Modes of periodic waveguides

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Received March 6, 1997

We calculate exactly the two bound Floquet modes of a periodic linear waveguide induced in a medium by a second-order soliton of the nonlinear Schrödinger equation. The modes are degenerate at the writing frequency, having the same quasi-propagation constant, which suggests applications of our method to spectral filtering.

Periodic waveguides are used in many practical applications such as dispersion compensation in wavelength-division multiplexing systems[1] and pulse compression.[2] From the point of view of perturbation theory,[3] a small periodic modulation of a channel waveguide generally serves to couple power between bound and unbound modes of the unperturbed channel. Now it can be shown that there is an operating frequency at which the coupling between any given bound mode and the continuum vanishes to leading order. This fact suggests that true bound modes may exist to all orders for the periodic structure. However, it is not at all clear *a priori* whether more than one such bound mode can coexist within the periodic waveguide.

The mathematical setting for beam propagation in planar waveguides with slow periodic modulations is the Floquet theory of the nonstationary linear Schrödinger equation

$$i\phi_z + \frac{1}{2}\phi_{xx} + \Delta n(x,z)\phi = 0, \qquad \Delta n(x, z + L) = \Delta n(x, z) . \tag{1}$$

The period map $T(z){:}\phi(x,z) \mapsto \phi(x, z + L)$ is a unitary linear operator whose discrete eigenfunctions are bound states of the periodic waveguide specified by the refractive-index distribution $\Delta n(x,z)$. The corresponding eigenvalues are the Floquet multipliers, which can be written in the form $\exp(i\beta L)$; the numbers β are then called the quasi-propagation constants. This theory reduces to the usual modal theory if the refractive index does not depend on z; the Floquet modes and their quasi-propagation constants go over to bound modes of the waveguide and their propagation constants, respectively.

In general the construction of the period map T and its subsequent spectral decomposition require numerical methods. However, certain aspects of the theory of nonlinear optics permit the design of periodic waveguides with useful properties in a self-consistent way[4] that sidesteps these calculations. For example, the induced refractive-index distribution of a second-order soliton in a Kerr medium is a self-induced periodic waveguide for which all Floquet eigenfunctions can be found explicitly. Below we show that this periodic linear waveguide has exactly two bound Floquet modes. Of course, superpositions of these two modes, which are true modes of the structure, have zero scattering loss. Our main tools are the linearization theorems,[5] which are intimately related to the integrability of the nonlinear Schrödinger equation (NLSE), and which we use to construct exact solutions of Eq. (1). The two bound Floquet modes are degenerate in the sense that their quasi-propagation constants are equal, and so the symmetry properties of an initial condition built up from these modes will be preserved along the waveguide. There are also implications of this degeneracy for the behavior of the waveguide when the operating frequency is varied. The techniques that we describe can be used to calculate the bound modes of quasi-periodic higher-order soliton waveguides, although we concentrate here on the two-mode case, as it admits of simple exact solutions in closed form.

Since the linear waveguides that we study have index profiles that coincide with the intensity distributions of fields that satisfy the NLSE, there are two interpretations of our results. On the one hand, the index profile can be considered to be permanently written in the planar medium, say, by a photolithographic process. Such a device is completely passive. On the other hand, we can exploit the fact that the NLSE itself describes the propagation of intense light in a Kerr medium and use a pump beam to induce the device that then guides a weak probe beam of a nearby frequency or in the orthogonal polarization. Such a device would be active, since one can change the shape of the waveguide dynamically by controlling the pump. Slow nonlinear effects such as the photorefractive effect can also be used to write waveguides with light that can later be used passively, as the medium retains some memory of previous excitations.

