Ercolani and Miller [3].

3.2 Focusing and Defocusing Solutions

Imposing *reality* means isolating data leading to solutions satisfying $R = \pm\overline{Q}$ for real t. These solve the restricted problem of interest, Eq. (3). The potentials as given by formulas like Eq. (18) may not be periodic functions of each of the θ_j individually unless reality is imposed on the data. We want to impose reality in order to have a family of multiphase wavetrains.

The difficulty in restricting the class of solutions found above to the focusing and defocusing subclasses is that the simple condition $R = \pm\overline{Q}$ must be translated into conditions on the data that we used to generate the potentials Q and R, namely the branch points z_k, the points P_j, the scaling parameter ξ, and the choice of the asymptotic behavior of $\mathbf{u}$ over $z = 0$ and $z = \infty$. A convenient tool for carrying out this translation is the system of squared eigenfunctions associated to the function $\mathbf{u}$. These squared eigenfunctions can be imagined as a kind of intermediate level between the potentials (Q, R), where the reality conditions are trivial, and the data for specifying the function $\mathbf{u}$, where the corresponding conditions are transcendental. In fact, at the level of the squared eigenfunctions, the reality conditions are algebraic.

Let us introduce the squared eigenfunctions. Recall the notation $\mathbf{u}^{\pm}(z)$ for

the two vector solutions to the Lax pair obtained by stereographic projection from the function $\mathbf{u}(P)$ on Γ. Define

$$\begin{aligned} \varphi(z) &= u_1^+(0,0,z)u_1^-(0,0,z)\,, \\ \chi(z) &= u_2^+(0,0,z)u_2^-(0,0,z)\,, \\ f(z) &= \frac{1}{2}[u_1^+(0,0,z)u_2^-(0,0,z) + u_2^+(0,0,z)u_1^-(0,0,z)]\,. \end{aligned} \tag{20}$$

It is possible to deduce from properties of $\mathbf{u}(P)$ described in the last subsection that the squared eigenfunctions are polynomials that can be normalized so that f is monic. Thus, they take the form

$$\begin{aligned} \varphi(z) &= \varphi_g z^g + \varphi_{g-1}z^{g-1} + \cdots + \varphi_0\,, \\ \chi(z) &= z[\chi_g z^g + \chi_{g-1}z^{g-1} + \cdots + \chi_0]\,, \\ f(z) &= z^{g+1} + f_g z^g + \cdots + f_1 z + f_0\,. \end{aligned} \tag{21}$$

This construction can be viewed as a transformation from the problem data to the $3g+3$ complex coefficients of these polynomials. However, it is also possible to invert the mapping by operations no more complicated than root finding.

The advantage of this change of variables is that the reality conditions on the polynomial coefficients are easy to deduce from the linear problems Eq. (7) and Eq. (9). Namely for focusing $(-)$ and defocusing $(+)$ potentials,

$$f(z) = \frac{z^{g+1}}{\overline{f_0}}\overline{f(\overline{z}^{-1})}\,, \tag{22}$$

$$\chi(z) = \pm\frac{z^{g+1}}{\overline{f_0}}\overline{\varphi(\overline{z}^{-1})}\,. \tag{23}$$

These relations imply that in both the focusing and defocusing cases, the branch points must, as a set, be symmetric in reflection through the unit circle in the z-plane. In fact, more detailed investigations[6] suggest the following conjectures[1]:

Conjecture 1 *In the focusing case, the branch points cannot lie on the unit circle unless they have multiplicity greater than 1.*

Conjecture 2 *In the defocusing case, the branch points all lie on the unit circle as long as $|Q(n,t)| < 1$ for all n.*

[1]The first conjecture has been proved[6] when Q and R are periodic in n. The second conjecture is suggested by the fact that the defocusing system preserves the sign of $|Q(n,t)| - 1$ for each n, and that only the potentials for which $|Q(n,t)| < 1$ for all n contribute to the continuum limit, in which all branch points are fixed by the antiholomorphic involution $z \mapsto \overline{z}^{-1}$.

