Physica Scripta. Vol. 61, 518–525, 2000

A Coupled Korteweg-de Vries System and Mass Exchanges Among Solitons

Peter D. Miller*[,1] and Peter L. Christiansen†[,2]

[1] Department of Mathematics and Statistics, Monash University, Clayton, Victoria 3168 Australia
[2] Department of Mathematical Modelling, The Technical University of Denmark, DK2800 Lyngby, Denmark

Received August 20, 1999; accepted November 10, 1999

PACS Ref: 03.40.Kf, 68.10.-m

Abstract

We study an N-component, symmetrically coupled system of Korteweg-de Vries (KdV) equations that is integrable in the context of the $sl(N+1)$ AKNS hierarchy. We show how the coupled system can be solved through a combination of the well-known inverse-scattering transform for (one-component) KdV and the solution of a linear equation with nonconstant coefficients. The coupled KdV system may be viewed as a phenomenological model for the sharing of mass among interacting solitons of the (one-component) KdV equation. Results for the scattering theory of solutions of the nonconstant coefficient linear equation arising in the solution of the coupled system are used to quantify the redistribution of mass during soliton collisions within the framework of the coupled KdV model.

1. Introduction

This paper is about the system of coupled equations:

$$\partial_t u_k + \partial_x \left[\frac{u_k}{2} \sum_{j=1}^{N} u_j + \partial_x^2 u_k \right] = 0, \quad k = 1, \ldots, N. \tag{1}$$

For $N > 1$ this system is a completely symmetrical multicomponent generalization of the Korteweg-de Vries equation. In this paper, we will show that this system is integrable by giving the Lax pair, which itself is a natural generalization of that for the scalar KdV equation. We will then show that for this coupled system of equations, unlike for other members of the corresponding hierarchy, there is an efficient solution technique that combines two established methods for scalar equations: the inverse-scattering transform for the KdV equation [1] and the analysis of a certain linear PDE having a given solution of KdV as a nonconstant coefficient [2].

A very interesting application of the coupled system (1) is to the old question of soliton identity in collisions. An observer of two KdV solitons involved in a collision is easily able to distinguish the two individual solitons before their interaction, and again after their interaction. But during the collision this clear identity becomes confused. If one looks at pictures of interacting solitons, one gets the impression that when the relative velocity is large, the faster soliton simply passes through the slower soliton. There does not seem to be any difficulty distinguishing the solitons in this case because they spend so little time interacting. On the other hand, if the relative velocity is small, the two solitons appear to slow down as they approach and ‘‘bounce off’’ each other with an exchange of mass through their tails. It looks like the slower soliton has siphoned off some energy from the faster soliton and run off with it. In this case, the fast soliton that emerges might be said to be made up of some mixture of contributions from the two solitons that originally entered the collision, and in this sense soliton identity is not preserved.

There is some evidence for each of these points of view in the mathematical structure of the multisoliton solutions of the KdV equation. The idea that solitons *always* maintain their identity throughout all interactions (corresponding to the first scenario above) is given a mathematical framework in a series of papers by Moloney and Hodnett [3]. Using algebraic properties of multisoliton solutions $w(x, t)$ of KdV suggested by the Hirota formalism, they broke $w(x, t)$ into a sum of components $u_k(x, t)$, each of which has a constant mass. Moreover, the fields $u_k(x, t)$ each had the form of a solitary wave with a time-dependent amplitude and phase shift. If each function $u_k(x, t)$ is associated with a soliton of $w(x, t)$ in this way, then it may be said that each soliton has a well-defined amplitude and center of mass for each t, even during the complicated interaction. This gives the solitons identity throughout their interaction. This point of view has been further developed in several works of Fuchssteiner [4] who places the division of $w(x, t)$ into noninteracting parts on a Lie-theoretic footing, representing the Lie algebra of multisoliton interactions in KdV as a direct sum in such a way that each direct summand is isomorphic to the symmetry algebra of an isolated soliton. The approach is quite general, and for KdV the resulting decomposition turns out to be the same as that obtained in [3].

The notion that during collisions the solitons ‘‘bounce off’’ each other with some exchange of mass can also be argued, as shown by Bowtell and Stuart [5]. They studied soliton interactions by analyzing the singularity structure of multisoliton solutions. For fixed t, a two-soliton solution $w(x, t)$ of the KdV equation is meromorphic in the finite complex x-plane. As $t \to \pm\infty$, the singularities are double poles that are periodically distributed on two lines parallel to the imaginary x-axis moving to the right at constant speeds. The motion of the poles may be traced for finite t, revealing that the poles of each soliton remain in vertical lines that undergo a repulsive interaction as the solitons approach each other. This model suggests that solitons of the KdV equation do not pass through each other at all but rather exchange their identities during interactions, with the larger and faster soliton giving up just enough of its mass to the smaller and slower soliton to effect this exchange.

The reason for describing these approaches to soliton interactions is to point out that one interesting application

* e-mail: millerpd@mail.maths.monash.edu.au
† e-mail: plc@imm.dtu.dk

of the coupled KdV system (1) is to provide a new, quantitative way to interpret soliton identity during interactions. The main idea is to focus on a conserved quantity that is transported by the equation. The KdV equation

$$\partial_t w + \partial_x \left[\frac{1}{2} w^2 + \partial_x^2 w \right] = 0 \,, \tag{2}$$

is in the form of a local conservation law for the mass density $w(x, t)$. That is, for appropriate boundary conditions, the total mass

$$M[w] \doteq \int_{-\infty}^{\infty} w \, \mathrm{d}x \tag{3}$$

is a conserved quantity. In a multisoliton solution, the individual solitons can really only be identified in the limits $t \to \pm\infty$, where their velocities, amplitudes, and wave shapes are the same before and after their nonlinear interaction. In particular, this means that if a soliton is traveling with velocity c as $t \to -\infty$ and is carrying some amount of mass M, then as $t \to +\infty$, the soliton traveling at speed c will also carry M units (or "quanta") of mass.

