ELSEVIER

Physica D 101 (1997) 17–26

PHYSICA D

Vector nonlinear Schrödinger hierarchies as approximate Kadomtsev–Petviashvili hierarchies

P.D. Miller [1]

Australian Photonics Cooperative Research Centre, Optical Sciences Centre, Research School of Physical Sciences and Engineering, The Australian National University, Canberra, ACT 0200, Australia

Received 3 June 1996; accepted 26 September 1996
Communicated by C.K.R.T. Jones

Abstract

The Kadomtsev–Petviashvili (KP) hierarchy, a collection of compatible nonlinear equations, each in 2 + 1 independent variables, can be consistently constrained in many different ways to yield hierarchies of equations in 1 + 1 independent variables. In particular, the N-component vector nonlinear Schrödinger (VNLS) hierarchies are contained within the KP hierarchy in this way. These hierarchies approximate the KP hierarchy in the limit of large N, and this permits the equations of the KP hierarchy to be approximated by nonlinear equations in 1 + 1 dimensions.

Keywords: Integrable hierarchies; Baker–Akhiezer functions; NLS; KP

1. Introduction

Virtually every numerical method or approximation scheme applicable to partial differential equations works by reducing the dimension of the problem in some way. In some cases, it is actually the spatio-temporal dimension (that is, the number of continuous independent variables) that is reduced by the scheme. The so-called "method of lines" is an example of this kind of dimension reduction. With this method a partial differential equation in 1 + 1 dimensions is spatially discretized to yield a system of differential equations in 0 + 1 dimensions. A further discretization in time then reduces the problem to algebraic steps easily carried out by a computer.

I want to describe an approximation scheme that applies to the nonlinear equations of the Kadomtsev–Petviashvili (KP) hierarchy, simulating the behavior of equations in 2 + 1 dimensions with smaller problems in 1 + 1 dimensions. Although it is not a discretization of one of the independent variables as in the method of lines, this scheme preserves the integrability of the hierarchy at every level of approximation. And of course, the approximate hierarchies in 1 + 1 dimensions will have solutions that converge to solutions of the fully 2 + 1-dimensional hierarchy. Dimensional

[1] E-mail: pdm124@rsphysse.anu.edu.au.

PII S0167-2789(96)00225-4

reductions of the KP hierarchy have indeed been known for some time, but to my knowledge the issue of how closely these reduced hierarchies resemble their parent has not been adequately addressed.

The KP hierarchy is a collection of nonlinear partial differential equations that can be expressed as compatibility conditions for a sequence of linear problems or *flows*. Let us describe this hierarchy using the formalism of pseudo-differential operators [1]. Take $\boldsymbol{x} = (x_1, x_2, x_3, \ldots)^{\mathrm{T}}$ to be a vector[2] of independent variables, and denote by ∂_k the derivative with respect to x_k. Introduce a pseudo-differential operator of the form

$$L = \partial_1 + w^{(1)}(\boldsymbol{x})\partial_1^{-1} + w^{(2)}(\boldsymbol{x})\partial_1^{-2} + \cdots \tag{1}$$

with arbitrary coefficients $w^{(n)}(\boldsymbol{x})$, and for each $k = 2, 3, \ldots$, define the associated differential operator

$$B_k = (-2\mathrm{i})^{1-k}(L^k)_+ , \tag{2}$$

where the subscript + indicates the part of the expression that includes only nonnegative powers of ∂_1. Then, the system of linear equations, or flows, for a function $\phi(\boldsymbol{x})$,

$$\partial_k \phi = B_k \phi , \tag{3}$$

will be consistent only if the coefficients $w^{(n)}(\boldsymbol{x})$ satisfy certain nonlinear partial differential equations. In particular, the pairwise compatibility conditions

$$\partial_l B_k - \partial_k B_l + [B_k, B_l] = 0 \tag{4}$$

are, upon separating the various powers of ∂_1, nonlinear partial differential equations in 2 + 1 independent variables, x_1, x_k, and x_l, for a finite number of the coefficients $w^{(n)}(\boldsymbol{x})$. They are the equations of the KP hierarchy. The simplest case is $k = 2$ and $l = 3$, which yields equations for $w^{(1)}(\boldsymbol{x})$ and $w^{(2)}(\boldsymbol{x})$. From these, $w^{(2)}(\boldsymbol{x})$ may be eliminated to yield the equation for $V(\boldsymbol{x}) = w^{(1)}(\boldsymbol{x})$:

$$\tfrac{3}{4}\partial_2^2 V = \partial_1[\partial_3 V + \tfrac{1}{16}\partial_1^3 V + \tfrac{3}{4} V \partial_1 V], \tag{5}$$

which is the KP equation of water wave theory from which the hierarchy gets its name. If the compatibility conditions are all satisfied, then the evolution of ϕ with respect to $x_2, x_3, \ldots$ is an isospectral deformation of the *formal* eigenvalue equation