The NLSE is

$$i\psi_z + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0 , \tag{2}$$

where x is the transverse variable and z is the axis of propagation. The two-soliton solution[6] of Eq. (2) with solitons of unequal amplitudes b_1 and b_2 that are bound together at $x = 0$ propagating along the z axis is

$$\psi(x,z) = \frac{4i({b_2}^2 - {b_1}^2)[b_1 \cosh(2b_2x)\exp(2i{b_1}^2z) - b_2\cosh(2b_1x)\exp(2i{b_2}^2z)]}{(b_1 - b_2)^2C_+(x) + (b_1 + b_2)^2C_-(x) - 4b_1b_2\cos\phi(z)} , \tag{3}$$

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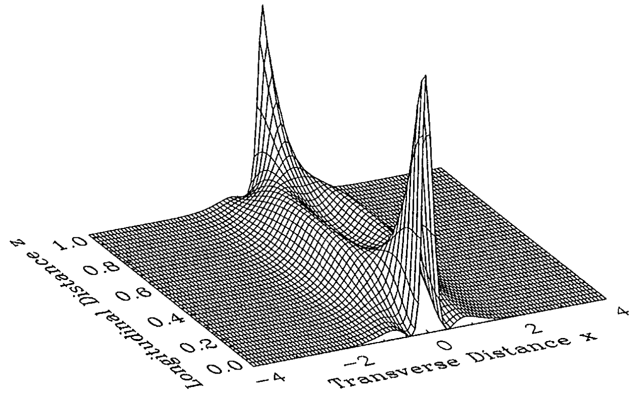

Fig. 1. Periodic waveguide-squared refractive index and even-mode intensity distribution.

where $C_\pm(x) = \cosh[2(b_2 \pm b_1)x]$ and $\phi(z) = 2({b_2}^2 - {b_1}^2)z$. Without loss of generality, we take $b_2 > b_1$. This solution has the Floquet form in z. This means that the solution can be written in the form

$$\psi(x,z) = u(x,z)\exp(i\beta z)\,, \tag{4}$$

where $u(x,z)$ is periodic in z with period $L = \pi/({b_2}^2 - {b_1}^2)$ and the quasi-propagation constant is $\beta = 2{b_1}^2 \equiv 2{b_2}^2 \pmod{2\pi/L}$. The periodicity of the function $u(x,z)$ can be interpreted as the nonlinear interference between the two soliton components of the solution. The periodic refractive-index profile $|\psi(x,z)|^2$ self-induced by the two-soliton bound state is shown in Fig. 1. The two free parameters, b_1 and b_2, can be varied to meet design specifications of the periodic waveguide.

We wish to find the bound Floquet modes of a periodic linear waveguide with a refractive-index distribution given by $\Delta n(x,z) = |\psi(x,z)|^2$. To find the explicit expressions for these modes we use a technique related to the integrability of the NLSE. The NLSE is the compatibility condition for the set of two linear differential equations[7]

$$\begin{pmatrix} r \\ s \end{pmatrix}_x = \begin{bmatrix} -i\lambda & \psi \\ -\psi^* & i\lambda \end{bmatrix}\begin{pmatrix} r \\ s \end{pmatrix},$$

$$\begin{pmatrix} r \\ s \end{pmatrix}_z = \begin{bmatrix} -i\lambda^2 + i\,\dfrac{1}{2}|\psi|^2 & \lambda\psi + i\,\dfrac{1}{2}\,\psi_x \\ -\lambda\psi^* + \dfrac{i}{2}\,{\psi_x}^* & i\lambda^2 - \dfrac{i}{2}|\psi|^2 \end{bmatrix}\begin{pmatrix} r \\ s \end{pmatrix}. \tag{5}$$

These two linear problems, with the arbitrary complex parameter λ, are said to form a Lax pair for the NLSE. By compatibility, two linearly independent simultaneous solutions can be found for all complex λ whenever $\psi(x,z)$ satisfies the NLSE and in particular when $\psi(x,z)$ is the two-soliton bound-state solution given by Eq. (3).