Now we would like to allow for the possibility that the solitons exchange some mass during their interactions, and this means that the M quanta of mass transported away from the interaction in the soliton traveling with velocity c may not be the same M quanta as entered the interaction region in the soliton with velocity c (see the schematic diagram in Fig. 1).

To address this problem quantitatively, we must introduce internal degrees of freedom into the KdV equation. We need to consider the mass density field $w(x, t)$ to be composed of a number of distinguishable contributions. Thus, like Moloney and Hodnett, we write:

$$w(x, t) = \sum_k u_k(x, t) \,. \tag{4}$$

The $u_k(x, t)$ are the internal degrees of freedom. Rather than determining them using properties of special solutions of KdV (as in [3]) or equivalently in an attempt to give the solitons unambiguous identity at the Lie-theoretic level (as in [4]), we seek a more physically motivated description by postulating equations of motion[1] for a system of unknowns $u_k(x, t)$ that imply the KdV equation for $w(x, t)$ subject to (4). We impose several physical constraints on the system of equations:

1 **Conservation.** The individual fields $u_k(x, t)$ are conserved local densities.
2 **Symmetry.** The system of equations is invariant under any permutation of the individual fields $u_k(x, t)$.
3 **Homogeneity.** The evolution equations admit the reduction obtained by setting any subset of the $u_k(x, t)$ identically equal to zero.

These three criteria are intended to model a physical division of mass into different quantities. The conservation property means that no kind of mass can be created or destroyed. The symmetry property means that the different kinds of quanta are merely *labeled* differently and thus are all interchangeable from the point of view of their dynamics. Finally, the homogeneity property guarantees that it is physically reasonable to presume that some species of mass may not be present at all. In view of the relation (4) we will consider the homogeneity condition replaced by the stronger condition:

3′ **Linearity.** The evolution equation for $u_k(x, t)$ is linear once the KdV equation has been solved for the sum $w(x, t)$.

A splitting (4) of the KdV mass density according to these criteria will thus result in a nonlinear coupled model for a number of fields $u_k(x, t)$. Such a model can be interpreted as a way of giving internal degrees of freedom to the KdV mass density field $w(x, t)$; these internal degrees of freedom are essentially linear, since all the nonlinearity is contained in the scalar KdV equation for the sum $w(x, t)$.

The linearity of the equations for $u_k(x, t)$ given the corresponding solution $w(x, t)$ of the KdV equation (which could be obtained for given initial conditions $u_k(x, 0)$ by summing and applying the inverse-scattering transform to the resulting initial condition for w) does not necessarily make them amenable to analysis. Only very special linear equations connected with the KdV equation can be studied using techniques related to the integrability of KdV. More typically, linear equations with coefficients depending on a solution $w(x, t)$ of KdV can have solutions that are scattered from the soliton trajectories before, during, or after interactions [2]. Worse yet, the initial value problem for the linear equation may be ill-posed. In view of the integrability of the KdV equation itself, we might wish to further constrain the dynamics of the internal degrees of freedom by imposing

4 **Integrability.** The coupled nonlinear system for the fields $u_k(x, t)$ should be integrable, having a Lax pair representation, and an associated method of exact solution.

The integrability property would imply that the quantitative details of the mass redistribution process could be worked out exactly.

It is a direct matter to check that the coupled KdV system (1) satisfies the conservation, symmetry, and linearity criteria, and that the sum of the components (4) satisfies KdV (2). We will show below that it also satisfies the integrability condition. This makes the coupled KdV system

[1] The special soliton components $u_k(x, t)$ studied by Moloney and Hodnett [3] in fact do satisfy the coupled system of equations

$$\partial_t u_k + w\partial_x u_k + \partial_x^3 u_k = 0 \,,$$

which are also the "interacton" equations of Fuchssteiner [4] for KdV. Although the total mass of each component $u_k(x, t)$ turns out to be conserved for the particular solutions considered in [3], the masses are not individually conserved for general solutions. We will impose general mass conservation on models we consider. The point is that the interacton equations are only intended to have meaning when $w(x, t)$ is a multisoliton solution of KdV, and then only when special solutions are considered (the "interactons") corresponding to the decoupled solitary waves $u_k(x, t)$ studied by Moloney and Hodnett. The theory we propose differs from that in [3] and [4] in that the models we will consider will describe the interaction of several mass density fields that in principle can superpose to form an arbitrary solution of KdV. Furthermore, for a given solution of $w(x, t)$ of KdV for the sum of the degrees of freedom, we will want to admit the most general possible solution of the coupled system. In particular, the number N of fields $u_k(x, t)$ need not have anything to do with the number M of solitons in the KdV field $w(x, t)$.

a good model for quantifying mass exchanges among solitons.