$$L\phi = -2\mathrm{i}\lambda\phi . \tag{6}$$

This equation is only formal because the action of the pseudo-differential operator L cannot be defined on a sufficiently broad class of functions $\phi(\boldsymbol{x})$. There are, however, special conditions under which this eigenvalue problem makes sense, as we will see below. Proceeding formally, one finds that the compatibility of the mth power of Eq. (6) with the individual flows (3) gives rise to a hierarchy of Lax equations

$$\partial_k L^m = [B_k, L^m] , \tag{7}$$

where m is arbitrary. Now, unlike the nonlinear equations (4), these Lax equations are equations in only 1 + 1 independent variables, x_1 and x_k, and thus one might prefer to solve them rather than the 2 + 1-dimensional equations (4). However, the Lax equations (7) generally involve *all* of the coefficients $w^{(n)}(\boldsymbol{x})$ together. Thus, if one only wants to consider a finite number of the coefficients, the KP hierarchy remains a system of genuinely 2 + 1-dimensional equations.

---

[2] Throughout we will denote vector quantities with small boldface letters, matrices with capital boldface letters, with all other symbols representing scalars or scalar operators.

In some special cases, it is possible for both sides of the Lax equation (7) to be pure differential operators, in which case only a finite number of the coefficients $w^{(n)}(x)$ are involved; thus the KP hierarchy "comes apart" into a number of 1 + 1-dimensional problems. The typical constraint imposed to achieve this reduction [1] is that the operator L^m is, for some integer m, a differential operator. Then, $B_m = L^m$, and the hierarchy consists of 1 + 1-dimensional equations of the form

$$\partial_k B_m = [B_k, B_m], \tag{8}$$

where m is considered fixed as k varies. Since the operators are differential, only a finite number of the coefficients $w^{(n)}(x)$ are involved in each equation. There is a generalization of this technique due to Krichever [2] in which m is allowed to be rational. One question that arises regarding these 1 + 1-dimensional reductions is: Can they be considered to approximate the true 2 + 1-dimensional dynamics of the unconstrained hierarchy in any precise way?

I think that the answer is yes. One sequence of 1 + 1-dimensional hierarchies that can be shown to approximate the KP hierarchy is the sequence of N-component vector nonlinear Schrödinger (VNLS) hierarchies which we now describe using a generalization of the procedure employed in [3]. Let $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ be two complex vector functions of $\boldsymbol{x}$, with N components each. Further, introduce a complex scalar u and an N-component vector $\boldsymbol{v}$ that satisfy the linear equation

$$\partial_1 \begin{bmatrix} u \\ \boldsymbol{v} \end{bmatrix} = \boldsymbol{M}^{(1)} \begin{bmatrix} u \\ \boldsymbol{v} \end{bmatrix} \doteq \begin{bmatrix} -\mathrm{i}\lambda & \boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}} \\ \boldsymbol{r}(\boldsymbol{x}) & \mathrm{i}\lambda \mathbb{I}_N \end{bmatrix} \begin{bmatrix} u \\ \boldsymbol{v} \end{bmatrix}, \tag{9}$$

where λ is a parameter and $\mathbb{I}_N$ is the $N \times N$ identity matrix. One may introduce a sequence of linear flows in the remaining independent variables x_k by choosing matrices $\boldsymbol{M}^{(k)}$ that are polynomials in λ of degree k, with leading term

$$\boldsymbol{M}^{(k)} = \begin{bmatrix} -\mathrm{i}\lambda^k & \boldsymbol{0}^{\mathrm{T}} \\ \boldsymbol{0} & \mathrm{i}\lambda^k \mathbb{I}_N \end{bmatrix} + \mathrm{O}(\lambda^{k-1}), \tag{10}$$

where $\boldsymbol{0}$ is the N-component zero vector, and taking

$$\partial_k \begin{bmatrix} u \\ \boldsymbol{v} \end{bmatrix} = \boldsymbol{M}^{(k)} \begin{bmatrix} u \\ \boldsymbol{v} \end{bmatrix}. \tag{11}$$

The remaining coefficients of the various powers of λ in $\boldsymbol{M}^{(k)}$ are then uniquely determined in terms of derivatives with respect to x_1 of $\boldsymbol{q}$ and $\boldsymbol{r}$ by insisting that the compatibility condition of (11) and (9),

$$\partial_k \boldsymbol{M}^{(1)} - \partial_1 \boldsymbol{M}^{(k)} + [\boldsymbol{M}^{(1)}, \mathbf{M}^{(k)}] = \boldsymbol{0}, \tag{12}$$

not involve the arbitrary parameter λ. These latter equations (12) are then the 1 + 1-dimensional nonlinear equations of the hierarchy. In particular, taking $k = 2$, the system (12) takes the form

$$\mathrm{i}\partial_2 \boldsymbol{q} + \tfrac{1}{2}\partial_1^2 \boldsymbol{q} - (\boldsymbol{q}^{\mathrm{T}}\boldsymbol{r})\boldsymbol{q} = \boldsymbol{0}, \qquad -\mathrm{i}\partial_2 \boldsymbol{r} + \tfrac{1}{2}\partial_1^2 \boldsymbol{r} - (\boldsymbol{q}^{\mathrm{T}}\boldsymbol{r})\boldsymbol{r} = \boldsymbol{0}, \tag{13}$$

which under the consistent constraint $\boldsymbol{r} = \pm\bar{\boldsymbol{q}}$ becomes an N-component generalization of the nonlinear Schrödinger equation that gives the hierarchy its name.

