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The $\overline{\partial}$ Steepest Descent Method and the Asymptotic Behavior of Polynomials Orthogonal on the Unit Circle with Fixed and Exponentially Varying Nonanalytic Weights

K. T.-R. McLaughlin and P. D. Miller

1 Introduction

1.1 Asymptotic analysis of Riemann-Hilbert problems

The steepest descent method for asymptotic analysis of matrix Riemann-Hilbert problems was introduced by Deift and Zhou in 1993 [14]. A matrix Riemann-Hilbert problem is specified by giving a triple $(\Sigma, \mathbf{v}, \mathcal{N})$ consisting of an oriented contour Σ in the complex z-plane, a matrix function $\mathbf{v} : \Sigma \to SL(N)$ which is usually taken to be continuous except at self-intersection points of Σ where a certain compatibility condition is required, and a normalization condition $\mathcal{N}$ as $z \to \infty$. If Σ is not bounded, certain asymptotic conditions are required of $\mathbf{v}$ in order to have compatibility with the normalization condition. Consider an analytic function $\mathbf{M} : \mathbb{C} \setminus \Sigma \to SL(N)$ taking continuous boundary values $\mathbf{M}_+(z)$ (resp., $\mathbf{M}_-(z)$) on Σ from the left (resp., right). The Riemann-Hilbert problem $(\Sigma, \mathbf{v}, \mathcal{N})$ is then to find such a matrix $\mathbf{M}(z)$ satisfying the normalization condition $\mathcal{N}$ as $z \to \infty$ and the jump condition $\mathbf{M}_+(z) = \mathbf{M}_-(z)\mathbf{v}(z)$ whenever z is a non-self-intersection point of Σ (so the left and right boundary values are indeed well defined). The steepest descent method of Deift and Zhou applies to certain Riemann-Hilbert problems where the jump matrix $\mathbf{v}(z)$ depends on an auxiliary control parameter, and is a method for extracting asymptotic properties of the solution $\mathbf{M}(z)$ (and indeed proving the existence and

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uniqueness of solutions along the way) when the control parameter tends to a singular limit of interest.

The original method put forth in [14] bears a striking resemblance to the well-known steepest descent method or saddle-point method for analyzing contour integrals with exponential integrands. A distinguished point on Σ is identified (analogous to a point of stationary phase) and an explicit change of variables of the form $\mathbf{N}(z) = \mathbf{M}(z)\mathbf{t}(z)$ where $\mathbf{t}(z)$ is a piecewise analytic matrix is introduced in the vicinity of this point and it is observed that (i) the matrix $\mathbf{N}(z)$ satisfies an equivalent Riemann-Hilbert problem with a new contour $\Sigma_\mathbf{N}$ and a new jump matrix $\mathbf{v}_\mathbf{N}$, and (ii) the jump matrix $\mathbf{v}_\mathbf{N}$ converges to the identity matrix in the singular limit of interest for all z bounded away from the stationary-phase point. One therefore expects that a good approximation to $\mathbf{N}(z)$ can be constructed by an explicit local analysis near the stationary-phase point. With the explicit local approximant $\dot{\mathbf{N}}(z)$ constructed, one uses it in a final change of variables $\mathbf{H}(z) = \mathbf{N}(z)\dot{\mathbf{N}}(z)^{-1}$ and observes (i) that $\mathbf{H}(z)$ satisfies a Riemann-Hilbert problem with a possibly new contour $\Sigma_\mathbf{H}$ and a new explicit jump matrix $\mathbf{v}_\mathbf{H}$ and (ii) that the new jump matrix $\mathbf{v}_\mathbf{H}$ is now *uniformly* close to the identity matrix in the limit of interest. This allows one to construct $\mathbf{H}(z)$ by iteration of certain singular integral equations that are equivalent to any given Riemann-Hilbert problem, and to show that $\mathbf{H}(z)$ is uniformly close to the identity matrix in any region bounded away from $\Sigma_\mathbf{H}$. With additional work, it may in some circumstances be shown that $\mathbf{H}(z)$ is close to the identity uniformly right up to the contour $\Sigma_\mathbf{H}$ (this usually requires more detailed information about the jump matrix). In this way, one obtains a formula $\mathbf{M}(z) = \mathbf{H}(z)\dot{\mathbf{N}}(z)\mathbf{t}(z)^{-1}$ for the solution that can be used to compute *directly* an asymptotic expansion of $\mathbf{M}(z)$ valid in the singular limit of interest.

Since the introduction of the steepest descent method for Riemann-Hilbert problems, there have been several key developments. In [12, 15], a technique was established in which one makes a change of variables involving a matrix constructed from a single unknown scalar function $g(z)$ analytic in $\mathbb{C}\setminus\Sigma$. The transformation modifies the jump matrix in a way involving the boundary values $g_\pm(z)$ taken on Σ. One then chooses relations between the boundary values of $g(z)$ such that the transformed Riemann-Hilbert problem becomes asymptotically simple. The desired conditions amount to a *scalar* Riemann-Hilbert problem for $g(z)$, which is easily solved in many circumstances. A crucial feature of this method is that the dominant contribution to the solution typically comes from subintervals of the contour Σ of finite length rather than from isolated points. Here, we therefore see an important difference between singular limits of matrix Riemann-Hilbert problems and evaluation of saddle-point integrals. In the contributing intervals, the transformed jump matrix has a factorization (see (1.1), (3.43), and (4.15)) whose factors admit analytic continuation to the left and right of each such interval. A further change of variables based on this analytic factorization is carried out in lens-shaped regions surrounding each contributing interval. Ultimately, a model problem is solved (typically in terms of Riemann theta functions of genus related to the number of contributing intervals) and along with local analysis near the endpoints of the intervals a model for $\mathbf{M}(z)$ is built and compared with $\mathbf{M}(z)$ to obtain a Riemann-Hilbert problem for the error $\mathbf{H}(z)$. When the method is successful, the jump matrix for $\mathbf{H}(z)$ is uniformly close to the identity, and thus $\mathbf{H}(z)$ may be constructed via iteration of integral equations. Significantly, the conditions imposed on the boundary values of $g(z)$ can often be viewed as the Euler-Lagrange conditions for a certain variational problem (see [13] as well as [11, 10] and Appendix A of this paper).

A further development emerged from problems in which it was recognized that no appropriate function $g(z)$ can be found relative to the given contour Σ. In [19] and later in [3], it was shown how analyticity of the jump matrix could be exploited to effectively deform arcs of the contour Σ to alternative locations in the complex plane such that the jump matrix maintains the same functional form; specific locations of the arcs are determined such that there exists an appropriate function $g(z)$ as above. These selected arcs are the closest relatives in the noncommutative theory to the paths of steepest descent from saddle points in the asymptotic theory of contour integrals. The contour selection principle was also encoded into a variational problem in [19].

More recently [4, 19, 21], new techniques have been added to the framework of the steepest descent method that are adapted for determining the asymptotic contribution to the solution of a coalescence of a large number of poles in the unknown matrix (this is strictly speaking not a Riemann-Hilbert problem in the sense described above due to the polar singularities, however the problem is first converted into a standard Riemann-Hilbert problem by explicit transformations). The key idea here is to exploit certain analytic interpolants of given residues at the poles.

For the fundamentally nonlinear cases in which the dominant contribution comes from subintervals of a contour, a central feature is that the analytical methods rely on piecewise analyticity of the given jump matrix and of the boundary values of the scalar function $g(z)$. For some of the cases of long-time asymptotics of integrable nonlinear partial differential equations [9, 14], as well as the recent long-time asymptotic analysis for perturbations of the defocusing nonlinear Schrödinger equation [16], the dominant contribution comes from isolated points of the contour Σ, and while analyticity is not fundamental, the asymptotic calculations proceed by an approximation argument, in which an analytic part is deformed away, and a (small) residual contribution is handled by technical and analytical prowess. The approximation argument is delicate and requires detailed analysis that depends sensitively on the geometry of the particular contour Σ.

Far from being a mere pursuit of abstraction, a simple asymptotic technique that applies to Riemann-Hilbert problems regardless of whether the jump matrix is analytic or not would have immediate application in a number of important areas. For example, a unified treatment of the asymptotic theory of orthogonal polynomials on the real line with general nonanalytic weights would allow the resolution of universality conjectures from random matrix theory in the most natural and general context (see [11, 10] for the analytic case, and [20] for an application of the $\overline{\partial}$ steepest descent method described in this paper to the nonanalytic but convex case). As another example, if it were possible to treat systematically problems with a large number of poles that accumulate in a very regular but nonanalytic fashion, then the important problem of semiclassical asymptotics for the focusing nonlinear Schrödinger equation with general nonanalytic initial data could begin to be addressed.

In this paper, we present a new generalization of the steepest descent method for Riemann-Hilbert problems that applies in absence of analyticity of the jump matrix, and yet does not depend on an approximation argument for the jump matrix. While we believe that the ideas we will develop in this paper are useful in very general contexts, we have chosen to focus on a particular application of the steepest descent method in order to demonstrate the technique.

1.2 The essence of the $\overline{\partial}$ steepest descent method

As mentioned above, after changing variables using an appropriate scalar function $g(z)$, the jump matrix is converted into a form that is well suited for further asymptotic analysis. A common "target" form for the jump matrix in certain arcs of Σ is the following form:

$$\mathbf{v}(x) = \begin{pmatrix} e^{i\kappa(x)/\epsilon} & 1 \\ 0 & e^{-i\kappa(x)/\epsilon} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-i\kappa(x)/\epsilon} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{i\kappa(x)/\epsilon} & 1 \end{pmatrix}, \tag{1.1}$$

where x is a real parameter along the arc of Σ which for simplicity here we assume that it lies on the real axis (for a representation, see Figure 1.2), $\epsilon > 0$ is the control parameter tending to zero in the singular limit of interest, and $\kappa(x)$ is a strictly increasing real function of x that is related to the boundary values of $g(z)$ on Σ. Suppose $\mathbf{N}(z)$ is the unknown satisfying $\mathbf{N}_+(x) = \mathbf{N}_-(x)\mathbf{v}(x)$. With the assumption of analyticity of $\kappa(x)$, one may transform the Riemann-Hilbert problem by introducing as a new unknown a matrix

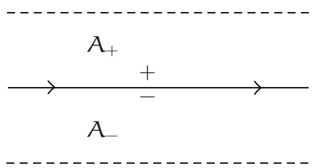

Figure 1.1 The complex plane in the vicinity of a contour Σ (here coincident with the real axis) supporting a factorized jump matrix.

$\mathbf{O}(z)$ defined in terms of $\mathbf{N}(z)$ by the following scheme: in some region lying on the minus side of the arc (the region labeled A_- in Figure 1.2), set

$$\mathbf{O}(x+iy) := \mathbf{N}(x+iy)\begin{pmatrix} 1 & 0 \\ e^{-i\kappa(x+iy)/\epsilon} & 1 \end{pmatrix} \tag{1.2}$$

and in some region lying on the plus side of the arc (the region labeled A_+ in Figure 1.2), set

$$\mathbf{O}(x+iy) := \mathbf{N}(x+iy)\begin{pmatrix} 1 & 0 \\ -e^{i\kappa(x+iy)/\epsilon} & 1 \end{pmatrix}. \tag{1.3}$$

Elsewhere, set $\mathbf{O}(z) = \mathbf{N}(z)$. One has thus introduced two new jump contours, one on either side of the arc (these are the two dashed lines in Figure 1.2). However, the monotonicity of the real analytic function $\kappa(x)$ implies via the Cauchy-Riemann equations that the induced jump matrix relating the boundary values of $\mathbf{O}(z)$ on these two contours is exponentially close to the identity matrix in the limit $\epsilon \downarrow 0$. On the original arc, the matrix $\mathbf{O}(z)$ satisfies the constant jump relation

$$\mathbf{O}_+(x) = \mathbf{O}_-(x)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.4}$$

which can be subsequently dealt with in terms of special functions. In this paper, we show how this procedure can be carried out effectively when one does not have the option of extending $\kappa(x)$ from the contour Σ because it is not assumed to be an analytic function. We choose to extend $\kappa(x)$ in a way that does not assume any analyticity (see (1.5)). The price that must be paid is that the analogue of the matrix $\mathbf{O}(z)$ above is no longer analytic in the regions to the left and right of the arc; therefore this matrix cannot be the solution of any Riemann-Hilbert problem. It can, however, be the solution of a matrix $\overline{\partial}$ problem (or more generally, a mixed Riemann-Hilbert-$\overline{\partial}$ problem). It is into this framework that we extend the steepest descent method. This explains the terminology of the "$\overline{\partial}$ steepest descent method."