In Section 2, we briefly describe the integrability of the coupled KdV system (1). This will serve to at once place the system in a familiar context, the $sl(N+1)$ AKNS hierarchy, and also to introduce the solution method we choose. This method is not based directly on an inverse-scattering transform for (1), but rather on the simpler transform for KdV and the analysis of an associated linear equation. Then, in Section 3, we use the solution method and results from [2] to explicitly quantify the exchange of mass among KdV solitons according to the coupled KdV model. We briefly illustrate the results with a numerical experiment in Section 4 before concluding with a few remarks.

2. Integrability of the coupled KdV system

In this section, we identify the coupled KdV system (1) as a member of a well-known integrable hierarchy, and then use this fact and some additional structure to offer a specialized solution technique that under some circumstances suffices to solve the initial value problem for (1). First, we show that the system (1) can be obtained as a real reduction of one of the nonlinear equations in the $sl(N+1)$ AKNS [6] or vector nonlinear Schrödinger hierarchy. Let $\boldsymbol{q} \in C^N$ and $\boldsymbol{r} \in C^N$ be N-component vector potentials, $\boldsymbol{f} \in C^{N+1}$ an $N+1$-component vector, and $\lambda \in C$ a complex spectral parameter. The vector nonlinear Schrödinger hierarchy is the set of nonlinear equations that are obtained as pairwise compatibility conditions of the linear problem

$$\partial_x \boldsymbol{f} = \begin{bmatrix} \frac{\lambda}{2\mathrm{i}} & \boldsymbol{q}^T \\ \boldsymbol{r} & -\frac{\lambda}{2\mathrm{i}} I \end{bmatrix} \boldsymbol{f}, \tag{5}$$

where I is the $N \times N$ identity matrix, with linear problems of the form

$$\partial_{t_n} \boldsymbol{f} = \left\{ \frac{\lambda^{n+1}}{2\mathrm{i}} \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & -I \end{bmatrix} + \sum_{m=0}^{n} \lambda^m \mathbf{V}^{(nm)} \right\} \boldsymbol{f}, \tag{6}$$

where $\boldsymbol{0}$ is the N-component zero vector. The matrix coefficients $\mathbf{V}^{(nm)}$ are taken to be traceless[2] and then are uniquely chosen so that the compatibility condition between (5) and (6) is independent of λ. If we restrict attention to even values of n, then for all $k = 1, \ldots, N$ we may consistently set $q_k = 1$ and take $r_k = u_k/6$ to be real-valued for real x and t_n. As functions of x and $t = t_2$, the compatible potentials $u_k(x, t)$ satisfy the coupled KdV equations (1).

[2] Since the trace of the matrices appearing in the linear problems (5) and (6) is always contributed by the leading term which is independent of the potentials $\boldsymbol{q}$ and $\boldsymbol{r}$, the trace can always be removed by the elementary scalar gauge transformation

$$\boldsymbol{f} = \boldsymbol{g} \exp\left(\frac{\mathrm{i}}{2}\frac{N-1}{N+1}\left(\lambda x + \sum_{n=1}^{\infty} \lambda^{n+1} t_n\right)\right),$$

where in formulas of this type it is assumed that all but a finite number of the variables t_n are zero to avoid convergence questions. This shows that the vector fields for $\boldsymbol{g}$ are all in $sl(N+1)$ and define local flows for $\boldsymbol{g}$ on $SL(N+1)$.

In principle, this integrable structure can be used to find exact solutions of (1) through Bäcklund transformation methods and also to solve the initial value problem for (1) through an inverse-scattering transform constructed from the spectral theory of the linear problem (5). However, there are certain technical difficulties in the inverse-scattering theory for the problem (5) connected with the fact that the diagonal matrix proportional to λ on the right-hand side of (5) has nondistinct eigenvalues [7] for $N > 1$.

So, rather than pursuing this direction, let us point out an alternative method of constructing solutions of (1), which in many cases leads to the general solution of the initial value problem. This method makes use of the fact that the sum of the individual fields $u_k(x, t)$ solves the scalar KdV equation (2), which is well understood, and also the fact that once the KdV equation is solved for the sum $w(x, t)$, the equations for the individual fields $u_k(x, t)$ are linear. Thus, given the solution $w(x, t)$ of the KdV equation corresponding to the initial data $w(x, 0) = u_1(x, 0) + \ldots + u_N(x, 0)$, we want to construct solutions of the linear equation

$$\partial_t u + \partial_x \left[\frac{1}{2} w u + \partial_x^2 u \right] = 0, \tag{7}$$

in which $w(x, t)$ is now interpreted as a given nonconstant coefficient. Each component field $u_k(x, t)$ satisfies this linear equation.

Fortunately, the linear equation (7) can be solved using essentially the same machinery as was involved in solving the scalar KdV equation for the sum $w(x, t)$. Particular solutions of (7) can be obtained from the following fact. Due to the block structure of the coefficient matrices in the linear problems, the first component of the vector $\boldsymbol{f}$ plays a distinguished role. Thus, it can be shown that whenever the vector $\boldsymbol{f} \in C^{N+1}$ satisfies the linear equations of the vector nonlinear Schrödinger hierarchy (5) and (6), the scalar function[3]

$$\phi = f \exp\left(-\frac{1}{2\mathrm{i}} \left(\lambda x + \sum_{n=1}^{\infty} \lambda^{n+1} t_n \right) \right), \tag{8}$$

where f is the first component of $\boldsymbol{f}$, satisfies all of the linear equations of a certain Kadomtsev–Petviashvili (KP) hierarchy, in which the scalar KP potentials are built from $\boldsymbol{q}$ and $\boldsymbol{r}$ [8]. In the special case of $q_k = 1$ and $r_k = u_k/6$, one of the linear equations in that KP hierarchy is

$$\partial_t \phi + \frac{1}{2} w \partial_x \phi + \partial_x^3 \phi = 0, \tag{9}$$

where $t \doteq t_2$ and $w = \boldsymbol{q}^T \boldsymbol{r} = u_1 + \ldots + u_N$. This equation is exactly the potential form of (7), as seen by setting $u = \partial_x \phi$. So we know how to find solutions of (7) from the first component of $\boldsymbol{f}$. This procedure becomes more effective when we observe that under the restriction $q_k = 1$ and $r_k = u_k/6$, the first component f of a vector $\boldsymbol{f}$ satisfying the Lax pair (5) and (6) decouples and satisfies the scalar