It is interesting to observe that the nonstationary scalar linear Schrödinger equation with potential $V(\boldsymbol{x})$

$$\mathrm{i}\partial_2 \phi + \tfrac{1}{2}\partial_1^2 \phi + V\phi = 0 \tag{14}$$

plays an important role in both the KP and N-component VNLS hierarchies. On one hand, this problem appears obviously in the KP hierarchy as the linear flow (3) for $k = 2$. On the other hand, in the VNLS hierarchies it arises as the partial linearization of (13) in the following way. Let $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ be solutions of the $N+1$-component hierarchy, with the final component of the vector $\boldsymbol{q}(\boldsymbol{x})$ being of order $\epsilon \ll 1$. Then, to leading order in ϵ, this small component satisfies an equation of the form (14) with

$$V(\boldsymbol{x}) = -\boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{r}(\boldsymbol{x})|_{\epsilon=0}\,. \tag{15}$$

Only the first N field components of $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ contribute to this potential function; to leading order these components satisfy the smaller N-component hierarchy.

In [4] it is shown how to find solutions to this partially linearized $N+1$-component VNLS problem in terms of the auxiliary functions u and $\boldsymbol{v}$ connected through the linear flows (11) with the solution $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ of the N-component VNLS hierarchy that arises at leading order. In particular, if one sets

$$\phi = u\exp(-\mathrm{i}(\lambda x_1 + \lambda^2 x_2))\,, \tag{16}$$

then ϕ is a solution of (14) with $V(\boldsymbol{x}) = -\boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{r}(\boldsymbol{x})$. In this paper, we will see how to reinterpret this relation as an embedding of the N-component VNLS hierarchy within the KP hierarchy. Among other things, it will follow that the function $V(\boldsymbol{x}) = -\boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{r}(\boldsymbol{x})$, constructed out of solutions to the N-component VNLS hierarchy, is a solution of the KP equation (5). Such an embedding has been previously reported by Freeman and West [5] in the scalar case of $N=1$. Because the embedding involves what we will see to be a concrete analytic object, namely the function ϕ, or alternatively u, we will be able to promote the notion of algebraic reductions of the KP hierarchy to an analytic level, where we will prove that a large class of solutions of the fully $2+1$-dimensional KP hierarchy can be approximated pointwise in $\boldsymbol{x}$ to arbitrary accuracy by solutions of the $1+1$-dimensional N-component VNLS hierarchy in the limit of large N.

2. Alternative definition of the hierarchies via Baker–Akhiezer functions

The KP and N-component VNLS hierarchies are defined above using purely formal calculations of operator algebra; this reasoning manages to avoid any consideration of the types of functions u, $\boldsymbol{v}$, and ϕ on which these operators act. However, to make the link between the hierarchies transparent, it is useful to develop an alternative, equivalent definition. Essentially, rather than starting first from linear differential equations themselves, one begins with a set of well-defined functions, called Baker–Akhiezer functions, and then proceeds to find a set of linear equations satisfied by them. These turn out to be the linear flows whose compatibility conditions are the nonlinear equations of the hierarchy. For our purposes, it will be more useful to think of the hierarchies as being *defined* by their Baker–Akhiezer functions.

Let us first describe the Baker–Akhiezer functions of the N-component VNLS hierarchy. Let Γ be the $N+1$ sheeted covering of the complex λ-plane on which the function r defined by

$$r^{N+1} + P_N(\lambda)r^N + P_{N-1}(\lambda)r^{N-1} + \cdots + P_0(\lambda) = 0\,, \tag{17}$$

where $P_k(\lambda)$ are polynomials in λ, is single valued. Assume that the polynomials are such that there is no branching over $\lambda=\infty$, so that there are $N+1$ points over $\lambda=\infty$, which we denote by ∞_i, for $i=0,1,\cdots,N$. Denote the genus of Γ by g.

We now introduce some functions on Γ using methods of algebraic geometry [6]. Begin by specifying $\mathcal{D} = P_1 + \cdots + P_g$, a nonspecial integral divisor of degree g on Γ. Consider a linear space of scalar functions $u(\boldsymbol{x}, P)$ depending on complex parameters $\boldsymbol{x}$, all but a finite number of which are assumed to be equal to zero, as well as a point $P \in \Gamma$. Let these functions be meromorphic on $\Gamma \setminus \bigcup_{i=0}^{N}\{\infty_i\}$, with all poles confined for all $\boldsymbol{x}$ to the divisor

$\mathcal{D}$. Further specify that at $P = \infty_0$, these functions $u(\boldsymbol{x}, P)$ have essential singularities modeled by asymptotic expansions of the form

$$u(\boldsymbol{x}, P) = \left(A + \sum_{j=1}^{\infty} u^{(0,j)}(\boldsymbol{x})\lambda(P)^{-j}\right)\exp\left(-\mathrm{i}\sum_{n=1}^{\infty} x_n\lambda(P)^n\right), \tag{18}$$

where A is some constant, while at $P = \infty_i$, for $i = 1, 2, \ldots, N$, they have essential singularities modeled by expansions of the form

$$u(\boldsymbol{x}, P) = \left(\sum_{j=0}^{\infty} u^{(i,j)}(\boldsymbol{x})\lambda(P)^{-j}\right)\exp\left(\mathrm{i}\sum_{n=1}^{\infty} x_n\lambda(P)^n\right). \tag{19}$$