As the fundamental contour Σ in this paper is the unit circle S^1, we can now be very specific about what we mean by an extension of a nonanalytic function in this context. Suppose that $f(\theta)$ is a $C^{m-1}(S^1)$ function, $m = 1, 2, 3, \ldots$. Then, we define an extension operator $E_m : C^{m-1}(S^1) \to C(\mathbb{R}^2 \setminus \{0\})$ as follows:

$$E_m f(r,\theta) := \sum_{p=0}^{m-1} \frac{f^{(p)}(\theta)}{p!} \big(-i \log(r) \big)^p, \tag{1.5}$$

where (r, θ) are the standard polar coordinates for $\mathbb{R}^2$. Note that this indeed defines a continuous extension to any annulus $r_+ \le r \le r_-$, where $0 < r_+ < 1 < r_- < \infty$ since $E_m f(1,\theta) = f(\theta)$. Also, since $z = re^{i\theta}$ and $\overline{z} = re^{-i\theta}$, the fundamental differential operators of complex variable theory are represented in polar coordinates as

$$\overline{\partial} := \frac{\partial}{\partial \overline{z}} = \frac{e^{i\theta}}{2} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right), \qquad \partial := \frac{\partial}{\partial z} = \frac{e^{-i\theta}}{2} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right), \tag{1.6}$$

and therefore we see that if $f^{(m-1)}(\theta)$ is Lipschitz, then in particular it has a derivative almost everywhere that is uniformly bounded, and we have

$$\overline{\partial} E_m f(r,\theta) = \frac{ie^{i\theta}}{2r} \frac{f^{(m)}(\theta)}{(m-1)!} \big(-i \log(r) \big)^{m-1}, \tag{1.7}$$

$$\partial E_m f(r,\theta) = -\frac{ie^{-i\theta}}{2r} \left(\frac{f^{(m)}(\theta)}{(m-1)!} \big(-i \log(r) \big)^{m-1} + 2 \sum_{p=0}^{m-2} \frac{f^{(p+1)}(\theta)}{p!} \big(-i \log(r) \big)^p \right) \tag{1.8}$$

both holding for all $r \ge 0$ and almost all $\theta \in S^1$ (these formulae hold at every point of the plane if f is of class $C^m(S^1)$). It follows that $\overline{\partial} E_m f(r,\theta)$ vanishes to order $m-1$ as $r \to 1$ uniformly in θ. In fact, if $f(\theta)$ is analytic for all θ, then the infinite series $E_\infty f(r,\theta)$ converges uniformly in some annulus containing the unit circle $r = 1$ and represents the unique analytic extension of $f(\theta)$.

Generally speaking, Riemann-Hilbert problems with rapidly oscillatory jump matrices are equivalent to systems of singular integral equations with Cauchy kernel and rapidly oscillatory densities. Such equations can in principle be analyzed asymptotically [27]. This approach requires delicate arguments of harmonic analysis. On the other hand, the $\overline{\partial}$ steepest descent method we will develop in this paper avoids such complicated reasoning. Indeed, by extending contour integration into integration over two-dimensional regions, the Cauchy kernel becomes less singular, and the analysis becomes correspondingly more straightforward.

In the analytic case described briefly above, the asymptotic analysis is in general complicated by the fact that the procedure is valid in the neighborhood of certain intervals of Σ, and it turns out that a different analysis must be carried out in the vicinity of the endpoints of the intervals. The same would be expected to be true in the general nonanalytic case. In order to have the clearest possible presentation, we have chosen to describe in this paper the $\overline{\partial}$ steepest descent method in the context of a problem where there are nontrivial cases without endpoint issues, namely the asymptotic behavior of polynomials orthogonal with respect to weights on the unit circle. While the presence of endpoints complicates the analysis, they do not present an insurmountable obstruction, and the reader is referred to [20] for a description of the more general theory.

1.3 Polynomials orthogonal on the unit circle

Let $\phi(\theta)$ be an integrable 2π-periodic function satisfying $\phi(\theta) > 0$ for almost all θ. For two complex-valued functions $f(\theta)$ and $g(\theta)$, there is an associated inner product

$$\langle f, g\rangle_\phi := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\phi(\theta)d\theta = \frac{1}{2\pi i}\oint_\Sigma f\big(\arg(z)\big)\overline{g\big(\arg(z)\big)}\phi\big(\arg(z)\big)\frac{dz}{z}. \tag{1.9}$$

This inner product leads to a system $\{p_n(z)\}_{n=0}^{\infty}$ of orthogonal polynomials in the complex variable z:

$$p_n(z) = \gamma_n z^n + \sum_{j=0}^{n-1} c_{n,j} z^j, \quad \gamma_n > 0, \tag{1.10}$$

and the defining relation is the orthonormality condition

$$\big\langle p_m, p_n\big\rangle_\phi = \delta_{mn}, \quad 0 \le m < \infty,\ 0 \le n < \infty. \tag{1.11}$$

Here the polynomials $p_n(z)$ are considered as complex-valued functions of θ by restriction to the unit circle: $z = e^{i\theta}$. The constants γ_n have the interpretation of normalization constants, and the corresponding system $\{\pi_n(z)\}_{n=0}^{\infty}$ of monic orthogonal polynomials is defined by rescaling:

$$\pi_n(z) = \frac{1}{\gamma_n} p_n(z), \quad 0 \le n < \infty. \tag{1.12}$$

The orthogonal polynomials satisfy recurrence relations of the form

$$\begin{aligned} \pi_{n+1}(z) &= z\pi_n(z) + \alpha_{n+1} z^n \overline{\pi_n\left(\frac{1}{\overline{z}}\right)}, \\ z^{n+1}\overline{\pi_{n+1}\left(\frac{1}{\overline{z}}\right)} &= \overline{\alpha_{n+1}} z\pi_n(z) + z^n \overline{\pi_n\left(\frac{1}{\overline{z}}\right)}, \end{aligned} \tag{1.13}$$

for $n = 0, 1, 2, 3, \ldots$. Here, the complex constants $\{\alpha_n\}_{n=1}^{\infty}$ are the *recurrence coefficients* associated with the weight ϕ (also known as the *Schur parameters* or *Verblunsky coefficients*). By setting $z = 0$ in the first equation of (1.13), it is easy to see that

$$\alpha_n = \pi_n(0). \tag{1.14}$$

For a general discussion of properties of polynomials orthogonal on the unit circle, see Chapter XII of Szegő's monograph [26], in which (among other things) the asymptotic behavior of $\pi_n(z)$ for $n \to \infty$ is discussed. The extraction of asymptotic formulae for quantities related to the orthogonal polynomials of large degree with a fixed weight ϕ on the unit circle is the type of asymptotic problem in the theory of general orthogonal polynomials (i.e., beyond particular cases involving classical special functions) for which results have been known for the longest time. This can be traced to the fact that if $\phi(\theta)^{-1}$ happens to be a positive trigonometric polynomial, then there is a closed-form expression for the orthonormal polynomial $p_n(z)$ that is convenient for analysis, as long as n is sufficiently large compared to the degree of $\phi(\theta)^{-1}$, see [26, Section 11.2]. In other words, for certain special fixed weights $\phi(\theta)$, the asymptotic formulae one obtains become *exact* as long as n is large enough. This leads to a general strategy for asymptotic analysis of orthogonal polynomials on the unit circle based on approximating an arbitrary given positive function $\phi(\theta)^{-1}$ by positive trigonometric polynomials.

The asymptotics described in the monograph of Szegő are of a rather general character and hold whenever $\log(\phi(\theta))$ is an integrable real-valued function. In the years since the origin of Szegő's methods, there have been many further developments in the asymptotic theory. These developments move both in the direction of generalizing the class of weights for which the Szegő asymptotics are valid (perhaps in a weaker form) and also in the direction of trading generality of the weight for detail of the asymptotics. It seems that certain problems remain difficult to treat by these methods; in particular, it is difficult to verify convergence in a uniform sense, and it is difficult to characterize the detailed asymptotic behavior of zeros. There is a vast literature on this subject; we refer the interested reader to the memoir of Nevai [23] and the two-volume monograph of Simon [24, 25].

Beyond being a source of classical information about orthogonal polynomials on the unit circle, Simon's monograph describes a different viewpoint of the theory of these polynomials. Namely, Simon and his school have made great progress by exploiting the connection between orthogonal polynomials and spectral theory for operators that encode the recurrence relations that all orthogonal polynomials satisfy (see also [18]). This theory is capable of establishing a number of very general results relevant to asymptotics in the limit of large degree. An important point is that the hypotheses required to establish results of this kind involve assumptions about the asymptotic behavior of the sequence $\{\alpha_n\}_{n=1}^{\infty}$ of recurrence coefficients. Indeed, the fundamental problem of spectral theory in this context is the construction of the spectral measure $\phi(\theta)d\theta$ from the finite difference operator involving the recurrence coefficients $\{\alpha_n\}_{n=1}^{\infty}$.

On the other hand, the recovery of the polynomials $\{p_n(z)\}_{n=0}^{\infty}$ and of the recurrence coefficients $\{\alpha_n\}_{n=1}^{\infty}$ from the spectral measure $\phi(\theta)d\theta$ is the fundamental problem of inverse spectral theory. A general approach to asymptotic problems in the theory of orthogonal polynomials in which the measure of orthogonality is the given data therefore involves the translation of the orthogonality conditions into the conditions making up a Riemann-Hilbert problem for sectionally analytic matrices. A Riemann-Hilbert formulation for polynomials orthogonal with respect to a measure on the unit circle was described in [1] and follows closely the well-known Riemann-Hilbert formulation for polynomials orthogonal with respect to a measure on $\mathbb{R}$ discovered in [17]. In [1], and in a number of papers which followed (see, e.g., [2, 5, 6]), the polynomial of degree n orthogonal with respect to a specific family of weights of the form $\phi(\theta) = e^{-nV(\theta)}$, where $V(\theta) = \gamma\cos(\theta)$, was studied in the limit $n \to \infty$. Note that this is a joint limit as the degree n of the polynomial in question appears in the measure of orthogonality as well; see Section 4 for a general discussion of such exponentially varying weights. For the large n asymptotics carried out in [1], and in subsequent works with this measure, as well as closely related measures (see, e.g., [3]), analyticity of the weight $\phi(\theta)$ played a central role in the analysis.

In [8], Deift used polynomials orthogonal with respect to a measure on the unit circle to give an example of his theory of integrable operators. Specifically, he introduced a one-parameter family of positive, analytic functions $\phi(\theta; t)$ and related solutions of Riemann-Hilbert Problem 2.1 in Section 2 below (with $\phi(\theta)$ replaced by $\phi(\theta; t)$) to Toeplitz determinants. The $n \to \infty$ asymptotic behavior of the corresponding solution $\mathbf{M}^n(z; t)$ to Riemann-Hilbert Problem 2.1 then yields asymptotics for the associated Toeplitz determinants. Exploiting the analyticity of $\phi(\theta; t)$, Deift outlined how one obtains an asymptotic description for $\mathbf{M}^n(z; t)$. The calculations which we will present in Section 3 may be viewed as complementary to this asymptotic calculation of [8], in that we will establish asymptotics for orthogonal polynomials under the much weaker assumption that $\phi(\theta)$ is a continuous function satisfying a Lipschitz condition. Furthermore, we show how the error estimates depend on smoothness properties of $\phi(\theta)$.

1.4 Outline and summary of results

The polynomials orthogonal with respect to a weight given on the unit circle in the complex plane can be characterized in terms of the solution of a matrix Riemann-Hilbert problem in which the contour Σ is the unit circle. In Section 2, we describe this Riemann-Hilbert problem, and then in Sections 3 and 4 we study the singular limit in which the degree of the polynomials tends to infinity. In Section 3, we consider the weight function to be held fixed as the degree tends to infinity, while in Section 4 we study the joint limit when the degree becomes large while the weight function is exponentially varied. A summary of the relevant logarithmic potential theory referred to in Section 4 is given in Appendices A and B.

The key results we obtain in the fixed-weights case are described in Section 3.1. To the best of our knowledge, the uniform nature of the asymptotics we obtain is new to the field, as is our detailed characterization of the zeros. While there exist several classical methods available for the asymptotic analysis of orthogonal polynomials on the unit circle with fixed-weight function $\phi(\theta)$, with the degree n of the polynomial in question tending to infinity, this problem is the ideal context in which to introduce the $\overline{\partial}$ steepest descent method.