[3] See the remark in the above footnote regarding the interpretation of this infinite sum.

equations

$$\partial_x^2 f = -\frac{\lambda^2}{4} f - \frac{1}{6} w f, \qquad \partial_t f = \frac{1}{6}\partial_x w \cdot f + \left(\lambda^2 - \frac{1}{3} w\right)\partial_x f. \tag{10}$$

These are of course the well-known Lax representation of the scalar KdV equation (2) itself, and in solving the KdV equation for the sum $w(x, t)$ by inverse-scattering, we have already obtained f as a by-product! In summary, solutions of the linear equation (7), where $w(x, t)$ is a solution of the KdV equation (2), are constructed from simultaneous solutions $f(x, t; \lambda)$ of the KdV Lax pair (10) by the formula

$$u(x, t) = \partial_x \left[f(x, t; \lambda) \exp\left(-\frac{\lambda}{2\mathrm{i}} x - \frac{\lambda^3}{2\mathrm{i}} t \right) \right]. \tag{11}$$

After the fact, it is of course easy to verify directly that this formula represents a large family of solutions of (7) indexed by the complex parameter λ and the integration constants arising from the simultaneous solution of (10).

At this point it is perhaps useful to draw a comparison between the equation (7) and the well-known *linearized KdV equation*[4]

$$\partial_t \varepsilon + \partial_x \left[w\varepsilon + \partial_x^2 \varepsilon \right] = 0, \tag{12}$$

which is obtained from (2) and a solution w thereof by making the subsitution $w \to w + \varepsilon$ and keeping only the linear terms in ε. The only difference between (7) and (12) is the factor of 2, but this cannot be removed by rescaling x and t because w depends on x and t according to (2). In fact, the equation (7) is not a linearized KdV equation for *any* solution $w(x, t)$. A formula similar to (11) for solving (12) was known to the discoverers of the integrability of the KdV equation [9]. Their formula is

$$\varepsilon(x, t) = \partial_x \left[f(x, t; \lambda)^2 \right]. \tag{13}$$

The parametric dependence on λ and the integration constants used to specify f means that there are many such solutions of (12). In fact, they form a complete set [10] and they thus span the tangent space of the manifold of KdV solutions. By contrast with the formula (13), which is quadratic in the eigenfunction f, the formula (11) is only linear in f.

One explanation for this linearity in the solution formula (11) is that, whereas the linear equation (7) is *not* a linearized KdV equation for any solution $w(x, t)$ of (2), it *does* arise in the linearization of the coupled system (1). Substituting $u_k + \varepsilon_k$ for u_k in (1) and keeping only terms linear in the ε_k gives

$$\partial_t \varepsilon_k + \partial_x \left[\frac{1}{2} \varepsilon_k \sum_{j=1}^{N} u_j + \frac{1}{2} u_k \sum_{j=1}^{N} \varepsilon_j + \partial_x^2 \varepsilon_k \right] = 0. \tag{14}$$

The homogeneity property of (1) guarantees that we can linearize around a particular solution for which $u_n(x, t) \equiv 0$ for some n, and for which the remaining fields satisfy a coupled system of the same form. In such a linearization, the equation for ε_n decouples and takes the form of (7). In general, solutions of the linearization (14) can be expressed in terms of quadratic forms in solutions of the compatible system (5) and (6). This is in complete analogy with the way one solves the linearized KdV equation (12). Now, if $u_n(x, t) \equiv 0$, then one of the factors in the quadratic expression for $\varepsilon_n(x, t)$ becomes trivial, making its appearance in the solution formula (11) as the exponential function. This linear formula can thus be interpreted in the context of quadratic eigenfunction expansions, with the eigenfunctions coming from the vector Lax pair (5) and (6) for the coupled system (1), rather than from the KdV Lax pair (10).

Just as the squared eigenfunctions form a complete set [10] in which to expand the solution of the initial value problem for (12), it has been shown [2] that when $w(x, t)$ is a *multisoliton solution* of KdV, the functions defined by (11) are also complete and can be used to solve the initial value problem for (7) and therefore for the coupled system (1). The completeness relation established in [2] probably holds in some form for more general solutions of KdV, and of course the formula (11) *always* provides a large number of exact solutions of (7) even if it is not known how to superpose these solutions to satisfy given side conditions. In the next section, we will use the results of time-dependent scattering theory carried out for the linear problem (7) in [2] to study the problem of redistribution of mass quanta among solitons according to the coupled KdV model (1).

3. Mass exchanges among solitons

Here, we use the solution method for the coupled KdV system (1) described in Section 2 along with some long-time asymptotic results for the linear equation (7) established in [2] to describe the exchanges of mass among interacting solitons that are predicted by the model (1).