We now set $\Delta n(x,z) = |\psi(x,z)|^2$, where $\psi(x,z)$ is the particular solution of Eq. (3). The task at hand is to find solutions of linear Schrödinger equation (1). Comparison of this linear equation with Eq. (2) shows immediately that there is always a particular solution ϕ that coincides with $\psi(x,z)$. This solution is the even Floquet mode of the periodic waveguide. It turns out that there is also an odd bound Floquet mode; let us now find an expression for this mode. It has been shown[5] that many solutions of linear equation (1) can be found from simultaneous solutions of the Lax pair. The solution formula is simple. Take the first component $r(x,z,\lambda)$ of any simultaneous solution of Eqs. (5) and set

$$\phi(x,z,\lambda) = r(x,z,\lambda)\exp(-i\lambda x - i\lambda^2 z)\,. \tag{6}$$

For Eq. (6) to yield a solution to Eq. (1) it is sufficient that r satisfy

$$ir_z + \frac{1}{2}\,\lambda^2 r + \frac{1}{2}\,r_{xx} - i\lambda r_x + r|\psi|^2 = 0\,. \tag{7}$$

But Eq. (7) holds for all complex λ, as a consequence of the compatible systems of Eqs. (5), so that the function ϕ given by Eq. (6) is a solution of Eq. (1) for all λ. This result makes it possible to solve linear Schrödinger equation (1) for a large class of nonstationary potentials of the form $\Delta n(x,z) = |\psi(x,z)|^2$, where ψ solves the NLSE. The Floquet theory of the periodic waveguide of interest in this Letter is a special case of this general theory.

Exact simultaneous solutions of Eqs. (5) corresponding to the second-order soliton solution (3) can be obtained by algebraic techniques, after which finding a particular solution $\phi(x,z)$ of Eq. (1) then amounts to substituting $r(x,z,\lambda)$ into Eq. (6), instantiating Eq. (6) on particular values of λ, and then applying linear superposition.[5] We choose $\lambda = ib_1$ and $\lambda = ib_2$ and then use superposition to isolate the odd and the even parts of the field. This procedure yields an even Floquet mode $\phi_{b_1,b_2}^{\text{even}}(x,z)$, which is proportional to solution (3) of the NLSE that generates the periodic waveguide, and an odd Floquet mode proportional to

$$\phi_{b_1,b_2}^{\text{odd}}(x,z) = \frac{({b_2}^2 - {b_1}^2)[\sinh(2b_2x)\exp(2i{b_1}^2z) - \sinh(2b_1x)\exp(2i{b_2}^2z)]}{(b_2 - b_1)^2C_+(x) + (b_2 + b_1)^2C_-(x) - 4b_1b_2\cos\phi(z)}\,. \tag{8}$$

The intensity profiles of these two modes are shown in Figs. 1 and 2.

Since these are Floquet modes their field profiles are recovered periodically along the waveguide, modulo phase factors (the Floquet multipliers) that can be read off from the exact solution formulas. Examination of these Floquet multipliers reveals that the quasi-propagation constants β^{even} and β^{odd} of the two Floquet modes are equal, modulo $2\pi/L$. To our knowledge this is the first example in the optics literature of a periodic waveguide with degenerate Floquet modes, or indeed of a planar waveguide with degenerate bound modes. Now, as in any linear problem, the general bound solution of the waveguide problem is given by a superposition of the individual bound modes, in this case one even and one odd mode:

$$\phi(x,z) = A^{\text{even}}\phi_{b_1,b_2}^{\text{even}}(x,z) + A^{\text{odd}}\phi_{b_1,b_2}^{\text{odd}}(x,z)\,, \tag{9}$$

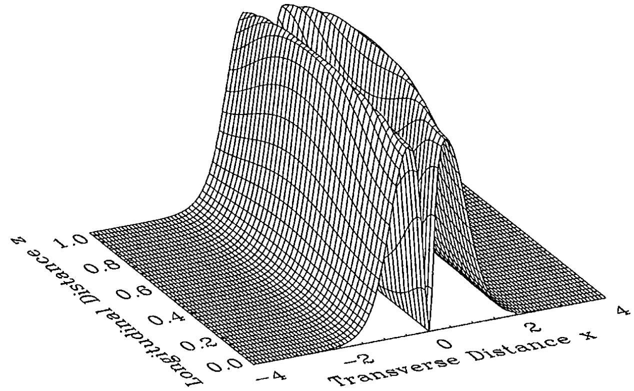

Fig. 2. Odd-mode intensity distribution.