On the other hand, the asymptotic behavior of polynomials orthogonal with respect to a varying-weight on the unit circle, considered in Section 4, is more challenging to obtain by more classical techniques. The results we obtain with the use of the $\overline{\partial}$ steepest descent method are stated in Section 4.1. A point we wish to emphasize is that with the use of the $\overline{\partial}$ steepest descent method, the analysis in the varying-weights case is no more difficult than in the fixed-weights case. This fact distinguishes the $\overline{\partial}$ steepest descent method from more classical techniques.

1.5 Notation

Throughout, we assume a fixed norm $\|\cdot\|$ on 2×2 matrices. For $p = 0, 1, 2, \ldots$, we use the following induced norm on sufficiently smooth matrix functions $\mathbf{F} : \mathbb{R}^2 \to \mathbb{C}_{2\times 2}$:

$$|||\mathbf{F}|||_p := \sum_{\alpha+\beta\leq p} \sup_{\mathbb{R}^2} \left\| \frac{\partial^{\alpha+\beta}\mathbf{F}}{\partial x^\alpha \partial y^\beta} \right\|. \tag{1.15}$$

Here x and y are Cartesian coordinates in $\mathbb{R}^2$. For $p = 0$, we may apply this norm on all functions F in the space $L^\infty(\mathbb{R}^2)$, whereas for $p > 0$ we may apply this norm to a subset of functions F in the space $C^{p-1,1}(\mathbb{R}^2)$ of functions with Lipschitz continuous mixed partial derivatives of all orders up to and including $p - 1$. The finiteness of the norm indicates the uniformity of the Lipschitz condition. Since for functions F in the class $C^{p-1,1}(\mathbb{R}^2)$ the mixed partial derivatives of order p exist almost everywhere, the condition $|||F|||_p < \infty$ can be equivalently expressed as saying that F have all derivatives of total order at most p in the space $L^\infty(\mathbb{R}^2)$. We also use the notations $C_0^{p-1,1}(\mathbb{R}^2 \setminus \{0\})$ and $L_0^\infty(\mathbb{R}^2 \setminus \{0\})$ for spaces of functions, respectively, in $C^{p-1,1}(\mathbb{R}^2)$ and $L^\infty(\mathbb{R}^2)$ that vanish identically for $|\log(x^2+y^2)|$ large enough (i.e., outside some annulus).

For functions $V(\theta)$ defined on the circle S^1 (i.e., V is defined for $-\pi \leq \theta < \pi$), we also say that V is of class $C^{k-1,1}(S^1)$ if the periodic extension of V to $\theta \in \mathbb{R}$ has $k-1$ Lipschitz continuous derivatives, or equivalently, has k derivatives in $L^\infty(\mathbb{R})$. A suitable norm for such functions is given by

$$|||V|||_{\circ,k} := \sup_{-\pi<\theta<\pi} \left|V(\theta)\right| + \sup_{-\pi<\theta<\pi} \left|V^{(k)}(\theta)\right|, \tag{1.16}$$

since it is easy to establish that for all m satisfying $1 \leq m \leq k-1$,

$$\sup_{-\pi<\theta<\pi} \left|V^{(m)}(\theta)\right| \leq (2\pi)^{k-m} \sup_{-\pi<\theta<\pi} \left|V^{(k)}(\theta)\right|. \tag{1.17}$$

If $V(\theta)$ satisfies a Hölder continuity condition, there exists a unique function $N(z)$, analytic for $|z| > 1$, decaying as $z \to \infty$, and taking Hölder continuous boundary values on $|z| = 1$, such that

$$V(\theta) = N\left(e^{i\theta}\right) + V_0 + \overline{N\left(e^{i\theta}\right)}, \tag{1.18}$$

and V_0 is the average value of V. Thus, $N(e^{i\theta})$ is the negative frequency component of the Fourier series for $V(\theta)$:

$$V(\theta) = \sum_{j=-\infty}^{\infty} V_j e^{ij\theta}, \quad \text{with coefficients } V_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\theta) e^{-ij\theta}\, d\theta, \tag{1.19}$$

and we have

$$N(z) := \sum_{j=1}^{\infty} \frac{V_{-j}}{z^j}, \quad \text{for } |z| \geq 1. \tag{1.20}$$

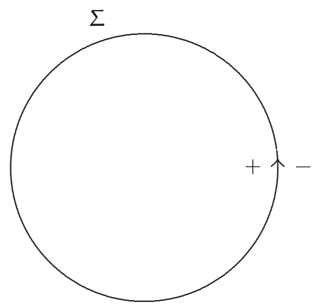

Figure 2.1 The contour Σ of the Riemann-Hilbert problem for polynomials orthogonal on the unit circle is the unit circle itself $|z| = 1$ oriented in the counterclockwise direction.

We also introduce the function $\Omega : S^1 \to \mathbb{R}$ by the formula

$$\Omega(\theta) := 2\Im\big(N\big(e^{i\theta}\big)\big). \tag{1.21}$$

Note that Ω and V are functions that are related by the Cauchy transform.

Throughout the paper, we will use a "bump" function $B : \mathbb{R} \to [0, 1]$ with the properties that B is infinitely differentiable, $B(l) \equiv 1$ for $|l| < \log(2)/2$, and $B(l) \equiv 0$ for $|l| > \log(2)$.

2 The Riemann-Hilbert problem for polynomials orthogonal on the unit circle

Consider the contour Σ illustrated in Figure 2. Let n be a positive integer. Relative to the contour Σ we pose, for each $n = 0, 1, 2, 3, \ldots$, the following Riemann-Hilbert problem for a 2×2 matrix $\mathbf{M}^n(z)$.

Riemann-Hilbert Problem 2.1. Find a 2×2 matrix $\mathbf{M}^n(z)$ with the following properties.

Analyticity. $\mathbf{M}^n(z)$ is analytic for $|z| \neq 1$, and takes continuous boundary values $\mathbf{M}^n_+(z)$, $\mathbf{M}^n_-(z)$ as w tends to z with $|z| = 1$ and $|w| < 1$, $|w| > 1$.

Jump condition. The boundary values are connected by the relation

$$\mathbf{M}^n_+\big(e^{i\theta}\big) = \mathbf{M}^n_-\big(e^{i\theta}\big)\begin{pmatrix} 1 & \phi(\theta)e^{-in\theta} \\ 0 & 1 \end{pmatrix}. \tag{2.1}$$

Normalization. The matrix $\mathbf{M}^n(z)$ is normalized at $z = \infty$ as follows:

$$\lim_{z\to\infty} \mathbf{M}^n(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = \mathbb{I}. \tag{2.2}$$

Proposition 2.2. Suppose that the positive weight function $\phi(\theta)$ satisfies a uniform Hölder condition $|\phi(\theta_2) - \phi(\theta_1)| \leq K|\theta_2 - \theta_1|^\nu$ for some $\nu \in (0, 1]$ and with some K independent of θ_1 and θ_2. Then Riemann-Hilbert Problem 2.1 has a unique solution for each integer $n \geq 0$, namely if $n > 0$,

$$\mathbf{M}^n(z) = \begin{pmatrix} \pi_n(z) & \dfrac{1}{2\pi i}\displaystyle\oint_\Sigma \dfrac{\pi_n(s)s^{-n}}{s-z}\phi(\arg(s))\,ds \\ -\gamma_{n-1}^2 z^{n-1}\overline{\pi_{n-1}\left(\dfrac{1}{\overline{z}}\right)} & -\dfrac{\gamma_{n-1}^2}{2\pi i}\displaystyle\oint_\Sigma \dfrac{\overline{\pi_{n-1}(s)}s^{-1}}{s-z}\phi(\arg(s))\,ds \end{pmatrix}, \tag{2.3}$$

and if $n = 0$,

$$\mathbf{M}^0(z) = \begin{pmatrix} 1 & \dfrac{1}{2\pi i}\displaystyle\oint_\Sigma \dfrac{1}{s-z}\phi(\arg(s))\,ds \\ 0 & 1 \end{pmatrix}. \tag{2.4}$$

In particular, $M_{11}^n(0) = \alpha_n$ and $M_{21}^n(0) = -\gamma_{n-1}^2$ for $n \geq 1$. Here, $\{\pi_n\}_{n=0}^\infty$ is the sequence of monic orthogonal polynomials with respect to the weight ϕ and the inner product (1.9), and $\{\alpha_n\}_{n=1}^\infty$ is the sequence of associated recurrence coefficients (see (1.13)) while $\{\gamma_n\}_{n=0}^\infty$ is the sequence of associated normalization constants. □

Proof. If $n = 0$, then the Riemann-Hilbert problem is triangular with identity asymptotics and is trivially solved in closed form by a Cauchy integral, yielding (2.4). Thus from now on, we consider $n \geq 1$.

The uniqueness of the solution for $n \geq 1$ can be seen from the following argument. Continuity of the boundary values taken on Σ implies that the ratio of any two solutions of Riemann-Hilbert Problem 2.1 is an entire function of z that tends to the identity matrix as $z \to \infty$. Uniqueness thus follows by Liouville's theorem.

To derive (2.3), first note that if $\mathbf{M}^n(z)$ solves Riemann-Hilbert Problem 2.1, then the first column of $\mathbf{M}^n(z)$ must be analytic throughout the z-plane. From the normalization condition (2.2), it is then clear that $M_{11}^n(z)$ is a monic polynomial of degree n while $M_{21}^n(z)$ is a polynomial of degree at most $n-1$ (the leading coefficient of $M_{21}^n(z)$ is not determined from the normalization condition alone). The jump condition for the second column reads

$$\begin{aligned} M^n_{12+}(e^{i\theta}) - M^n_{12-}(e^{i\theta}) &= M^n_{11}(e^{i\theta})e^{-in\theta}\phi(\theta), \\ M^n_{22+}(e^{i\theta}) - M^n_{22-}(e^{i\theta}) &= M^n_{21}(e^{i\theta})e^{-in\theta}\phi(\theta). \end{aligned} \tag{2.5}$$

In particular, since $\phi(\theta)$ satisfies a Hölder condition, we may express $M^n_{12}(z)$ as a Cauchy-type integral using (2.5). Thus,

$$M^n_{12}(z) = \frac{1}{2\pi i}\oint_\Sigma \frac{M^n_{11}(s)s^{-n}}{s-z}\phi\big(\arg(s)\big)\,ds, \tag{2.6}$$

and the normalization condition (2.2) then requires that this Cauchy integral be of order z^{-n-1} as $z \to \infty$ for each fixed $n \in \mathbb{Z}_+$. Expanding the Cauchy kernel in a geometric series, we see that the following conditions must be satisfied:

$$\oint_\Sigma M^n_{11}(z)z^{k-n}\phi\big(\arg(z)\big)\,dz = 0 \tag{2.7}$$

for $k = 0, 1, 2, \ldots, n-1$. Since $d\theta = dz/(iz)$ for an angular coordinate θ on the contour Σ, this proves that $M^n_{11}(z)$ is orthogonal to the monomials $1, z, z^2, \ldots, z^{n-1}$ with respect to the inner product (1.9). The existence of such a monic polynomial of degree n follows from the Gram-Schmidt algorithm. Thus, $M^n_{11}(z) = \pi_n(z)$, the nth monic orthogonal polynomial with respect to the weight $\phi(\theta)$ on the unit circle.

A similar argument applies to the second row of $\mathbf{M}^n(z)$. Indeed, $M^n_{21}(z)$ is a polynomial of degree at most $n-1$. Using (2.5), we may express $M^n_{22}(z)$ as a Cauchy integral:

$$M^n_{22}(z) = \frac{1}{2\pi i}\oint_\Sigma \frac{M^n_{21}(s)s^{-n}}{s-z}\phi\big(\arg(s)\big)\,ds, \tag{2.8}$$

and then the normalization condition (2.2) requires that $M^n_{22}(z) = z^{-n} + O(z^{-n-1})$ as $z \to \infty$. Expanding the Cauchy kernel in a geometric series, one sees that $M^n_{21}(z)$ is required to satisfy the following conditions:

$$\big\langle M^n_{21}(e^{i\theta}), 1\big\rangle_\phi = -1, \qquad \big\langle M^n_{21}(e^{i\theta}), e^{ik\theta}\big\rangle_\phi = 0, \quad k = 1, \ldots, n-1. \tag{2.9}$$

Equivalently, these relations may be written in the form

$$\begin{aligned} &\Big\langle e^{i(n-1)\theta}, e^{i(n-1)\theta}\overline{M^n_{21}(e^{i\theta})}\Big\rangle_\phi = -1, \\ &\Big\langle e^{ik\theta}, e^{i(n-1)\theta}\overline{M^n_{21}(e^{i\theta})}\Big\rangle_\phi = 0, \quad k = 0, \ldots, n-2. \end{aligned} \tag{2.10}$$

Clearly, the degree $n-1$ polynomial $z^{n-1}\overline{M^n_{21}(1/\overline{z})}$ is orthogonal to the monomials $1, z, \ldots, z^{n-2}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\phi$, and the normalization condition then fixes the leading coefficient: $z^{n-1}\overline{M_{21}^{n}(1/\overline{z})} = -\gamma_{n-1}^{2}z^{n-1} + \ldots$. In other words, we have found that $z^{n-1}\overline{M_{21}^{n}(1/\overline{z})} = -\gamma_{n-1}^{2}\pi_{n-1}(z)$, or equivalently, $M_{21}^{n}(z) = -\gamma_{n-1}^{2}z^{n-1}\overline{\pi_{n-1}(1/\overline{z})}$. ■

Thus, the conditions of Riemann-Hilbert Problem 2.1 serve to define the orthogonal polynomials as an alternative to other representations that may be available, possibly including explicit contour integral formulae (for special families of weights).