Using the coupled KdV model (1) to describe the exchange of mass that occurs during the interactions of solitons of the KdV equation requires finding some exact solutions of (1). In order to represent a decomposition of the KdV mass density during such interactions, these solutions need to have the property that the sum of the fields $w(x, t)$ is a multisoliton solution of the KdV equation. It follows that the individual fields $u_k(x, t)$ will be particular solutions of the linear equation (7), where $w(x, t)$ is a multisoliton solution. The transformation formula (11) gives us a large number of such solutions if we can write down the corresponding simultaneous solutions of (10).

Finding these eigenfunctions leads us to briefly summarize the famous reflectionless potential theory of Kay and Moses [11]. For some integer $M > 0$ (the number of solitons), a simultaneous solution of (10) is assumed in the form

$$f_+(x, t; \lambda) = \left(1 + \sum_{n=0}^{M-1} \lambda^{n-M} f_n(x, t) \right) \exp\left(\frac{\lambda}{2\mathrm{i}} x + \frac{\lambda^3}{2\mathrm{i}} t \right). \tag{15}$$

The coefficients $f_n(x, t)$ are determined by choosing M positive real numbers $\eta_1 > \eta_2 > \ldots > \eta_M$ and M real numbers $\alpha_1, \ldots, \alpha_M$ and insisting that

$$f_+(x, t; 2\mathrm{i}\eta_n) = (-1)^{n+1} \exp(2\eta_n \alpha_n) f_+(x, t; -2\mathrm{i}\eta_n), \tag{16}$$

for $n = 1, \ldots, M$. The function $f_+(x, t; \lambda)$ is then a simul-

[4] The linearized KdV equation plays roughly the same role in the interacting particle picture of Moloney and Hodnett [3] and Fuchssteiner [4] that equation (7) plays in our theory.

taneous solution of the two linear problems of the Lax pair for the KdV equation as long as one takes as a definition

$$w(x,t) \doteq 6\mathrm{i}\partial_x f_{M-1}(x,t)\,. \tag{17}$$

By compatibility of the equations of the Lax pair, this latter function solves the KdV equation (2). It corresponds to the interaction of M solitons, which means that as $t \to \pm\infty$, it decouples into a sum of the form

$$\begin{aligned} w(x,t) &\sim \sum_{n=1}^{M} w_n^{\pm}(x,t) \\ w_n^{\pm}(x,t) &\doteq 12\eta_n^2 \mathrm{sech}^2(\eta_n(x-\alpha_n^{\pm}) - 4\eta_n^3 t)\,, \end{aligned} \tag{18}$$

where the asymptotic phase constants $\alpha_n^{\pm}$ are functions of the η_n and α_n. Another linearly independent solution of the Lax pair for the same potential $w(x,t)$ is then given by $f_-(x,t;\lambda) = f_+(x,t;-\lambda)$.

Note that, by comparing (7) with (2) it is clear that one obvious solution of (7) is in fact $u(x,t) = w(x,t)$. Since the equation (7) is local and linear, it follows that when $w(x,t)$ is an M-soliton solution of KdV, there exists for any $k = 1,\ldots,M$ a (unique) particular solution $u = u_k(x,t)$ of the linear equation (7) satisfying $u_k(x,t) \to w_k^-(x,t)$ as $t \to -\infty$ (or more precisely, $u_k(x-4\eta_k^2 t, t) - w_k^-(x-4\eta_k^2 t, t) \to 0$ in some appropriately weighted L^p norm in x). Thus, we may represent $w(x,t)$ as a sum over k of these solutions of (7) (see Fig. 2 for a schematic diagram).

Taken together, these M functions are solutions of the coupled KdV model (1). In this configuration, the conserved local density $u_k(x,t)$ has the interpretation of the mass density that is unambiguously carried by the KdV soliton with velocity $c = 4\eta_k^2$ for large negative times. One expects some redistribution of the mass to occur as the solitons begin to interact, such that after the interaction each soliton emerges carrying some mixture of the masses. We can use scattering theory results [2] for the linear equation (7) to calculate the behavior of the component fields $u_k(x,t)$ as $t \to +\infty$ and therefore determine the mixture of the masses carried by each outgoing soliton.

An important feature that makes this analysis possible is the integrability condition satisfied by the coupled KdV equation as mass transport model. The integrability ensures that the functions $u_k(x,t)$ that are defined to be asymptotically confined to the (incoming) soliton trajectories as $t \to -\infty$ are also asymptotically confined to the (outgoing) soliton trajectories as $t \to \infty$. In typical nonintegrable coupled models satisfying the conservation, symmetry, and linearity criteria, radiation is generated in each of the individual fields $u_k(x,t)$ during soliton collisions, which propagates away from all of the soliton trajectories. In this case, the integrability of KdV simply implies the rather weak constraint that the pointwise sum of this radiation over all M fields must be zero, because the sum is necessarily an M-soliton solution of KdV which has no radiation component. If radiation not asymptotically confined to