Here, $\lambda(P)$ is the "sheet projection" function on Γ. In these expansions, the coefficients $u^{(i,j)}(\boldsymbol{x})$ are just "place holders" that do not need to be given any particular values at this time. We will soon see that they are, however, well-defined. It is a consequence of the complex structure on Γ that these characteristics are sufficient to restrict the space of functions $u(\boldsymbol{x}, P)$ to be one-dimensional, the space being swept out by the parameter A. This follows from the Riemann–Roch theorem; see [7] for details. We choose to set $A = 1$ and thus obtain a unique function $u(\boldsymbol{x}, P)$. Likewise, we consider the space of vector functions $\boldsymbol{v}(\boldsymbol{x}, P)$, whose components are also meromorphic away from ∞_k with poles in $\mathcal{D}$, and that at $P = \infty_0$ have essential singularities with expansions of the form

$$\boldsymbol{v}(\boldsymbol{x}, P) = \left(\sum_{j=1}^{\infty} \boldsymbol{v}^{(0,j)}(\boldsymbol{x})\lambda(P)^{-j}\right)\exp\left(-\mathrm{i}\sum_{n=1}^{\infty} x_n\lambda(P)^n\right), \tag{20}$$

and at $P = \infty_i$, $i = 1, 2, \ldots, N$, have essential singularities with expansions of the form

$$\boldsymbol{v}(\boldsymbol{x}, P) = \left(c_i\boldsymbol{e}_i\lambda + \sum_{j=0}^{\infty} \boldsymbol{v}^{(i,j)}(\boldsymbol{x})\lambda(P)^{-j}\right)\exp\left(\mathrm{i}\sum_{n=1}^{\infty} x_n\lambda(P)^n\right), \tag{21}$$

where $c_1, \ldots, c_N$ are complex constants, and $\boldsymbol{e}_i$ are the standard unit vectors in $\mathbb{C}^N$. Once again, $\boldsymbol{v}^{(i,j)}(\boldsymbol{x})$ represent undetermined coefficients. In this case, these conditions restrict the space of such functions $\boldsymbol{v}(\boldsymbol{x}, P)$ to be N-dimensional, the space being parametrized by the constants c_i. As was the case with $u(\boldsymbol{x}, P)$, fixing the values of these constants determines $\boldsymbol{v}(\boldsymbol{x}, P)$ uniquely. We thus have two functions, a scalar $u(\boldsymbol{x}, P)$ and an N-component vector $\boldsymbol{v}(\boldsymbol{x}, P)$ that are uniquely specified by the data set $(\Gamma, \mathcal{D}, c_1, \ldots, c_N)$. They can be explicitly constructed from these data in terms of the Riemann theta function canonically associated with Γ [7]. These functions are the Baker–Akhiezer functions for the N-component VNLS problem. In particular, the data set $(\Gamma, \mathcal{D}, c_1, \ldots, c_N)$ maps to explicit formulas for the coefficients $u^{(i,j)}(\boldsymbol{x})$ and $\boldsymbol{v}^{(i,j)}(\boldsymbol{x})$. To obtain them one first builds $u(\boldsymbol{x}, P)$ and $\boldsymbol{v}(\boldsymbol{x}, P)$ from the data and then expands about $P = \infty_i$.

Let us now see how the Baker–Akhiezer functions determine the equations of the N-component VNLS hierarchy. Let $\boldsymbol{M}^{(k)}$ be a matrix, depending polynomially on $\lambda(P)$, with leading behavior as given in (10). We want to determine the matrix coefficients so that we can deduce that

$$\partial_k \begin{bmatrix} u \\ \boldsymbol{v} \end{bmatrix} - \boldsymbol{M}^{(k)} \begin{bmatrix} u \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{0} \end{bmatrix}. \tag{22}$$

First, note that since $\boldsymbol{M}^{(k)}$ depends polynomially on $\lambda(P)$, each component of the left-hand side of this equation is meromorphic on Γ away from the points ∞_i, and has poles confined to the g points of the divisor $\mathcal{D}$. These properties follow directly from the corresponding properties of $u(\boldsymbol{x}, P)$ and $\boldsymbol{v}(\boldsymbol{x}, P)$. It still remains to determine the coefficients of the various powers of $\lambda(P)$ in the matrix $\boldsymbol{M}^{(k)}$. This is done by insisting that the left-hand side of