Riemann-Hilbert problems like this one frequently arise as a consequence of the application of the Fourier transform or z-transform to certain types of linear integral equations (i.e., the Wiener-Hopf technique, see [7]). Reversing this sort of reasoning, the representation of the orthogonal polynomials in terms of Riemann-Hilbert Problem 2.1 immediately yields integral equations for certain auxiliary unknowns. Of particular interest are the Marchenko equations obtained in [18], in which the unknowns are Fourier coefficients of functions explicitly related to the orthogonal polynomial $\pi_n(z)$, and the relevant operator is of the form $I - K$, where K is an integral operator acting in $\ell^2(n+1, n+2, \ldots, \infty)$ with a kernel that depends explicitly on the weight ϕ but not otherwise on the degree n of the polynomial in question. Furthermore, the kernel does not depend in any crucial way on the smoothness of the weight. Therefore, in principle, this formulation makes possible the calculation of asymptotics for the polynomials (indeed, this is one of the applications of the Marchenko equations discussed in [18]) in a way that is relatively insensitive to the analyticity properties of the weight function ϕ. However, the correction terms that appear in such a scheme are necessarily in terms of infinite Fourier series (see, e.g., [18, equation (VI.7)]) that while having known coefficients are not convenient for detailed analysis of zeros of the polynomials in regions of the complex plane where these zeros necessarily arise from a competition between different terms in an expansion. To provide details of the asymptotics, it is more advantageous to work with the Riemann-Hilbert problem directly.

The uniqueness of $\mathbf{M}^n(z)$ coupled with symmetry of the Riemann-Hilbert problem under reflection through the unit circle leads to the following result.

Proposition 2.3. The matrix $\mathbf{M}^n(z)$ satisfying Riemann-Hilbert Problem 2.1 satisfies the symmetry relation

$$\mathbf{M}^n(z) = i^{\sigma_3}\overline{\mathbf{M}^n(0)^{-1}}\overline{\mathbf{M}^n\left(\frac{1}{\overline{z}}\right)}(-iz)^{\sigma_3}. \tag{2.11}$$

In particular, the identity

$$\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 = 1 - |\alpha_n|^2. \tag{2.12}$$

follows by taking $z = 0$ in (2.11). □

3 Fixed weights

3.1 Asymptotic behavior of orthogonal polynomials and related quantities as $n \to \infty$

In this section, we describe several asymptotic results in the theory of orthogonal polynomials with fixed weights on the unit circle that we will obtain as a fundamental illustration of our method. It will be convenient to introduce the real-valued function $V : S^1 \to \mathbb{R}$ such that

$$\phi(\theta) = e^{-V(\theta)}, \quad \forall \theta \in S^1. \tag{3.1}$$

The fundamental object of the asymptotic theory for the fixed weight ϕ is the so-called Szegő function:

$$S_\phi(z) := \exp\left(-\frac{1}{2\pi i}\oint_\Sigma \frac{V\big(\arg(s)\big)\,ds}{s-z}\right), \quad z \notin \Sigma. \tag{3.2}$$

This is a function analytic for $|z| \neq 1$ that decays to zero as $z \to \infty$. Its value at $z = 0$ has the interpretation of the geometric mean of the weight $\phi(\theta)$:

$$S_\phi(0) = \exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi} V(\theta)\,d\theta\right) = e^{-V_0}. \tag{3.3}$$

If $\phi(\theta)$ satisfies a Hölder continuity condition, then by strict positivity so does $V(\theta)$. In this case, by the Plemelj formula [22], we have

$$\lim_{r\uparrow 1} S_\phi\left(re^{i\theta}\right) = \phi(\theta)\lim_{r\downarrow 1} S_\phi\left(re^{i\theta}\right). \tag{3.4}$$

Furthermore, recalling the negative frequency component $N(z)$ of V defined by (1.20), we can obtain the following equivalent characterization of $S_\phi(z)$. Since the function $\overline{N(1/\overline{z})}$ is analytic for $|z| < 1$, it follows that

$$S_\phi(z) = \begin{cases} e^{N(z)}, & |z| > 1, \\ e^{-V_0 - \overline{N(1/\overline{z})}}, & |z| < 1. \end{cases} \tag{3.5}$$

Recalling the function $\Omega : S^1 \to \mathbb{R}$ defined by (1.21), we have

$$\lim_{r\uparrow 1} S_\phi\left(re^{i\theta}\right) \cdot \lim_{r\downarrow 1} S_\phi\left(re^{i\theta}\right) = e^{-V_0}e^{i\Omega(\theta)}. \tag{3.6}$$

3.1.1 General theorems. The following results hold for weights where V is of class $C^{k-1,1}(S^1)$ with $k \geq 1$.

Theorem 3.1. Suppose that $\phi(\theta) = e^{-V(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 1$. Then, for each fixed integer p and for each $\rho > 1$, there is a constant $K_{p,\rho} > 0$ such that the estimate

$$\sup_{|z|\geq\rho} \left| \frac{d^p}{dz^p} \left[\pi_n(z) z^{-n} e^{-N(z)} - 1 \right] \right| \leq K_{p,\rho} \frac{\log(n)}{n^{2k}} \tag{3.7}$$

holds for all n sufficiently large. □

The constant $K_{p,\rho}$ typically blows up as $\rho \to 1$, and only a finite number of derivatives can be controlled. More generally, we have the following result.

Theorem 3.2. Let $p \geq 0$ be a fixed integer. Suppose that $\phi(\theta) = e^{-V(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 2p+1$. Then there exists a constant $K_p > 0$ such that the estimate

$$\sup_{|z|\geq 1} \left| \frac{d^p}{dz^p} \left[\pi_n(z) z^{-n} e^{-N(z)} - 1 \right] \right| \leq K_p \frac{\log(n)}{n^{k-2p}} \tag{3.8}$$

holds for all n sufficiently large. □

Remark 3.3. Note that as a special case of the estimate (3.8), we obtain the following estimate (under the same conditions) characterizing the polynomials on the unit circle:

$$\sup_{-\pi<\theta<\pi} \left| \left(-ie^{-i\theta} \frac{d}{d\theta} \right)^p \left[\pi_n(e^{i\theta}) e^{-in\theta} e^{-N(e^{i\theta})} - 1 \right] \right| \leq K_p \frac{\log(n)}{n^{k-2p}}. \tag{3.9}$$

In fact, the proof of Theorem 3.2 is to first establish (3.9), from which the estimate (3.8) follows (with the same constant K_p) via the maximum modulus principle.

The weakest conditions under which the above theorem provides large-degree asymptotics are that ϕ is a strictly positive weight that is Lipschitz continuous. Theorem 3.2 may be compared with results reported in the classic monograph of Szegő [26, Section 12.1]. While asymptotics of $\pi_n(z)$ have been established by other methods under weaker conditions than Lipschitz continuity and strict positivity of the weight ϕ, Theorem 3.2 exhibits clearly the dependence of the rate of decay of the error on the smoothness of ϕ, and the number of derivatives desired.

To our knowledge, the results of Theorem 3.2 are stronger than those previously known in that they establish the convergence in a uniform sense. This leads to the following corollary.

Corollary 3.4. Let $p \geq 0$ be a fixed integer. Suppose that $\phi(\theta) = e^{-V(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 2p+1$. Then

$$\lim_{n\to\infty} \frac{1}{n^p} \cdot \frac{\|\pi_n^{(p)}\|_\phi}{\|\pi_n\|_\phi} = 1. \tag{3.10}$$

□

Proof. This follows directly from **Theorem 3.2**. Indeed, upon carrying out the differentiation in (3.9), and combining this estimate with its analogues for all smaller values of p, we learn that $|\pi_n^{(p)}(e^{i\theta})|/n^p$ converges uniformly to $|\pi_n(e^{i\theta})|$ for $-\pi < \theta < \pi$. The proof is then complete since on S^1 uniform convergence implies convergence in L^2. ■

Remark 3.5. Notice that **Theorem 3.2** immediately implies the following formula valid for all z with $|z| < 1$:

$$\pi_n(z) = \frac{1}{2\pi i} \oint_{|s|=1} \frac{s^n e^{N(s)} + h_n(s)}{s - z} ds, \tag{3.11}$$

where

$$\sup_{|s|=1} |h_n(s)| \leq K_p \frac{\log(n)}{n^k}. \tag{3.12}$$

While in principle this could be used to compute asymptotics for $\pi_n(z)$ in this region, more detailed analysis gives the following improved results.

Theorem 3.6. Suppose that $\phi(\theta) = e^{-V(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 1$. Then for each ρ satisfying $0 < \rho < 1$, there are constants $K_\rho^\pm > 0$ such that the estimates

$$\sup_{\rho<|z|<1} \left|\pi_n(z) - z^n e^{-V_0 - \overline{N(1/\overline{z})}} e^{E_k V(r,\theta)}\right| \leq K_\rho^- \frac{\log(n)}{n^k}, \tag{3.13}$$

$$\sup_{|z|<\rho} |\pi_n(z)| \leq \frac{K_\rho^+}{n^k} \tag{3.14}$$

hold for all n sufficiently large. □

An immediate corollary is that there exists an annulus inside the unit circle that asymptotically contains no zeros. That the result we are about to state in this direction is in a sense sharp will be made clear when we consider more specific weights below in Section 3.1.2 (in particular, see Corollary 3.12).

Corollary 3.7 (zero-free regions)**.** Suppose that $\phi(\theta) = e^{-V(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 1$. Let $\delta > 0$ be an arbitrarily small number. Then there are no zeros of $\pi_n(z)$ in the region

$$\left\{ z \mid \log\left(|z|\right) > -(k-\delta)\frac{\log(n)}{n} \right\} \tag{3.15}$$

as long as n is sufficiently large. □

Proof. This follows immediately from the estimate (3.13). Indeed, since $\overline{N(1/\overline{z})}$ and $E_k V(r,\theta)$ are bounded for $\rho < r < 1$, zeros of $\pi_n(z)$ in the region $\rho < |z| < 1$ necessarily arise from a balance between z^n and a term of uniform size $\log(n)/n^k$. However, z^n is large compared with $\log(n)/n^k$ in the region where the inequality $\log(|z|) > -(k-\delta)\log(n)/n$ holds. ■

A second corollary is an immediate consequence of (3.14).

Corollary 3.8 (recurrence coefficients)**.** Suppose that $\phi(\theta) = e^{-V(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 1$. Then there is a constant $K > 0$ such that the bound

$$\left|\alpha_n\right| \leq \frac{K}{n^k} \tag{3.16}$$

holds for sufficiently large n. □

Proof. This follows directly from (3.14) with the use of the identity $\alpha_n = \pi_n(0)$. ■

Finally, we have the following result concerning the asymptotic behavior of the normalization constants.

Theorem 3.9. Suppose that $\phi(\theta) = e^{-V(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 1$. Then there is a constant $K > 0$ such that the bound

$$\left|\gamma_n^2 e^{-V_0} - 1\right| \leq K\frac{\log(n)}{n^{2k}} \tag{3.17}$$

holds for sufficiently large n. □

We give a direct proof of this theorem based on the identity $\gamma_{n-1}^2 = -M_{21}^n(0)$ in Section 3.3.3. However, another proof with a less sharp error estimate may be based upon Theorem 3.2 because on S^1 uniform convergence implies convergence in L^2. Thus, since $\|p_n(z)\|_\phi = 1$ and $p_n(z) = \gamma_n \pi_n(z)$,

$$\gamma_n^2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\pi_n\left(e^{i\theta}\right)\right|^2 \phi(\theta)d\theta\right)^{-1}. \tag{3.18}$$

Using Theorem 3.2, one finds that

$$\gamma_n^2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| e^{N(e^{i\theta})} \right|^2 \phi(\theta) d\theta \right)^{-1} + O\left(\frac{\log(n)}{n^k} \right). \tag{3.19}$$

Next, using (3.5), we have

$$\left| e^{N(e^{i\theta})} \right|^2 = e^{N(e^{i\theta})} e^{\overline{N(e^{i\theta})}} = e^{V(\theta) - V_0} = \frac{e^{-V_0}}{\phi(\theta)}. \tag{3.20}$$

Substitution into (3.19) completes the alternate proof.