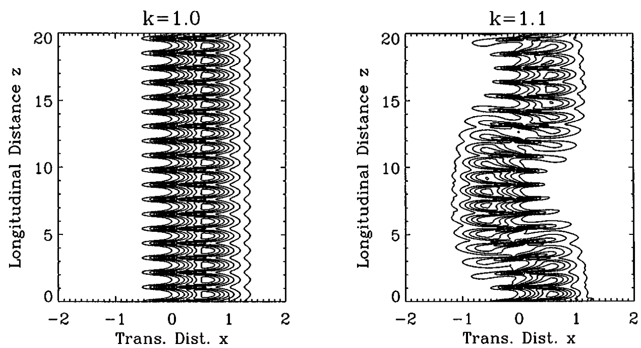

Fig. 3. Contour plot of the intensity distribution resulting from propagation of an initially asymmetric beam at $k = 1.0$ and $k = 1.1$.

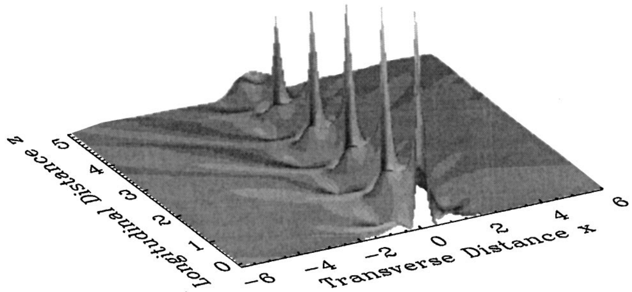

Fig. 4. Even mode-intensity distribution for $k = 1.25$.

with arbitrary coefficients A^{even} and A^{odd}. Owing to the degeneracy of the quasi-propagation constants, this bound solution has the property that any phase difference between the two components is reproduced exactly after one period of the waveguide, as is the field profile (modulo phase). Thus, the bound mode [Eq. (9)] propagates without losses, and its intensity has the same periodicity as the waveguide itself. An initially asymmetric pulse would therefore maintain its form along the device, as is shown in Fig. 3 for $k = 1.0$. This suggests applications of the periodic waveguide in optical imaging: The problem of beating between modes of a standard multimode waveguide is eliminated, and so it is possible to transmit extra information (i.e., the source asymmetry) along the waveguide without accumulative interference effects. Further, since one can also use the techniques described here to construct periodic and quasi-periodic waveguides with an arbitrary number of Floquet modes, this method of imaging could be extended to permit transmission of more-complicated information along the waveguide.

A second potential application of this periodic waveguide is related to spectral filtering. If we fix the induced refractive-index profile from a second-order soliton at one frequency, then linearly propagate a field at a different frequency, the diffraction coefficients in Eqs. (1) and (2) will no longer agree, and the above exact results will no longer be valid. The appropriate equation to solve is now the linear Schrödinger equation given by

$$ik\phi_z + \frac{1}{2}\phi_{xx} + k^2|\psi(x,z)|^2\phi = 0\,, \qquad (10)$$

where k is the ratio of the propagating frequency to the writing frequency. We have used perturbation methods and numerical techniques to study the effects of slightly changing the parameter k.[8] The effects take place on two scales. First, the degeneracy of the even and odd Floquet modes is broken, leading to beating phenomena on longitudinal length scales that are inversely proportional to $|k - 1|$, as illustrated in Fig. 3. This beating could be used as a sensitive method of frequency detection around the writing frequency ($k = 1$), where the beat length is infinite. Second, the Floquet modes decay, radiating power into the surrounding medium as shown in Fig. 4. The scattering of power is clearly visible over five periods of the device at $k = 1.25$ and in general will occur on length scales that are inversely proportional to $|k - 1|^2$. This decay process would permit the design of a spectral bandpass filter.

It should be said that, as we are using Schrödinger equation (10) to model propagation in periodic waveguides, backward reflections are neglected. Thus, although the above analysis shows that at the writing frequency there is no scattering of the light into forward-propagating radiation modes, a complete analysis of these structures requires the use of the scalar Helmholtz equation. However, the smooth and slow periodic profile of this waveguide, when transformed from paraxial coordinates back to unscaled laboratory coordinates, is one for which reflections are negligible.

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