4 Modulation Equations for Multiphase Waves in AL

When we were concerned with modulation of harmonic plane waves, we inserted the plane wave Ansatz into local conservation laws and passed to the continuum limit in the parameters of the motion. We have a large set of oscillatory solutions available, as constructed in the last section. These exact solutions generalize the plane wave Ansatz Eq. (4). Let us now present a compact representation of the infinite number of local conservation laws implied by Eq. (6). Define

$$F(n,t,P) = d\log\frac{u_1(n+1,t,P)}{u_1(n,t,P)}\,, \tag{24}$$

$$G(n,t,P) = \partial_t d\log u_1(n,t,P)\,. \tag{25}$$

Then, it follows from the compatibility of differentiation in t with spatial differencing in n that

$$\partial_t F(n,t,P) = G(n+1,t,P) - G(n,t,P)\,, \tag{26}$$

which has the form of a local conservation law with parameter $P \in \Gamma$. As trivial as it may appear, this expression contains an infinity of nontrivial local conservation laws obtained by expanding with respect to P.

The idea in obtaining modulation equations in integrable systems (as described by Flaschka, Forest, and McLaughlin [4] for KdV and Bloch and Kodama [2] for the Toda lattice) is to imagine a slowly deforming Riemann surface $\Gamma(X=\varepsilon n, T=\varepsilon t)$ for $\varepsilon \downarrow 0$. Let z_k depend on the slow scales X and T. Substitute the theta function representation of u_1 into the conservation law generator and use angled brackets to denote an average over n and t sufficient to remove dependence on n and t. Then one obtains the modulation equations for the finite genus solutions of Eq. (6) in *homology gauge form*[2]:

$$\begin{aligned}\partial_T\omega_{(3)} + \partial_T\left\langle d\log\frac{\Theta(A(P)-\mathbf{Z}+\mathbf{U}(n+1)+\mathbf{V}t)}{\Theta(A(P)-\mathbf{Z}+\mathbf{U}n+\mathbf{V}t)}\right\rangle = \\ \partial_X\omega_{(2)} + \partial_X\left\langle d\frac{\partial_t\Theta(A(P)-\mathbf{Z}+\mathbf{U}(n+1)+\mathbf{V}t)}{\Theta(A(P)-\mathbf{Z}+\mathbf{U}(n+1)+\mathbf{V}t)}\right\rangle\,.\end{aligned} \tag{27}$$

[2]We use this terminology because the appearance of these modulation equations may be altered by changing the basis of homology cycles on Γ.

The Abel mapping, $A(P)$, is a vector of integrals of normalized holomorphic differentials along a contour ending at the point P. The differentials $\omega_{(3)}$ and $\omega_{(2)}$ control the asymptotic behavior of the function $\mathbf{u}$ near the points $0^{\pm}$ and $\infty^{\pm}$. The vectors $\mathbf{U}$ and $\mathbf{V}$ contain wavenumbers and frequencies obtained from these two differentials. Only the vector $\mathbf{Z}$ depends on the initial phase information encoded in the points P_j.

In the focusing and defocusing cases, it is usually[3] possible to choose the homology cycles on Γ in such a way that the modulation equations take the simple form

$$\partial_T \omega_{(3)} = \partial_X \omega_{(2)} \,. \tag{28}$$

This kind of cosmetic improvement is an application of the homology gauge symmetry of the finite genus formulas mentioned above. All initial phase information in the vector $\mathbf{Z}$ is lost in this representation. It turns out that the branch points $z_k(X,T)$ are Riemann invariants of these equations, and that the characteristic speed of z_k is real if z_k is on the unit circle. Then, our conjectures of the previous section lead us to the following conclusions:

- In the focusing case, these modulation equations are never hyperbolic and there are no stable multiphase wavetrains.
- In the defocusing case, there exist both stable and unstable multiphase wavetrains. The stable wavetrains are those whose amplitudes are everywhere less than 1.

5 Resonances

If there are resonances in the underlying focusing or defocusing wavetrain, corresponding to commensurate frequencies or wavenumbers, then the modulation equations must be written in a form more general than Eq. (28). The terms involving the phase vector $\mathbf{Z}$ cannot be made to vanish by any choice of homology cycles, and in fact $\mathbf{Z}$ must be taken to depend on X and T, thus introducing the phenomenon of *phase modulation*. The modulation equations for the branch points $z_k(X,T)$ must be supplemented by modulation equations for certain functions of the pole positions P_j. In these resonant cases, the modulation equations need not have the stability properties given at the end of the last section. Indeed, Miller, Ercolani, and Levermore [7] discuss a simple modulationally stable resonance in the focusing Ablowitz-Ladik equations.

[3] Here, "usually" means in the case that none of the frequencies or wavenumbers of the underlying wavetrain are commensurate.

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