Remark 3.10. At this point, it is important to comment that the $\overline{\partial}$ method we develop below in Section 3.2 yields new formulae for the polynomial $\pi_n(z)$ (see, e.g., (3.95)). The formulae are semi-explicit, in that they are written in terms of the solution of a $\overline{\partial}$ problem (or, equivalently, in terms of the solution of an integral equation). This $\overline{\partial}$ problem is arrived at after a sequence of explicit transformations, and we prove that this problem has a unique solution, which possesses an asymptotic expansion for $n \to \infty$. In general, the terms in this expansion can be estimated (from above). Such estimations give rise to the general results described in this subsection. However, in the situation that some further information about the weight function e^{-V} is known, it is frequently possible to obtain much more precise information about the terms in the asymptotic expansion. To illustrate what can be obtained from an analysis of the terms of the expansion, we consider in the following subsection a slightly more specific family of weights, and present a rather complete description of the pointwise asymptotic behavior of the polynomials.

3.1.2 More specific weights. While the estimate (3.14) allows one to bound the recurrence coefficients, it does not provide an asymptotic description of the polynomial $\pi_n(z)$ for z bounded within the unit circle. In particular, (3.14) is insufficient for deducing the location of the zeros. With further assumptions on the regularity of $V(\theta)$, we can extract a leading term that paves the way for further analysis of $\pi_n(z)$ outside the zero-free region, but within the unit disk.

Theorem 3.11. Suppose that $\phi(\theta) = e^{-V(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 2$. Suppose further that $V^{(k)}(\theta)$ is piecewise continuous with $\ell < \infty$ jump discontinuities at points $-\pi \leq \theta_1 < \theta_2 < \cdots < \theta_\ell < \pi$, of magnitudes

$$\Delta_j^{(k)} := \lim_{\theta \downarrow \theta_j} V^{(k)}(\theta) - \lim_{\theta \uparrow \theta_j} V^{(k)}(\theta). \tag{3.21}$$

Let $V^{(k)}(\theta)$ have one Lipschitz continuous derivative between consecutive jump discontinuities. Then, for each $\epsilon > 0$, $\sigma > 0$, and $\delta > 0$, the estimate

$$\sup_{\log(|z|)<-(k-\sigma)\log(n)/n} \Bigg| n^{k+1} e^{-\overline{N(1/\overline{z})}} \pi_n(z) - n^{k+1} z^n e^{-V_0 - 2\overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} B\left(\frac{\log(|z|)}{\epsilon}\right) - f_n(z) \Bigg| \le \delta \tag{3.22}$$

holds with

$$f_n(z) := \frac{i^{k+1}}{2\pi} \sum_{j=1}^{\ell} \Delta_j^{(k)} e^{i\Omega(\theta_j)} \frac{e^{i(n+1)\theta_j}}{e^{i\theta_j} - z}, \tag{3.23}$$

for all n sufficiently large. □

Note that $f_n(z)$ is a rational function of z with poles at the ℓ points of discontinuity of $V^{(k)}(\theta)$ on the unit circle, and with $\ell - 1$ zeros which may lie anywhere in the complex plane, and fluctuate about as n is varied.

With this result, we can completely characterize the zeros of $\pi_n(z)$ under the same assumptions on $V(\theta)$. The simplest example of orthogonal polynomials on the unit circle is of course the case $\phi(\theta) \equiv 1$, in which case $\pi_n(z) = z^n$ for all $n \ge 0$. Here we see that all zeros of $\pi_n(z)$ lie exactly at $z = 0$. In particular, the zeros avoid the unit circle $|z| = 1$. This situation is typical for strictly positive analytic weights, in which case it is known that the zeros of $\pi_n(z)$ asymptotically lie within a smaller disk $|z| \le \rho < 1$. Here, the nearest singularity z_0 to the unit circle of the analytic continuation through $|z| = 1$ of the function $S_\phi(z)$ from the domain $|z| > 1$ determines the radius ρ by $\rho = |z_0|$. However, such confinement of the zeros within the circle is no longer typical once one leaves the analytic class. For example, discontinuities in any derivatives of $\phi(\theta)$ make it possible for at most a finite number of zeros to be bounded away from the unit circle while all remaining zeros converge to the unit circle, as the following corollaries of Theorem 3.11 show. For each $M > 0$, let

$$\begin{aligned} F_n^+(M) &:= \left\{z \in \mathbb{C} \text{ such that } \log\left(\left|f_n(z)\right|\right) > M\right\}, \\ F_n^-(M) &:= \left\{z \in \mathbb{C} \text{ such that } \log\left(\left|f_n(z)\right|\right) < -M\right\}, \\ F_n^0(M) &:= \left\{z \in \mathbb{C} \text{ such that } \left|\log\left(\left|f_n(z)\right|\right)\right| \le M\right\}. \end{aligned} \tag{3.24}$$

Corollary 3.12 (zeros near the unit circle)**.** Assume the same hypotheses as in Theorem 3.11. Let $A_n(\sigma)$ denote the annulus

$$A_n(\sigma) := \left\{ z \mid -(k+1)\frac{\log(n)}{n} - \frac{\sigma}{n} < \log\left(|z|\right) < -(k-\sigma)\frac{\log(n)}{n} \right\}. \tag{3.25}$$

Then, for each $\sigma > 0$, there is some $M > 0$ such that the region $A_n(\sigma) \cap F_n^-(M)$ contains no zeros of $\pi_n(z)$ for sufficiently large n.

For each $M > 0$, the zeros of $\pi_n(z)$ in the region $A_n(\sigma) \cap (F_n^0(M) \cup F_n^+(M))$ satisfy

$$|z| = 1 - (k+1)\frac{\log(n)}{n} + \frac{1}{n}\log\left(\left|f_n(z)\right|\right) + o\left(\frac{1}{n}\right), \tag{3.26}$$

$$\theta = -\frac{1}{n}\Omega(\theta) + \frac{1}{n}\arg\left(f_n(z)\right) + \frac{\pi}{n} + o\left(\frac{1}{n}\right) \tag{3.27}$$

modulo $2\pi/n$, where $\theta = \arg(z)$, and in both cases the error term is uniformly small in the specified region. It follows that the angular spacing between neighboring zeros of $\pi_n(z)$ in the specified region is $\Delta\theta = 2\pi/n + o(1/n)$.

For any fixed $M > 0$, (3.26) can be rewritten uniformly in the region $A_n(\sigma) \cap F_n^0(M)$ as

$$|z| = 1 - (k+1)\frac{\log(n)}{n} + O\left(\frac{1}{n}\right), \tag{3.28}$$

and consequently there exists some $\alpha \in (0, \sigma)$ such that the zeros of $\pi_n(z)$ in the region $A_n(\sigma) \cap F_n^0(M)$ asymptotically lie between the two circles $|z| = 1 - (k+1)\log(n)/n \pm \alpha/n$.

If $M > 0$ is sufficiently large, then the region $F_n^+(M)$ is contained in a disjoint union of small disks centered at the poles $e^{i\theta_j}$ of the rational function $f_n(z)$. In this situation, let $F_{n,j}^+(M)$ denote the component of $F_n^+(M)$ near the pole $e^{i\theta_j}$. Then from (3.26), it is seen that the zeros of $\pi_n(z)$ in the region $A_n(\sigma) \cap F_{n,j}^+(M)$ satisfy

$$|z| = 1 - (k+1)\frac{\log(n)}{n} - \frac{1}{n}\log\left|z - e^{i\theta_j}\right| + O\left(\frac{1}{n}\right), \tag{3.29}$$

where the error term is uniform in the specified region, which indicates that zeros are attracted to a curve that "bulges" outward from the circle $|z| = 1 - (k+1)\log(n)/n$ in a region of angular width proportional to $\log(n)/n$ centered at the point $e^{i\theta_j}$ to a maximum radius defined by the equation

$$1 - |z| = k\frac{\log(n)}{n} + \frac{\log\left(\log(n)\right)}{n} + O\left(\frac{1}{n}\right). \tag{3.30}$$

Note that this radius is just within the inner boundary of the zero-free annulus described by Corollary 3.7. □

Proof. The annulus $A_n(\sigma)$ converges toward the unit circle as $n \to \infty$, and therefore if z is a zero of $\pi_n(z)$ and n is large enough, Theorem 3.11 gives

$$n^{k+1} z^n e^{-V_0 - 2\overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} + f_n(z) = o(1), \tag{3.31}$$

as $n \to \infty$ because for such z, $B(\log(|z|)/\epsilon) = 1$ and $\pi_n(z) = 0$. The $o(1)$ error term is uniformly small for all zeros in $A_n(\sigma)$. Now, let

$$c := \inf_{|z| \le 1} \left| e^{-V_0 - 2\overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} \right| \tag{3.32}$$

and note that $c > 0$ due to the assumptions in force on V. Since

$$\inf_{z \in A_n(\sigma)} n^{k+1} |z|^n = e^{-\sigma}, \tag{3.33}$$

we see that (3.31) is inconsistent for large enough n if $|f_n(z)| < ce^{-\sigma}$. Therefore, given $\sigma > 0$, $A_n(\sigma) \cap F_n^-(M)$ contains no zeros as $n \to \infty$ as long as $M > \sigma - \log(c)$.

For any $M > 0$, we now consider those zeros z of $\pi_n(z)$ in the region $A_n(\sigma) \cap (F_n^0(M) \cup F_n^+(M))$, in which case we may divide through in (3.31) by $f_n(z)$ to obtain

$$\frac{n^{k+1} z^n}{f_n(z)} e^{-V_0 - \overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} + 1 = o(1). \tag{3.34}$$

Consistency requires that $n^{k+1} z^n / f_n(z) = O(1)$, and then since the n-independent exponential factors are continuous up to the unit circle and $A_n(\sigma)$ is converging to the unit circle, we may replace these factors by their limiting values on the unit circle without changing the error estimate. Therefore, (3.34) becomes

$$\frac{n^{k+1} z^n}{f_n(z)} e^{i\Omega(\theta)} + 1 = o(1), \tag{3.35}$$

as $n \to \infty$ uniformly for those zeros z of $\pi_n(z)$ that lie in the annulus $A_n(\sigma)$. This proves both (3.26) and (3.27). ∎

Remark 3.13. Note that the presence of the outward "bulges" in the zero curve near the points of discontinuity of $V^{(k)}(\theta)$ indicates the sharpness of the zero-free region established for more general weights in Corollary 3.7.

While most zeros of $\pi_n(z)$ move toward the unit circle as $n \to \infty$ under the hypotheses of Theorem 3.11, there may be at most $\ell - 1$ zeros further inside the unit circle, which correspond to zeros of $f_n(z)$. We refer to these as "spurious zeros."

Corollary 3.14 (spurious zeros)**.** Assume the same hypotheses as in Theorem 3.11. For each $M > 0$, there exists a $\sigma > 0$, such that the zeros of $\pi_n(z)$ lying in the disk $\log(|z|) \leq -(k+1)\log(n)/n - \sigma/n$ also lie in the set $F_n^-(M)$ for sufficiently large n. Moreover, whenever ϵ_n is a sequence of positive numbers such that $\epsilon_n \to 0$ as $n \to \infty$, the zeros of $\pi_n(z)$ in the disk $|z| \leq 1 - (k+1)\log(n)/n - 1/(n\epsilon_n)$ satisfy

$$f_n(z) = o(1) \tag{3.36}$$

as $n \to \infty$. In particular, $\pi_n(z)$ has exactly one zero for each zero of $f_n(z)$ in this region, making at most $\ell - 1$ spurious zeros. □

Proof. Define a constant $C > 0$ by

$$C := \sup_{|z|\leq 1} \left| e^{-V_0 - 2\overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} B\left(\frac{\log(|z|)}{\epsilon}\right)\right|. \tag{3.37}$$

Therefore,

$$\sup_{\log(|z|)\leq -(k+1)\log(n)/n - \sigma/n} \left| n^{k+1} z^n e^{-V_0 - 2\overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} B\left(\frac{\log(|z|)}{\epsilon}\right)\right| \leq Ce^{-\sigma}. \tag{3.38}$$

It then follows easily from (3.31) that all zeros of $\pi_n(z)$ lying in the disk where the inequality $\log(|z|) \leq -(k+1)\log(n)/n - \sigma/n$ holds will also lie in the set $F_n^-(M)$ for large enough n whenever $\sigma > M + \log(C)$.