(22) behave correctly near the points ∞_i. Specifically, one defines the elements of the coefficient matrices in terms of the expansion coefficients $u^{(i,j)}(\boldsymbol{x})$ and $\boldsymbol{v}^{(i,j)}(\boldsymbol{x})$ such that on the left-hand side of (22), the first component has asymptotic expansions about $P = \infty_i$ of the same form as does $u(\boldsymbol{x}, P)$, *but with the value* $A = 0$, and the vector of N remaining components has asymptotic expansions about $P = \infty_i$ of the same form as does $\boldsymbol{v}(\boldsymbol{x}, P)$ *but with the values* $c_i = 0$. By the dimension counting used to define the Baker–Akhiezer functions themselves, then, all components of the left-hand side of (22) vanish identically on Γ for all $\boldsymbol{x}$. We are almost finished. Relation (22), considered for $k = 1$ and expanded about the points $P = \infty_i$, now provides an infinite number (one for each power of $\lambda(P)$) of relations among the coefficients $u^{(i,j)}(\boldsymbol{x})$ and $\boldsymbol{v}^{(i,j)}(\boldsymbol{x})$ and their x_1 derivatives. These relations can be telescopically solved to express all the coefficients in terms of $u^{(i,0)}(\boldsymbol{x})$ for $i = 1, \ldots, N$ and $\boldsymbol{v}^{(0,1)}(\boldsymbol{x})$ and their x_1 derivatives. If we define

$$\boldsymbol{q}(\boldsymbol{x}) = 2\mathrm{i}\left(\frac{u^{(1,0)}(\boldsymbol{x})}{c_1}, \frac{u^{(2,0)}(\boldsymbol{x})}{c_2}, \ldots, \frac{u^{(N,0)}(\boldsymbol{x})}{c_N}\right)^{\mathrm{T}}, \quad \boldsymbol{r}(\boldsymbol{x}) = -2\mathrm{i}\boldsymbol{v}^{(0,1)}(\boldsymbol{x}), \tag{23}$$

then the coefficients of the different powers of $\lambda(P)$ in the matrices $\boldsymbol{M}^{(k)}$ can be expressed in terms of $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ and their x_1 derivatives. These matrices turn out to be exactly the same as those derived in Section 1. In this way, the N-component VNLS hierarchy is determined completely by its Baker–Akhiezer functions. As an added bonus, we have also an explicit construction of a large number of simultaneous solutions $u(\boldsymbol{x}, P)$ and $\boldsymbol{v}(\boldsymbol{x}, P)$ of the linear problems of the hierarchy. For generic λ, these amount to $N + 1$ linearly independent solutions of the linear problems, and thus the consistency of these problems is guaranteed regardless of the value of λ. The fact that the functions $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ derived from the data $(\Gamma, \mathcal{D}, c_1, \ldots, c_N)$ satisfy all the nonlinear equations of the hierarchy follows immediately.

Note that one particular consequence of Eq. (9) satisfied by the Baker–Akhiezer functions is the relation

$$-2\mathrm{i}\partial_1 u^{(0,1)}(\boldsymbol{x}) = \boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{r}(\boldsymbol{x}). \tag{24}$$

We will have use for this relation below.

An important feature of this construction is that the Riemann surface Γ is not completely arbitrary, but is one on which there exists a global meromorphic function $\lambda(P)$ of degree $N + 1$. The poles of this function on Γ are the points ∞_i. This constraint on the surface Γ is not present when one considers the construction of Baker–Akhiezer functions for the KP hierarchy, as we will now see.

To define the Baker–Akhiezer function for the KP hierarchy, we summarize the construction appearing in [7]. Let Γ be an *arbitrary* Riemann surface of genus g, and choose one point on Γ, calling it ∞_0. Now, let $\lambda(P)^{-1}$ be any local parameter, defined in a neighborhood U of ∞_0, that has a simple zero at ∞_0. In this construction, it is important that $\lambda(P)^{-1}$ be considered as much a part of the set of data used to build the Baker–Akhiezer function as is the surface Γ. Now, let $\mathcal{D}$ be, as before, a nonspecial integral divisor of degree g on Γ. The linear space of scalar functions $\phi(\boldsymbol{x}, P)$ of $P \in \Gamma$ and depending on parameters $\boldsymbol{x}$, meromorphic on Γ away from ∞_0 with poles confined to the points of $\mathcal{D}$, and having essential singularities at $P = \infty_0$ with expansions in terms of the local parameter $\lambda(P)^{-1}$ of the form

$$\phi(\boldsymbol{x}, P) = \left(A + \sum_{j=1}^{\infty}\phi^{(j)}(\boldsymbol{x})\lambda(P)^{-j}\right)\exp\left(-2\mathrm{i}\sum_{n=1}^{\infty}x_n\lambda(P)^n\right) \tag{25}$$

is guaranteed by the Riemann–Roch theorem to be exactly one-dimensional, and is parametrized by the parameter A. Taking $A = 1$, we obtain a unique function $\phi(\boldsymbol{x}, P)$, determined by the data $(\Gamma, \infty_0, \lambda(P)^{-1}, \mathcal{D})$, called the Baker–Akhiezer function of the KP hierarchy.