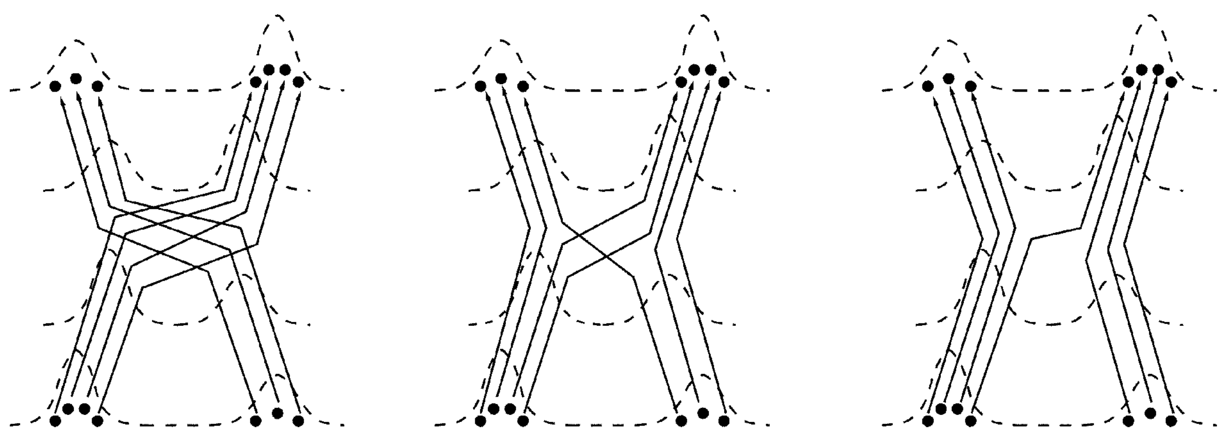

Fig. 1. Three ways that mass "quanta" might be redistributed during a soliton interaction.

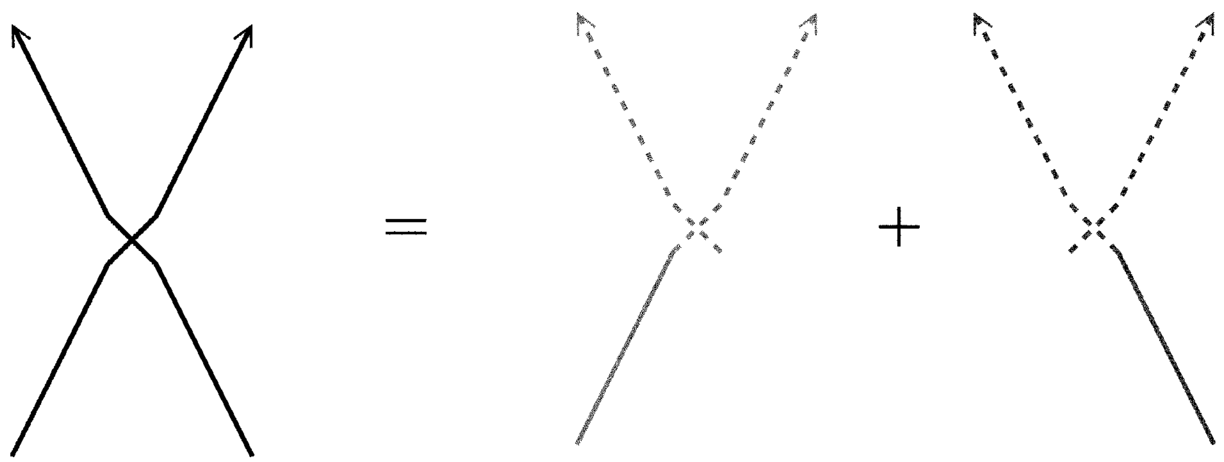

Fig. 2. Decomposition of a two-soliton solution $w(x,t)$ of KdV into two parts $u_1(x,t)$ and $u_2(x,t)$ satisfying the coupled KdV system and determined by their asymptotic behavior as $t \to -\infty$.

the soliton trajectories is present, it becomes necessary to include sources and sinks of various species of mass near $x = \pm\infty$ to complete the picture. This complicates the interpretation of solutions of the coupled system in the context of solitons exchanging mass.

In the paper [2], it is shown that if $w(x, t)$ is an M-soliton solution of KdV corresponding to the data set $\{\eta_1, \ldots, \eta_M; \alpha_1, \ldots, \alpha_M\}$, and if as $t \to -\infty$ a solution $u(x, t)$ of (7) has the form

$$u(x, t) \sim \sum_{n=1}^{M} \beta_n w_n^-(x, t)\,, \tag{19}$$

for some coefficients $\beta_1, \ldots, \beta_M$, then the same solution has the asymptotic description

$$u(x, t) \sim \sum_{n=1}^{M} \left[\sum_{m=1}^{M} T_{nm}\beta_m \right] w_n^+(x, t)\,, \tag{20}$$

as $t \to +\infty$ for some matrix $\mathbf{T}$ which turns out to depend only on the soliton eigenvalues $\eta_1, \ldots, \eta_M$. $\mathbf{T}$ is the *scattering matrix* for the linear equation (7). It is computed as follows. First, solve for Q_{mk} for $m = 1, \ldots, M$ and $k = 0, \ldots, M-2$ by solving the linear inhomogeneous system

$$\begin{aligned} &(-2i\eta_n)^{-1} + \sum_{k=0}^{M-2}(-2i\eta_n)^{k-M} Q_{mk} = 0\,, \quad n = 1, \ldots, m-1,\\ &(2i\eta_n)^{-1} + \sum_{k=0}^{M-2}(2i\eta_n)^{k-M} Q_{mk} = 0\,, \quad n = m+1, \ldots, N. \end{aligned} \tag{21}$$

Next, build the $M \times M$ matrices $\mathbf{G}^{\pm}$ by setting

$$G_{km}^- \doteq -\frac{1}{12\eta_k} + \frac{1}{6i}\sum_{n=0}^{M-2}(2i\eta_k)^{n-M} Q_{mn}^*. \tag{22}$$

$$G_{km}^+ \doteq -\frac{1}{12\eta_k} + \frac{1}{6i}\sum_{n=0}^{M-2}(2i\eta_k)^{n-M} Q_{mn}. \tag{23}$$