The term in (3.31) proportional to $n^{k+1}z^n$ is $o(1)$ as $n \to \infty$ uniformly for z in the disk delineated by the inequality $|z| \leq 1-(k+1)\log(n)/n-1/(n\epsilon_n)$ with $\epsilon_n \to 0$ as $n \to \infty$. Therefore, $f_n(z) = o(1)$ for zeros in this disk, and the one-to-one correspondence of zeros of $f_n(z)$ with spurious zeros of $\pi_n(z)$ in this region follows from the implicit function theorem. ■

Remark 3.15. Note that the zeros of $f_n(z)$ play an apparently contradictory role in the asymptotics. Indeed, zeros of $f_n(z)$ that occur near the unit circle repel zeros of $\pi_n(z)$, while each zero of $f_n(z)$ that occurs far enough within the unit circle attracts precisely one zero of $\pi_n(z)$.

Remark 3.16. A careful reading of the proof of Theorem 3.11 (see Section 3.3.3) shows that the $o(1)$ estimate that is stated in (3.22) can be improved to give a rate of decay, and that even better estimates can be obtained if one does not insist on uniformity. These simple improvements can provide, for example, decay rate information for the error terms in the description of the spurious zeros. We have opted not to give these slightly improved estimates in the interest of simplicity of presentation.

3.1.3 Numerical computation of zeros of $\pi_n(z)$ *when derivatives of* V *have jump discontinuities.* To illustrate the detailed asymptotic behavior of the zeros of $\pi_n(z)$ explained above, we have carried out some numerical experiments. Let us fix ℓ angles of discontinuity $\{\theta_1, \dots, \theta_\ell\} \subset (-\pi, \pi)$ by the formula

$$\theta_j := \frac{2\pi}{\ell}\left(j - \frac{1}{2} - \frac{\ell}{2}\right), \quad \text{for } j = 1, \dots, \ell. \tag{3.39}$$

Consider the family of weights $\phi(\theta)$ given by the formula

$$\phi(\theta) := \begin{cases} 1 + e^{w_j}\left|\sin\left(\frac{\ell}{2}(\theta - \theta_j)\right)\right|^k, & \text{for } \theta_{j-1} < \theta < \theta_j \text{ with } j = 2, 3, \dots, \ell, \\ 1 + e^{w_1}\left|\sin\left(\frac{\ell}{2}(\theta - \theta_1)\right)\right|^k, & \text{for } |\theta| > \frac{(\ell - 1)\pi}{\ell}. \end{cases} \tag{3.40}$$

The positive integer k and the real numbers $w_1, \dots, w_\ell$ are free parameters. This weight is of the form $\phi = e^{-V}$, where $V^{(k)}(\theta)$ has jump discontinuities at the points θ_j of magnitudes that can be adjusted by choice of the w_j. One advantage of this family from the point of view of numerical computation is that the Fourier coefficients of $\phi(\theta)$ can be evaluated symbolically. In a package such as Mathematica capable of arbitrary precision arithmetic, this leads to the possibility of computing the elements of the Toeplitz matrices (whose minors are assembled to yield the coefficients of the polynomial $\pi_n(z)$) with sufficient accuracy for subsequent numerical computation of the zeros when n is large. In practice, we computed the coefficients up to an overall factor by scaling the Fourier coefficients making up the Toeplitz matrix by e^{V_0}. This is necessary to avoid numerical overflow or underflow since according to the strong Szegő limit theorem, the Toeplitz determinant of $\phi = e^{-V}$ of size $n + 1$ scales as e^{-nV_0}. While the Fourier coefficients can be computed symbolically, we obtained the coefficients $\Delta_j^{(k)} e^{i\Omega(\theta_j)}$ appearing in the rational function $f_n(z)$ defined by (3.23) with the help of numerical integration.

The Mathematica code we wrote to carry out these computations is available from the companion website to this paper, which can be found by visiting http://www.hindawi.com and looking up this paper using the DOI reference 10.1155/IMRP/2006/48673. The code takes as input the number of jump discontinuities ℓ, the order k of the derivative experiencing the discontinuites, a vector $\boldsymbol{w}$ of length ℓ containing the parameters w_j, and the degree n of the polynomial $\pi_n(z)$. The output is a figure showing the unit circle (black) with exterior tick marks at the angles θ_j, $j = 1, \dots, \ell$, the zero-attracting circle $|z| = 1 - (k+1)\log(n)/n$ (green), the inner boundary circle $|z| = 1 - k\log(n)/n$ of the zero-free annulus (red), and the zeros of $f_n(z)$ that occur in a neighborhood of the unit

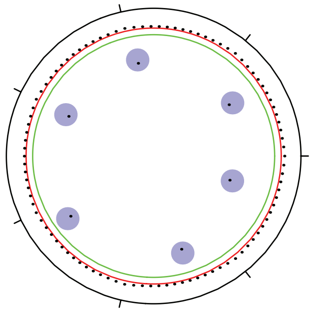

Figure 3.1 The output of the Mathematica code for $n = 104$, $k = 3$, $\ell = 7$, and $w^T = (-1, -1/2, -1/4, -1, -1/4, -1, -1/2)$.

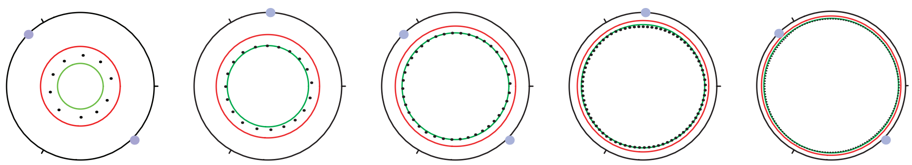

Figure 3.2 A sequence of plots showing convergence of the zeros of $\pi_n(z)$ toward the curve $|z| = 1 - (k+1)\log(n)/n$ as $n \to \infty$. The parameters are $\ell = 3$, $w^T = (-4, -2, -3)$, and $k = 2$. From left to right, $n = 10, 20, 40, 80$, and 160.

disk (large lavender dots). Superimposed on the figure are the zeros of $\pi_n(z)$ (small black dots). Sample output from the program is shown in Figure 3.1.3.

The first effect we would like to illustrate is the rate of convergence of the zeros of $\pi_n(z)$ to the unit circle with increasing n, see Figure 3.2. The annulus associated with the inequalities $1 - k\log(n)/n < |z| < 1$ is asymptotically zero-free, and the curve $|z| = 1-(k+1)\log(n)/n$ asymptotically attracts the zeros near the unit circle. The convergence to the zero-attracting circle is clear. More difficult to discern from the images is the outward "bulging" of the zeros near the angles θ_j of discontinuity toward the inner boundary of the zero-free region. The imaginative reader can see this effect beginning in the figure corresponding to $n = 160$, but larger values of n (and a rescaling of the figures near the unit circle) will be necessary to resolve the "bulging" completely.

Zeros of $f_n(z)$ play little role for the polynomials whose zeros are illustrated in Figure 3.2. Next, we would like to illustrate the effect zeros of $f_n(z)$ can have on $\pi_n(z)$; this is the phenomenon of spurious zeros. Note that in the present case of equally spaced

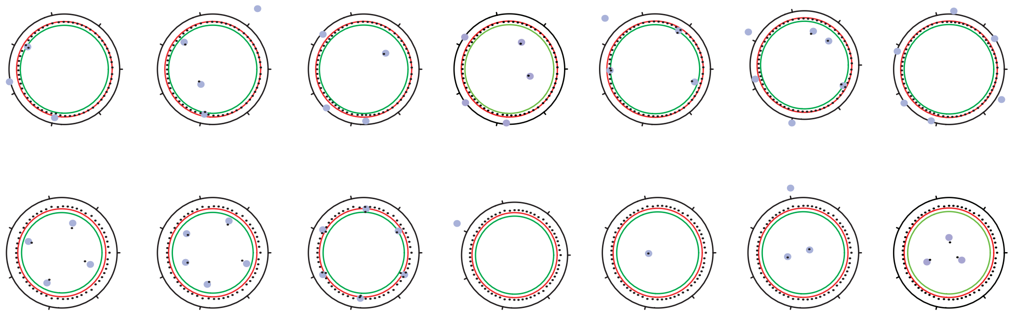

Figure 3.3 Periodic fluctuation of the spurious zeros with respect to n. Top: $k = 2$. Bottom: $k = 3$. In both cases, n varies from $n = 60$ to $n = 66$ from left to right, there are $\ell = 7$ equally spaced points of discontinuity, and $\boldsymbol{w}^T = (-1, -1/2, -1/4, 1/2, -1/4, 1/4, -1/2)$. Note that while for $k = 3$ the majority of zeros appear to lie in the zero-free annulus $1 - k\log(n)/n < |z| < 1$, this is a finite n effect.

angles $\theta_1, \ldots, \theta_\ell$, the function $f_n(z)$ is periodic in n with period ℓ. In Figure 3.3, we present images corresponding to one period of the function $f_n(z)$ in the case of discontinuities of the second and third derivatives of $V(\theta)$. Here it is clear that the $\ell - 1$ zeros of $f_n(z)$ fluctuate about rapidly with n, and can be either inside the unit circle or outside. Each zero of $f_n(z)$ inside the unit circle is an asymptote for exactly one zero of $\pi_n(z)$, while those outside the unit circle have little effect on the zeros of $\pi_n(z)$. The zeros near the unit circle, either inside or outside, have a repulsive effect on the zeros of $\pi_n(z)$.

From the images in Figure 3.3, it is not obvious that the zeros of $f_n(z)$ inside the unit disk attract corresponding spurious zeros of $\pi_n(z)$ in the limit $n \to \infty$. The images shown in Figure 3.4 show that this convergence indeed occurs. Here, we have used the periodicity of $f_n(z)$ to examine the asymptotic behavior of the zeros of $\pi_n(z)$ along a periodic subsequence of n-values along which the zeros of $f_n(z)$ remain fixed.

The effect of a zero of $f_n(z)$ upon those of $\pi_n(z)$ is the most subtle when it occurs near the unit circle. It should be stressed that the parameters w_j of the weight under consideration can be deformed in a continuous manner such that it may always be arranged that $f_n(z)$ has zeros near the unit circle. We fixed n and chose a one-parameter deformation of the $w_j = w_j(t)$ in order to continuously tune a zero of $f_n(z)$ through the unit circle from outside to inside. A movie of this deformation is available at the companion website to this paper (see http://www.hindawi.com, looking up this paper using the DOI reference 10.1155/IMRP/2006/48673) and several consecutive frames of this movie are shown below in Figure 3.5. Here it can be clearly seen that as a zero of $f_n(z)$ enters the

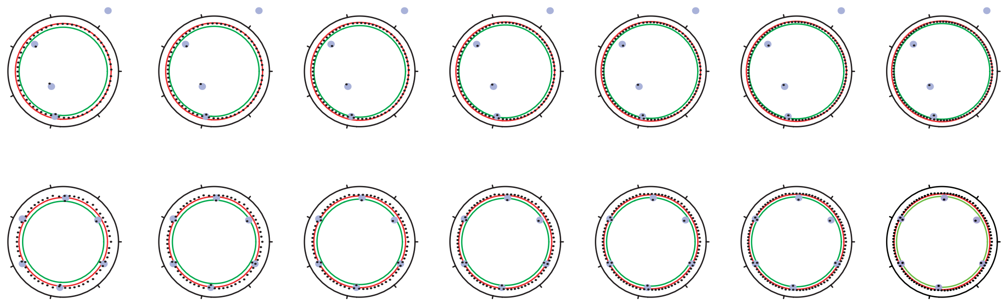

Figure 3.4 Convergence of spurious zeros with increasing n. Top: $k = 2$ and $n = 61, 68, 75, 82, 89, 96$, and 103 from left to right. Bottom: $k = 3$ and $n = 62, 69, 76, 83, 90, 97$, and 104 from left to right. In both cases there are $\ell = 7$ equally spaced points of discontinuity and $\boldsymbol{w}^T = (-1, -1/2, -1/4, 1/2, -1/4, 1/4, -1/2)$.

unit disk, it initially repels the zeros near the attracting circle of $|z| = 1 - (k+1)\log(n)/n$ by pushing them inwards. The zeros along the attracting curve then move apart to make way for the incoming zero of $f_n(z)$. Exactly one zero of $\pi_n(z)$ fails to get out of the way, however, and instead enters the orbit of the moving zero of $f_n(z)$. As the zero of $f_n(z)$ moves inside the attracting circle, it thus draws with it a spurious zero of $\pi_n(z)$.