As was the case with the N-component VNLS hierarchy, we can derive the linear flows of the KP hierarchy (3) directly from its Baker–Akhiezer function. Let B_k be a differential operator in x_1 of the form

$$B_k = (-2\mathrm{i})^{1-k}\left(\partial_1^k + \sum_{n=0}^{k-1} b_k^{(n)}(\boldsymbol{x})\partial_1^n\right), \tag{26}$$

and consider the scalar function

$$\psi_k(\boldsymbol{x}, P) = \partial_k\phi(\boldsymbol{x}, P) - B_k\phi(\boldsymbol{x}, P). \tag{27}$$

This function is meromorphic on Γ except at the point ∞_0, and its poles are confined to the points of the divisor $\mathcal{D}$. These properties follow from the form (26) of the operator B_k and the corresponding properties of the Baker–Akhiezer function $\phi(\boldsymbol{x}, P)$. Now, one uniquely defines the coefficients $b_k^{(j)}(\boldsymbol{x})$ in terms of the coefficients $\phi^{(j)}(\boldsymbol{x})$ so that $\psi_k(\boldsymbol{x}, P)$ has an expansion about $P = \infty_0$ of the form (25) but with $A = 0$. Since it then lies in a linear space of dimension 1 and has the coefficient $A = 0$, the function $\psi_k(\boldsymbol{x}, P)$ vanishes identically for all $\boldsymbol{x}$ and $P \in \Gamma$. We have thus found a sequence of linear flows of the form (3) satisfied by the Baker–Akhiezer function $\phi(\boldsymbol{x}, P)$. It may not appear immediately obvious that these equations are the same as the flows (3), since the former are written in terms of an infinite number of dependent variables $\phi^{(j)}(\boldsymbol{x})$ and the latter are written in terms of the dependent variables $w^{(j)}(\boldsymbol{x})$. The relation between the two sets of dependent variables can be found if desired by examining the asymptotic expansion about $P = \infty_0$ of the formal eigenvalue equation (6) in terms of the local parameter $\lambda(P)^{-1}$. This is one context in which the formal problem (6) makes sense: when asymptotically applied to a Baker–Akhiezer function $\phi(\boldsymbol{x}, P)$ in the neighborhood of ∞_0. However, it is not neccessary to do this in order to define the nonlinear equations of the KP hierarchy. For example, the operators B_2 and B_3 constructed in this manner involve only the coefficients $\phi^{(1)}(\boldsymbol{x})$ and $\phi^{(2)}(\boldsymbol{x})$. The compatibility conditions of the two corresponding linear equations satisfied by the Baker–Akhiezer function are two nonlinear equations in these two dependent variables, from which $\phi^{(2)}(\boldsymbol{x})$ may be eliminated to yield the KP equation (5) for the scalar function

$$V(\boldsymbol{x}) = 2\mathrm{i}\partial_1\phi^{(1)}(\boldsymbol{x}). \tag{28}$$

As was the case with the N-component VNLS hierarchies, we now have not only an alternate formulation of the KP hierarchy, but also a large family of analytic solutions $\phi(\boldsymbol{x}, P)$ of the flows (3).

3. Putting the VNLS hierarchies inside the KP hierarchy

The advantage of re-expressing the N-component VNLS hierarchies and the KP hierarchy in terms of their Baker–Akhiezer functions can now be made clear.

Theorem 1. Let Γ be an $N+1$ sheeted covering of the complex λ-plane of genus g, with some labeling $\infty_0, \ldots, \infty_N$ of the points on Γ above $\lambda = \infty$. Let $\mathcal{D}$ be a nonspecial integral divisor on Γ of degree g, and let $c_1, \ldots, c_N$ be complex constants. These data specify the Baker–Akhiezer functions $u(\boldsymbol{x}, P)$ and $\boldsymbol{v}(\boldsymbol{x}, P)$ of the N-component VNLS hierarchy. Now set

$$\phi(\boldsymbol{x}, P) \doteq u(\boldsymbol{x}, P)\exp\left(-\mathrm{i}\sum_{n=1}^{\infty} x_n\lambda(P)^n\right). \tag{29}$$

Then this function $\phi(\boldsymbol{x}, P)$ is the Baker–Akhiezer function of the KP hierarchy corresponding to the data $(\Gamma, \infty_0, \lambda(P)^{-1}, \mathcal{D})$, where $\lambda(P)$ is the global meromorphic function that identifies Γ with an $N+1$-fold covering of the complex λ-plane.

Remark. By a comparison of expansions (18) and (25), it is evident that (29) implies $\phi^{(j)}(\boldsymbol{x}) = u^{(0,j)}(\boldsymbol{x})$, and thus that all the nonlinear equations of the KP hierarchy are solved in terms of functions solving the N-component VNLS hierarchy. In particular, comparing (28) and (24), it is clear that whenever $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ are derived from Baker–Akhiezer functions according to (23), and thus satisfy the nonlinear equations of the N-component VNLS hierarchy, the function $V(\boldsymbol{x}) = -\boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{r}(\boldsymbol{x})$ is a solution of the KP equation (5). In fact, it is not difficult to verify that this statement holds for *arbitrary* solutions $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ of the N-component VNLS hierarchy, not only for those that can be derived from Baker–Akhiezer functions.