Finally, the scattering matrix is defined by

$$\mathbf{T} \doteq \left[(\mathbf{G}^-)^{-1}\mathbf{G}^+\right]^T. \tag{24}$$

The matrix $\mathbf{T}$ can now be used to compute the manner in which an amount of mass unambiguously traveling with the soliton moving with speed $c = 4\eta_m^2$ as $t \to -\infty$, is divided up by (7) during the soliton interaction to be redistributed in various portions traveling at speeds $c = 4\eta_n^2$ as $t \to +\infty$. Since the mass carried by each soliton is proportional to the corresponding eigenvalue:

$$M[w_n^{\pm}] \doteq \int_{-\infty}^{\infty} w_n^{\pm}(x, t)\,dx = 12\eta_n \int_{-\infty}^{\infty} \operatorname{sech}^2(y)\,dy\,, \tag{25}$$

we see by integrating (19) and (20) with respect to x that the *mass transfer matrix* $\mathbf{M}$, whose elements are defined by

$$M_{nm} \doteq \frac{\eta_n}{\eta_m} T_{nm}\,, \tag{26}$$

expresses the redistribution of mass by the linear problem (7). If Q quanta of mass are traveling in from $t = -\infty$ in the potential well of the mth soliton moving with speed $c = 4\eta_m^2$, then they will be divided by the soliton interaction in the potential function $w(x, t)$ so that, for each n, exactly $M_{nm}Q$ quanta are traveling out to $t = +\infty$ with speed $c = 4\eta_n^2$. By the relation of the linear equation (7) to the coupled nonlinear system (1), the mass transfer matrix encodes the redistribution process of mass during soliton interactions in the KdV equation, according to the coupled KdV model. For example, if $\mathbf{M}$ were the identity matrix, then the prediction would be that solitons do not share any mass, as suggested by the interacton theory in [3,4], and shown schematically in the left-hand diagram of Fig. 1. More generally, *the deviation of the mass transfer matrix from the identity matrix is a measure of how many quanta of mass are exchanged by solitons of the KdV equation during their interaction.* For the coupled KdV system, $\mathbf{M}$ is *never* the identity matrix.

Before proceeding to any examples, it is worth stressing a very important property of these matrices — that they are algebraically constructed out of the numbers η_k alone. There is no dependence on the numbers α_k, which determine the centers of mass of the various solitons at some time when they can be distinguished one from the other. Thus, the matrices $\mathbf{T}$ and $\mathbf{M}$ for an M-soliton solution with parameters η_k in which the solitons all collide more or less at the same time will be the same as the matrices for an M-soliton solution with the same parameters η_k in which the solitons collide only pairwise. This implies that all transfer matrices can be expressed in terms of those for the two-soliton interaction, for example using the procedure described in [12].

4. The two-soliton case. A numerical experiment

Since they are so fundamental, let us calculate explicitly the matrices $\mathbf{T}$ and $\mathbf{M}$ for the particular case of an interaction of two solitons. The result will depend only on the real positive values $\eta_1 > \eta_2$. First calculating the numbers Q_{mk} and then using (24) gives

$$\mathbf{T} = \frac{1}{\eta_1^2 - \eta_2^2}\begin{bmatrix} (\eta_1 - \eta_2)^2 & 2\eta_2(\eta_1 - \eta_2) \\ 2\eta_1(\eta_1 - \eta_2) & -(\eta_1 - \eta_2)^2 \end{bmatrix}. \tag{27}$$

Finally, applying (26) gives

$$\mathbf{M} = \frac{1}{\eta_1^2 - \eta_2^2}\begin{bmatrix} (\eta_1 - \eta_2)^2 & 2\eta_1(\eta_1 - \eta_2) \\ 2\eta_2(\eta_1 - \eta_2) & -(\eta_1 - \eta_2)^2 \end{bmatrix}. \tag{28}$$

Note that the law of conservation of mass is reflected in this matrix by the fact that for each k,

$$\sum_{j=1}^{2} M_{jk} = 1\,, \tag{29}$$

and thus all mass carried by a soliton before the interaction is divided *without loss* among the solitons emerging from the interaction. The analogous result holds for the $M \times M$ mass transfer matrix for the interaction of M solitons. This is one property that would be lost if integrability of the coupled model were not present, since it would be possible via radiation for mass to be exchanged at infinity.

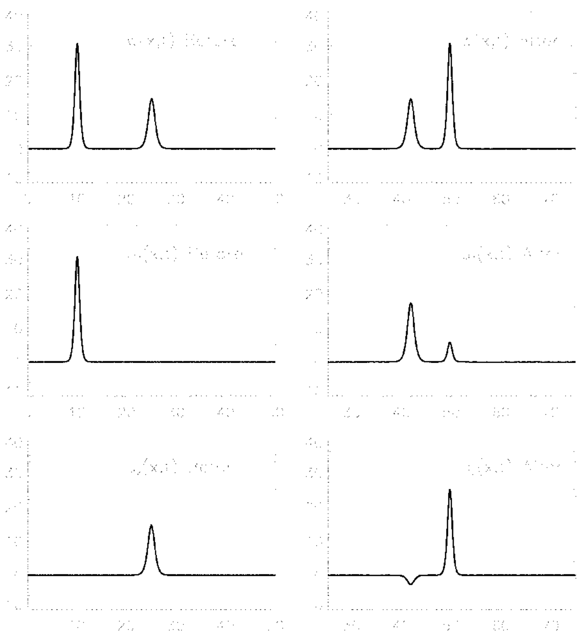

Fig. 3. KdV solitons exchanging mass according to the coupled KdV model. Left: before the interaction. Right: after the interaction. Bottom row: $u_2(x, t)$. Middle row: $u_1(x, t)$. Top row: $w(x, t) = u_1(x, t) + u_2(x, t)$.