3.2 The $\overline{\partial}$ steepest descent method for fixed weights

Here, we begin the task of proving the theorems stated in Section 3.1 by analyzing the behavior of the matrix $\mathbf{M}^n(z)$ solving Riemann-Hilbert Problem 2.1 for a fixed weight ϕ, in the limit $n \to \infty$. We recall the representation (3.1) of $\phi(\theta)$ in terms of $V : S^1 \to \mathbb{R}$. In force, in order that Riemann-Hilbert Problem 2.1 indeed describes the orthogonal polynomials with respect to $\phi(\theta)$, we have the following assumption.

Assumption 3.17. V is a real continuous function on the circle that, for some exponent $\nu \in (0, 1]$ and for some constant $K > 0$, satisfies a uniform Hölder continuity condition $|V(\theta_2) - V(\theta_1)| \le K|\theta_2 - \theta_1|^\nu$.

This guarantees that $\phi(\theta)$ is a strictly positive function that also satisfies a Hölder continuity condition with the same exponent, but with a possibly different constant K.

3.2.1 *Conversion to an equivalent $\overline{\partial}$ problem. Solution of the $\overline{\partial}$ problem in terms of integral equations.*

We proceed in several steps. First, let $\mathbf{N}^n(z)$ be a new unknown related

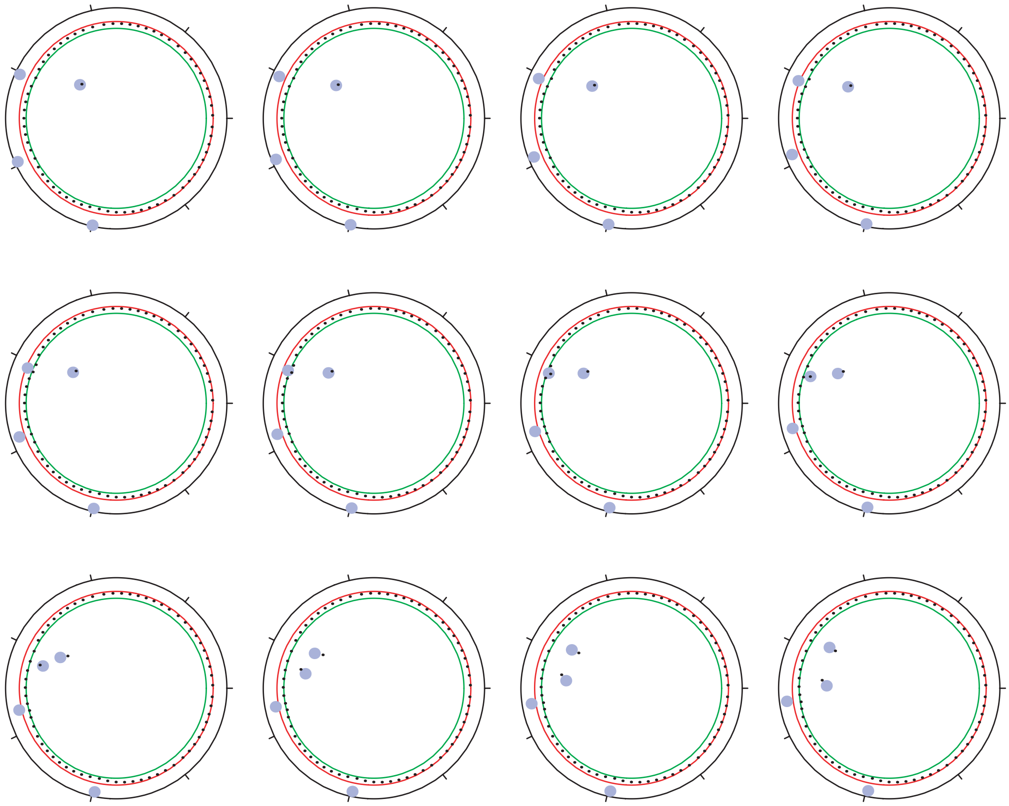

Figure 3.5 A continuous one-parameter deformation of a weight of the form (3.40) with $\ell = 7$ and $k = 2$. The vector of parameters is $w(t)^T = (0, 0, 0, 1/2, 0, 1/4, 0) - (1, 1/2, 1/4, 0, 1/4, 0, 1/2)t$ and t varies from $t = 1/40$ to $t = 13/80$ in steps of $\Delta t = 1/80$. The frames are ordered left to right and top to bottom.

to $\mathbf{M}^n(z)$ as follows:

$$\mathbf{N}^n(z) := \begin{cases} \mathbf{M}^n(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix}, & \text{for } |z| > 1, \\ \mathbf{M}^n(z), & \text{for } |z| < 1. \end{cases} \tag{3.41}$$

It follows from Riemann-Hilbert Problem 2.1 that the new unknown $\mathbf{N}^n(z)$ tends to the identity matrix as $z \to \infty$, and that $\mathbf{N}^n(z)$ is analytic for $|z| \neq 1$, with boundary values on the unit circle related by

$$\mathbf{N}^n_+(e^{i\theta}) = \mathbf{N}^n_-(e^{i\theta}) \begin{pmatrix} e^{in\theta} & \phi(\theta) \\ 0 & e^{-in\theta} \end{pmatrix}, \tag{3.42}$$

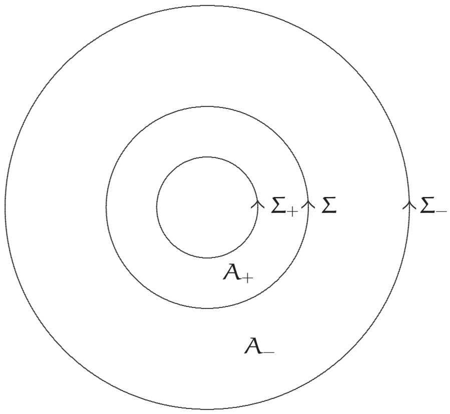

Figure 3.6 The annular domains $A_\pm$ of the equivalent $\overline{\partial}$ problem for polynomials orthogonal on the unit circle. For a given $\epsilon > 0$, the contour Σ_+ corresponds to $|z| = 2^{-\epsilon}$ and the contour Σ_- corresponds to $|z| = 2^{\epsilon}$.

where $\mathbf{N}^n_+(z)$ (resp., $\mathbf{N}^n_-(z)$) indicates the boundary value taken at the point z on the circle from the inside (resp., outside).

Next, observe the following factorization of the jump condition (3.42):

$$\mathbf{N}^n_+(e^{i\theta}) = \mathbf{N}^n_-(e^{i\theta}) \begin{pmatrix} 1 & 0 \\ e^{-in\theta}\phi(\theta)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & \phi(\theta) \\ -\phi(\theta)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{in\theta}\phi(\theta)^{-1} & 1 \end{pmatrix}. \tag{3.43}$$

To take advantage of this factorization, we introduce two new contours $\Sigma_\pm$ which together with Σ bound two concentric annular domains $A_\pm$ as shown in Figure 3.2.1.

We will now need some extension of the function $\phi(\theta)^{-1}$ defined for z on the unit circle with $\arg(z) = \theta$ to the annular domain $\overline{A_+ \cup A_-}$. To make use of the family of extensions defined in (1.5), we now make the following assumption about the weight $\phi(\theta)$.

Assumption 3.18. The function V is of class $C^{k-1}(S^1)$ for some $k = 1, 2, 3, \ldots$.

Note that when $k = 1$, this assumption is contained in Assumption 3.17, but when $k > 1$ it provides new information. Recall the "bump" function B with the properties listed in Section 1.5. Then, for any integer m in the range $1 \le m \le k$, and for any $\epsilon > 0$, we may apply the extension operator E_m to the function V and therefore define a matrix

$\mathbf{P}^n_{m,\epsilon}(r,\theta)$ as follows:

$$\mathbf{P}^n_{m,\epsilon}(r,\theta) := \begin{cases} \mathbf{N}^n(z)\begin{pmatrix} 1 & 0 \\ z^{-n}B\left(\dfrac{\log(r)}{\epsilon}\right)e^{E_m V(r,\theta)} & 1 \end{pmatrix}, & \text{for } z = re^{i\theta} \in A_-, \\ \mathbf{N}^n(z)\begin{pmatrix} 1 & 0 \\ -z^{n}B\left(\dfrac{\log(r)}{\epsilon}\right)e^{E_m V(r,\theta)} & 1 \end{pmatrix}, & \text{for } z = re^{i\theta} \in A_+, \\ \mathbf{N}^n(z), & \text{for } z = re^{i\theta} \notin \overline{A_+ \cup A_-}. \end{cases} \tag{3.44}$$

Thus, the factor $e^{E_m V(r,\theta)}$ appearing above is our selected extension of the function $\phi(\theta)^{-1}$ from the unit circle to the regions A_+ and A_-.

Unlike $\mathbf{M}^n(z)$ and hence $\mathbf{N}^n(z)$, the matrix $\mathbf{P}^n_{m,\epsilon}(r,\theta)$ is not piecewise analytic because the factors relating $\mathbf{P}^n_{m,\epsilon}(r,\theta)$ to $\mathbf{N}^n(z)$ in the domains $A_\pm$ are not analytic. Indeed, in view of (1.7), the exponent $E_m V(r,\theta)$ is not an analytic function. Note however that it follows from Assumption 3.18 and the analyticity of $\mathbf{M}^n(z)$ and hence of $\mathbf{N}^n(z)$ for $|z| \neq 1$ that the matrix $\mathbf{P}^n_{m,\epsilon}(r,\theta)$ is continuous for $r \neq 1$ as long as $1 \leq m \leq k$. In particular, the "bump" function factor $B(\log(r)/\epsilon)$ ensures that $\mathbf{P}^n_{m,\epsilon}(r,\theta)$ is continuous across the circles $\Sigma_\pm$. At the circle of discontinuity Σ, the boundary values taken by $\mathbf{P}^n_{m,\epsilon}(r,\theta)$ satisfy the jump condition

$$\lim_{r\uparrow 1}\mathbf{P}^n_{m,\epsilon}(r,\theta) = \lim_{r\downarrow 1}\mathbf{P}^n_{m,\epsilon}(r,\theta)\begin{pmatrix} 0 & \phi(\theta) \\ -\phi(\theta)^{-1} & 0 \end{pmatrix}. \tag{3.45}$$

Remark 3.19. Note that the approach to this problem taken in [8], where analyticity of ϕ is assumed, amounts to replacing $E_m V(r,\theta)$ with the analytic extension $E_\infty V(r,\theta)$ and omitting the "bump" function factor $B(\log(r)/\epsilon)$, with the latter being at the cost of an exponentially near-identity jump discontinuity across the inner and outer circles $\Sigma_\pm$.