Proof of Theorem 1. The transformation (29) has the effect of altering the behavior of $u(\boldsymbol{x}, P)$ over the points ∞_i without changing the divisor $\mathcal{D}$ of its poles. Specifically, the function $\phi(\boldsymbol{x}, P)$ defined by (29) is now holomorphic at $P = \infty_i$, for $i = 1, 2, \ldots, N$, so that the only essential singularity that remains is at the point ∞_0. Moreover its poles are contained in the divisor $\mathcal{D}$, and in the neighborhood of $P = \infty_0$ it has the expansion

$$\phi(\boldsymbol{x}, P) = \left(1 + \sum_{j=1}^{\infty} u^{(0,j)}(\boldsymbol{x})\lambda(P)^{-j}\right) \exp\left(-2\mathrm{i}\sum_{n=1}^{\infty} x_n \lambda(P)^n\right), \tag{30}$$

which is of the form (25). Thus, by uniqueness of the Baker–Akhiezer function, the function defined by (29) must coincide with the Baker–Akhiezer function for the KP hierarchy corresponding to the data $(\Gamma, \infty_0, \lambda(P)^{-1}, \mathcal{D})$. This completes the proof. □

The KP hierarchy thus contains all of the N-component VNLS hierarchies. It is likely that the latter hierarchies can all be considered at the level of the pseudo-differential operator formalism discussed in Section 1 to be rational reductions of the KP hierarchy, as this is true for the scalar case of $N = 1$ [2]. We are now led to the following question: Can the definition (29), when solved for $u(\boldsymbol{x}, P)$, be considered to define Baker–Akhiezer functions for the N-component VNLS hierarchy in terms of those for the KP hierarchy? That is, is the mapping of the N-component VNLS hierarchies to the KP hierarchy surjective? Or perhaps approximately so?

4. Approximation theory

To answer this question, we now take the data $(\Gamma, \infty_0, \lambda(P)^{-1}, \mathcal{D})$ to be given, so that we have a well-defined Baker–Akhiezer function $\phi(\boldsymbol{x}, P)$ of the KP hierarchy. A little thought shows that relation (29) can yield a function $u(\boldsymbol{x}, P)$ that is one of the Baker–Akhiezer functions for the N-component VNLS hierarchy only if the local parameter $\lambda(P)^{-1}$ has an analytic continuation on Γ to a global meromorphic function with simple zeros. One of these zeros is automatically at the point ∞_0, and the remaining zeros then *define* the points $\infty_1, \ldots, \infty_N$, and thus the number of components N. That is, when it exists, the global function $\lambda(P)$ serves to identify the Riemann surface Γ with an $N + 1$-fold covering of the complex λ-plane. From the function $u(\boldsymbol{x}, P)$ and the knowledge of $\lambda(P)^{-1}$ and its zeros, it is possible to reconstruct the function $\boldsymbol{v}(\boldsymbol{x}, P)$ and thus the potentials $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$.

Clearly, the obstruction to inverting the relation (29) is that not every local parameter $\lambda(P)^{-1}$ in the neighborhood of $P = \infty_0$ has such a global analytic continuation. However, every local parameter $\lambda(P)^{-1}$ can be written as a convergent (in the neighborhood U) power series, each partial sum of which is a global meromorphic function on Γ. This means that every local parameter $\lambda(P)^{-1}$ can be approximated to arbitrary accuracy by meromorphic functions. To show this, we make use of the fact that *every* Riemann surface has at least one nonconstant meromorphic function [8]. Without loss of generality, we can suppose that this function $\psi(P)$ vanishes at the point ∞_0 and that the differential $d\psi(P)$ does not vanish at ∞_0. Then, if the neighborhood U is sufficiently small, the local parameter $\lambda(P)^{-1}$ can be expressed in the form

$$\lambda(P)^{-1} = \sum_{n=1}^{\infty} \alpha_n \psi(P)^n, \tag{31}$$

which converges uniformly in U. If $\psi(P)$ is a function of degree d, then it is easy to see that the approximation $\lambda_M(P)^{-1}$ obtained by truncating at $n = M$ is a meromorphic function of degree Md, generically having simple zeros. If we use the approximate data $(\Gamma, \infty_0, \lambda_M(P)^{-1}, \mathcal{D})$ to build a Baker–Akhiezer function $\phi_M(\boldsymbol{x}, P)$ for the KP hierarchy, then relation (29) can be inverted to yield a Baker–Akhiezer function for the N-component VNLS hierarchy, where $N = Md - 1$. Arbitrary Baker–Akhiezer functions $\phi(\boldsymbol{x}, P)$ of the KP hierarchy can thus be approximated by Baker–Akhiezer functions $\phi_M(\boldsymbol{x}, P)$ that may be identified with Baker–Akhiezer functions of the N-component VNLS hierarchy by relation (29). Throughout this analysis, the vector $\boldsymbol{x}$ was considered to be a fixed parameter. This means that the convergence of $\phi_M(\boldsymbol{x}, P)$ to $\phi(\boldsymbol{x}, P)$ occurs *pointwise* in $\boldsymbol{x}$. Since the nonlinear hierarchies are solved in terms of the expansion coefficients that converge pointwise for all $\boldsymbol{x}$, and since at each level of approximation the Baker–Akhiezer functions of the KP hierarchy and an N-component VNLS hierarchy can be identified via (29), *arbitrary solutions of the KP hierarchy derived from Baker–Akhiezer functions may be approximated pointwise to arbitrary accuracy by solutions of N-component VNLS hierarchies, in the limit of large N.*

5. Conclusions

Two main ideas have been presented in the above paragraphs. First, the KP hierarchy contains the equations of the N-component VNLS hierarchies for all N. Second, solutions of the KP hierarchy that are obtained from Baker–Akhiezer functions can be approximated pointwise in $\boldsymbol{x}$ to arbitrary accuracy by solutions of these contained hierarchies.