The exchange of mass quantified precisely by the matrix **M** can be illustrated and confirmed by numerical simulations of the coupled KdV system (1).

The first row of Fig. 3 shows snapshots of a solution $w(x, t)$ of the KdV equation at two times, one before and one after the interaction of two solitons. Larger solitons overtake smaller ones as they travel to the right, and the horizontal scale has been translated to bring the solitons after their interaction back into the center of the picture. The KdV field $w(x, t)$ can be split into two mass density fields $u_1(x, t)$ and $u_2(x, t)$ that satisfy the two-component coupled KdV system (1). The second and third rows of Fig. 3 show the corresponding snapshots of particular solutions $u_1(x, t)$ and $u_2(x, t)$ each of which has the form of an isolated soliton as $t \to -\infty$. These mass density fields may be equivalently regarded as solutions of the linear equation (7) that satisfy $u_1(x, t) + u_2(x, t) = w(x, t)$ (again, the situation schematically represented in Fig. 2). Note that the field $u_2(x, t)$, which was initially confined to the potential well of the smaller of the two solitons making up $w(x, t)$, actually develops a *mass deficit* in one of the potential wells after the collision. This interesting feature is universal to all soliton interactions in the coupled system (1), and is predicted by the mass transfer matrix **M**, in which *the element M_{22} is always negative*, while all other matrix elements are positive.

The mass transfer matrix can also be used to describe solutions of the coupled KdV system (1) that have asymptotics for large t different from those illustrated in Fig. 2. For example, one might consider the problem of determining what mixture of the mass densities must be present in each KdV soliton as $t \to -\infty$ in order that each soliton emerges from the interaction carrying only one species of mass. This question is answered by considering other solutions $u_k(x, t)$ of the linear problem (7) that satisfy $u_k \sim w_k^+$ as $t \to +\infty$ and examining the asymptotic behavior as $t \to -\infty$. It is easy to see that the relevant mixtures in this limit are given by the columns of the inverse of the mass transfer matrix.

5. Conclusion

The coupled KdV system (1) is a fully symmetric integrable generalization of the KdV equation. It is a member of the $sl(N + 1)$ AKNS hierarchy, and by virtue of its representation as a collection of linear equations for N internal degrees of freedom coupled to an autonomous KdV equation for the sum, the system can be solved by an effective technique not directly requiring inverse-scattering for the $(N + 1) \times (N + 1)$ scattering problem (5).

One application of this coupled nonlinear system is as a phenomenological model that allows one to precisely quantify the process of redistribution of mass among interacting solitons of the KdV equation. This model gives a complete description of several mass density fields that are trapped by their own net dynamics, the latter being described by the KdV equation. The description holds for an arbitrary number of fields and prescribes them for all values of x and t, even during the complicated interaction of the KdV solitons. This fact allows "mass quanta" of different types to be traced as they are transported through the KdV field.

Using known scattering theory for the linear equation (7), it is possible to compute the large time asymptotics for all solutions of the coupled KdV system that are asymptotically suitably confined to the KdV soliton trajectories. Assuming that the number of fields is equal to the number of solitons in the solution of KdV, we can unambiguously place a distinguished type of mass in each distinct soliton for large negative times and determine the mixtures present in each soliton for large positive times. This leads to the algorithmic construction of a mass transfer matrix **M** whose elements give the mass fractions of each species ultimately captured by each KdV soliton. There are always some elements of the mass transfer matrix that are negative. It follows that the coupled KdV model makes the interesting prediction that some mass is always borrowed locally from the continuum during soliton collisions. These results contribute one more interpretation of soliton interactions to the several [3–5] already in the literature. The coupled KdV system (1) gives a new way of quantitatively thinking about soliton identity during interactions.

The integrability of the coupled KdV system (1) should provide a starting point for the study of a large class of physical problems. Indeed, coupled KdV systems of various kinds occur naturally in physical models when two or more distinct long wave modes with (nearly) degenerate phase speeds are weakly coupled in the presence of weak nonlinearity and dispersion (see, for example, Hirota and Satsuma [13], Gear and Grimshaw [14], and Gottwald *et al.* [15]). Of particular interest in the context of mass transport are some possible applications to the propagation of internal waves in deep water. In a sequence of impressive experiments, Davis and Acrivos [16] have shown that internal solitary waves of sufficiently large amplitude can in fact *trap a bubble of fluid*, carrying it along for some distance. More recent experiments by Stamp and Jacka [17] using a tracer fluid dye have shown that two such waves can be made to collide, *resulting in the exchange of some* *of the trapped fluid.* Analysis of the type presented above for the coupled KdV model (1) is prototypical of what will need to be carried out (in a more complicated model, no doubt) to describe effectively this observed mass exchange.

Acknowledgements

We would like to thank F. Campbell, J. Parkes and B. Fuchssteiner for useful discussions regarding interacting soliton equations. We also thank K. Lamb for pointing out the possible application to fluid trapping by internal waves.

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