The transformation (29) can be viewed as a kind of generalized Miura transformation, the classical version of which connects the Korteweg–de Vries equation, which is a special case of the KP equation, and the modified Korteweg–de Vries equation, which is contained within the 1-component VNLS hierarchy. A large class of generalized Miura transformations can be obtained at the level of Baker–Akhiezer functions by solving certain Riemann factorization problems [9]. A closely related point of view is that the relation (29) is part of a *gauge transformation* [10] linking the linear flows of two different hierarchies. In this case, the gauge transformation (29) has the unusual feature that it is not always invertible. It is also interesting that another N-component VNLS hierarchy, generally different from that considered in this paper for $N > 1$, is contained within a multiple-component version of the KP hierarchy itself [11].

Throughout the paper we have considered solutions of the KP and N-component VNLS hierarchies that are generally complex. It is interesting that the reality conditions on the algebro-geometric data required to impose that $\boldsymbol{r}(\boldsymbol{x}) = \pm\overline{\boldsymbol{q}(\boldsymbol{x})}$ are sufficient to also guarantee that the solutions of the KP hierarchy are real, when the two hierarchies may be identified via (29).

Although we have worked with the restricted class of solutions that are derived from Baker–Akhiezer functions, it seems likely that the explicit formulas that connect the various coefficients $w^{(n)}(\boldsymbol{x})$ in the pseudo-differential operator L to the solutions $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ of the N-component VNLS hierarchy and their x_1 derivatives should in fact hold for *arbitrary* solutions of the hierarchy. This is certainly the case for the coefficient $w^{(1)}(\boldsymbol{x}) = -\boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{r}(\boldsymbol{x})$. As mentioned above, this function satisfies the KP equation (5) when $\boldsymbol{q}(\boldsymbol{x})$ and $\boldsymbol{r}(\boldsymbol{x})$ are *any* solutions of the N-component VNLS hierarchy.

In this way, the genuinely 2+1-dimensional dynamics of the equations of the KP hierarchy can be approximated by the simpler 1 + 1-dimensional dynamics of the equations of a sequence of larger and larger VNLS hierarchies. Being integrable themselves, these approximate KP hierarchies could be used to develop powerful numerical methods for solving the KP equations. These approximate hierarchies are *not* discretizations of the KP hierarchy, although such integrable discretizations have been reported previously [12]. By contrast with the VNLS hierarchies, it is not clear whether these discretized KP hierarchies are contained within the KP hierarchy itself.

Acknowledgements

I would like to thank N.M. Ercolani and D. Pelinovsky for helpful discussions.

References

[1] E. Date, M. Kashiwara, M. Jimbo and T. Miwa, Transformation groups for soliton equations, in: Non-linear Integrable Systems – Classical Theory and Quantum Theory, Proc. RIMS Symp., Kyoto, Japan, 1981, eds. M. Jimbo and T. Miwa (World Scientific, Singapore, 1983).
[2] I.M. Krichever, Linear operators with self-consistent coefficients and rational reductions of KP hierarchy, Physica D 87 (1995) 14–19.
[3] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, The inverse scattering transform – Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 249–315.
[4] P.D. Miller and N.N. Akhmediev, Schrödinger equations with solvable time-dependent potentials and the initial value problem, preprint.
[5] M.D. Freeman and P. West, On the quantum KP hierarchy and its relation to the non-linear Schrödinger equation, Phys. Lett. B 295 (1992) 59–66.
[6] I.M. Krichever, Methods of algebraic geometry in the theory of non-linear equations, Russian Math. Surveys 32 (1977) 185–213.
[7] B.A. Dubrovin, Theta functions and non-linear equations, Russian Math. Surveys, 36 (1981) 11–92.
[8] H.M. Farkas and I. Kra, Riemann Surfaces, 2nd Ed. (Springer, New York, 1992).
[9] G. Wilson, Infinite-dimensional Lie groups and algebraic geometry in soliton theory, Philos. Trans. Roy. Soc. London ser. A 315 (1985) 393–404.
[10] L.D. Fadeev and L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons (Springer, Berlin, 1987).
[11] V.G. Kac and J.W. van de Leur, The *n*-component KP hierarchy and representation theory, in: Important Developments in Soliton Theory, eds. A.S. Fokas and V.E. Zakharov (Springer, Berlin, 1993).
[12] D. Gieseker, A lattice version of the KP equation, Acta Math. 168 (1992) 219–248.