Next, we may remove the jump discontinuity along the unit circle by introducing a model matrix $\dot{\mathbf{P}}(z)$ that is analytic for $|z| \neq 1$, tends to the identity matrix $\mathbb{I}$ as $z \to \infty$, and that takes on the unit circle continuous boundary values $\dot{\mathbf{P}}_+(z)$ (resp., $\dot{\mathbf{P}}_-(z)$) from the inside (resp., outside) that are related by

$$\dot{\mathbf{P}}_+(e^{i\theta}) = \dot{\mathbf{P}}_-(e^{i\theta})\begin{pmatrix} 0 & \phi(\theta) \\ -\phi(\theta)^{-1} & 0 \end{pmatrix}. \tag{3.46}$$

Such a matrix can be found in closed form. Setting

$$\dot{\mathbf{Q}}(z) = \dot{\mathbf{P}}(z)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.47}$$

for $|z| < 1$ and $\dot{\mathbf{Q}}(z) = \dot{\mathbf{P}}(z)$ for $|z| > 1$, one may equivalently seek a matrix $\dot{\mathbf{Q}}(z)$ that is analytic for $|z| \neq 1$, tends to the identity matrix as $z \to \infty$, and that takes on the unit circle continuous boundary values $\dot{\mathbf{Q}}_+(z)$ (resp., $\dot{\mathbf{Q}}_-(z)$) from the inside (resp., outside) that are related by the diagonal jump condition

$$\dot{\mathbf{Q}}_+(e^{i\theta}) = \dot{\mathbf{Q}}_-(e^{i\theta})\phi(\theta)^{\sigma_3}, \tag{3.48}$$

where σ_3 denotes the Pauli matrix

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.49}$$

Clearly, we may seek $\dot{\mathbf{Q}}(z)$ as a diagonal matrix. Assumption 3.17 guarantees that $\log(\phi(\theta)) = -V(\theta)$ is well defined on the circle that satisfies a uniform Hölder continuity condition with exponent $\nu \in (0, 1]$, and therefore we may obtain a matrix $\dot{\mathbf{Q}}(z)$ with the aforementioned properties in the explicit form $\dot{\mathbf{Q}}(z) = S_\phi(z)^{\sigma_3}$, where $S_\phi(z)$ is the Szegő function associated with the weight $\phi(\theta)$ as defined in (3.2). Going back to $\dot{\mathbf{P}}(z)$, the model matrix we will use to remove the jump discontinuity in $\mathbf{P}^n_{m,\epsilon}(r, \theta)$ for $r = 1$ is defined by the explicit formula:

$$\dot{\mathbf{P}}(z) := \begin{cases} \begin{pmatrix} S_\phi(z) & 0 \\ 0 & S_\phi(z)^{-1} \end{pmatrix}, & |z| > 1, \\ \begin{pmatrix} 0 & S_\phi(z) \\ -S_\phi(z)^{-1} & 0 \end{pmatrix}, & |z| < 1. \end{cases} \tag{3.50}$$

To actually remove the discontinuity, we introduce a new matrix function $\mathbf{H}^n_{m,\epsilon}(r, \theta)$ defined for $r \neq 1$ by the formula

$$\mathbf{H}^n_{m,\epsilon}(r, \theta) := \mathbf{P}^n_{m,\epsilon}(r, \theta)\dot{\mathbf{P}}(z)^{-1}. \tag{3.51}$$

By Assumptions 3.17 and 3.18, the matrix $\mathbf{H}^n_{m,\epsilon}(r, \theta)$ is continuous throughout the two regions $r < 1$ and $r > 1$. Moreover, a continuous extension to $r = 1$ is possible because

$\mathbf{P}^{n}_{m,\epsilon}(r,\theta)$ and $\dot{\mathbf{P}}(z)$ satisfy the same jump condition at $r = |z| = 1$. Thus, we see that for $1 \leq m \leq k$, and for any $\epsilon > 0$, the matrix function $\mathbf{H}^{n}_{m,\epsilon}(r,\theta)$ defined by (3.51) may be viewed as a continuous function on the whole plane with polar coordinates $-\pi \leq \theta < \pi$ and $0 \leq r < \infty$.

At this point, we can summarize the explicit transformations we have introduced and relate $\mathbf{H}^{n}_{m,\epsilon}(r,\theta)$ directly back to $\mathbf{M}^{n}(z)$. Combining (3.41), (3.44), (3.50), and (3.51), we have by definition

$$\mathbf{H}^{n}_{m,\epsilon}(r,\theta) := \begin{cases} \mathbf{M}^{n}(z)\begin{pmatrix} 0 & -S_{\phi}(z) \\ S_{\phi}(z)^{-1} & z^{n}S_{\phi}(z)B\left(\dfrac{\log(r)}{\epsilon}\right)e^{E_m V(r,\theta)} \end{pmatrix}, & 0 \leq r < 1, \\ \mathbf{M}^{n}(z)\begin{pmatrix} z^{-n}S_{\phi}(z)^{-1} & 0 \\ S_{\phi}(z)^{-1}B\left(\dfrac{\log(r)}{\epsilon}\right)e^{E_m V(r,\theta)} & z^{n}S_{\phi}(z) \end{pmatrix}, & r > 1, \end{cases} \tag{3.52}$$

where $z = re^{i\theta}$.

For $r < 2^{-\epsilon}$ and $r > 2^{\epsilon}$, we have $B(\log(r)/\epsilon) \equiv 0$, and in these regions the matrix $\mathbf{H}^{n}_{m,\epsilon}(r,\theta)$ clearly inherits analyticity from $\mathbf{M}^{n}(z)$; in other words, in these regions $\mathbf{H}^{n}_{m,\epsilon}(r,\theta)$ is a smooth function of the combination $z = re^{i\theta}$. However, for $z \in \Omega_{\pm}$, the matrix $\mathbf{H}^{n}_{m,\epsilon}(r,\theta)$ is certainly not analytic. In order to measure the deviation from analyticity in the case when $m = k$, we introduce a further assumption on $\phi(\theta)$.

Assumption 3.20. The function V is of class $C^{k-1,1}(S^1)$, that is, the function $V^{(k-1)}(\theta)$ is Lipschitz continuous.

Note that since $k \geq 1$, this condition implies in particular that $V(\theta)$ is Lipschitz, and thus the Hölder continuity part of Assumption 3.17 is subsumed. With Assumption 3.20 in force, we may compute the $\overline{\partial}$-derivative of $\mathbf{H}^{n}_{m,\epsilon}(r,\theta)$ for $r \neq 1$ and for all integer m in the range $1 \leq m \leq k$. Differentiation of (3.52) yields

$$\overline{\partial}\mathbf{H}^{n}_{m,\epsilon}(r,\theta) = \begin{cases} \mathbf{M}^{n}(z)\begin{pmatrix} 0 & 0 \\ 0 & z^{n}S_{\phi}(z)\overline{\partial}\left[B\left(\dfrac{\log(r)}{\epsilon}\right)e^{E_m V(r,\theta)}\right] \end{pmatrix}, & 0 \leq r < 1, \\ \mathbf{M}^{n}(z)\begin{pmatrix} 0 & 0 \\ S_{\phi}(z)^{-1}\overline{\partial}\left[B\left(\dfrac{\log(r)}{\epsilon}\right)e^{E_m V(r,\theta)}\right] & 0 \end{pmatrix}, & r > 1, \end{cases} \tag{3.53}$$

for almost all θ, and then elimination of $\mathbf{M}^n(z)$ in terms of $\mathbf{H}^n_{m,\epsilon}(r,\theta)$ using (3.52) again gives

$$\overline{\partial}\mathbf{H}^n_{m,\epsilon}(r,\theta) = \mathbf{H}^n_{m,\epsilon}(r,\theta)\mathbf{W}^n_{m,\epsilon}(r,\theta), \quad \text{for } r \neq 1 \text{ and almost all } \theta \in S^1, \tag{3.54}$$

where the matrix $\mathbf{W}^n_{m,\epsilon}(r,\theta)$ is given for $r \neq 1$ and almost all θ by the explicit formula

$$\mathbf{W}^n_{m,\epsilon}(r,\theta) := \begin{cases} \begin{pmatrix} 0 & z^n S_\phi(z)^2 \overline{\partial}\left[B\left(\dfrac{\log(r)}{\epsilon}\right)e^{E_m V(r,\theta)}\right] \\ 0 & 0 \end{pmatrix}, & 0 \le r < 1, \\ \begin{pmatrix} 0 & 0 \\ z^{-n} S_\phi(z)^{-2} \overline{\partial}\left[B\left(\dfrac{\log(r)}{\epsilon}\right)e^{E_m V(r,\theta)}\right] & 0 \end{pmatrix}, & r > 1. \end{cases} \tag{3.55}$$

In particular, we see that the matrix $\mathbf{H}^n_{m,\epsilon}(r,\theta)$ is a solution of the following $\overline{\partial}$-problem.

$\overline{\partial}$ Problem 3.21. Find a 2×2 matrix $\mathbf{U}(r,\theta)$ with the following properties.

Smoothness. $\mathbf{U}(r,\theta)$ is a Lipschitz continuous function throughout $\mathbb{R}^2$.

Deviation from analyticity. The relation

$$\overline{\partial}\mathbf{U}(r,\theta) = \mathbf{U}(r,\theta)\mathbf{W}^n_{m,\epsilon}(r,\theta) \tag{3.56}$$

holds for all points in $\mathbb{R}^2$ with the exception of a set of measure zero. The matrix $\mathbf{W}^n_{m,\epsilon}(r,\theta)$ is defined almost everywhere by (3.55) and is essentially compactly supported.

Normalization. The matrix $\mathbf{U}(r,\theta)$ is normalized at $r = \infty$ as follows:

$$\lim_{r\to\infty} \mathbf{U}(r,\theta) = \mathbb{I}. \tag{3.57}$$

In writing down this $\overline{\partial}$-problem, we have focused on just a few specific properties of the matrix $\mathbf{H}^n_{m,\epsilon}(r,\theta)$. However, it is important that in doing so, we have not introduced any spurious solutions.

Proposition 3.22. Suppose that $\phi = e^{-V}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k = 1,2,3,\ldots$. Then for all $n = 0,1,2,3,\ldots$, for $m = 1,2,\ldots,k$, and for all $\epsilon > 0$, the matrix $\mathbf{W}^n_{m,\epsilon}(r,\theta)$ is well defined almost everywhere by (3.55) and $\overline{\partial}$ Problem 3.21 has a unique solution, namely $\mathbf{U}(r,\theta) = \mathbf{H}^n_{m,\epsilon}(r,\theta)$. □

Proof. The existence of a solution follows from (3.52) and the existence of $\mathbf{M}^n(z)$ for $n = 0,1,2,\ldots$. To establish the uniqueness, we first consider the determinant of any solution of $\overline{\partial}$ Problem 3.21. Clearly, $\det(\mathbf{U}(r,\theta))$ is a Lipschitz continuous function that tends to 1 as $z \to \infty$. Moreover, the relation $\overline{\partial}\det(\mathbf{U}(r,\theta)) = \mathrm{tr}(\mathbf{W}^n_{m,\epsilon}(r,\theta))\det(\mathbf{U}(r,\theta))$ holds almost everywhere, and thus by (3.55) we see that $\overline{\partial}\det(\mathbf{U}(r,\theta)) = 0$ holds almost everywhere in the plane. It follows that $\det(\mathbf{U}(r,\theta))$ is not only Lipschitz continuous, but is in fact an entire function of $z = re^{i\theta}$ that tends to 1 as $z \to \infty$. Therefore from Liouville's theorem, we see that $\det(\mathbf{U}(r,\theta)) \equiv 1$. Next, consider the matrix ratio of any two solutions $\mathbf{U}(r,\theta)$ and $\widetilde{\mathbf{U}}(r,\theta)$ of $\overline{\partial}$ Problem 3.21; this is the matrix $\mathbf{R}(r,\theta)$ defined by

$$\mathbf{R}(r,\theta) := \mathbf{U}(r,\theta)\widetilde{\mathbf{U}}(r,\theta)^{-1}. \tag{3.58}$$

Since $\det(\widetilde{\mathbf{U}}(r,\theta)) \equiv 1$, it follows that $\mathbf{R}(r,\theta)$ is Lipschitz continuous throughout the plane. By direct calculation, we have

$$\begin{aligned}\overline{\partial}\mathbf{R}(r,\theta) &= \overline{\partial}\mathbf{U}(r,\theta)\cdot\widetilde{\mathbf{U}}(r,\theta)^{-1} - \mathbf{U}(r,\theta)\widetilde{\mathbf{U}}(r,\theta)^{-1}\overline{\partial}\widetilde{\mathbf{U}}(r,\theta)\cdot\widetilde{\mathbf{U}}(r,\theta)^{-1}\\ &= \mathbf{U}(r,\theta)\mathbf{W}^n_{m,\epsilon}(r,\theta)\widetilde{\mathbf{U}}(r,\theta)^{-1} - \mathbf{U}(r,\theta)\mathbf{W}^n_{m,\epsilon}(r,\theta)\widetilde{\mathbf{U}}(r,\theta)^{-1}\\ &= 0\end{aligned} \tag{3.59}$$

holding almost everywhere in the plane. It follows that $\mathbf{R}(r,\theta)$ is an entire function of $z = re^{i\theta}$ that tends to the identity matrix as $z \to \infty$, so again by Liouville's theorem we get $\mathbf{R}(r,\theta) \equiv \mathbb{I}$, or equivalently $\widetilde{\mathbf{U}}(r,\theta) \equiv \mathbf{U}(r,\theta)$. ■

The unique solution of $\overline{\partial}$ Problem 3.21 can also be expressed as a solution of an integral equation with Cauchy kernel.

Proposition 3.23. Suppose that $\phi = e^{-V}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k = 1, 2, 3, \ldots$. Then for all $n = 0, 1, 2, 3, \ldots$, for $m = 1, 2, \ldots, k$, and for all $\epsilon > 0$, the matrix $\mathbf{W}^n_{m,\epsilon}(r,\theta)$ is well defined almost everywhere by (3.55) and the corresponding solution $\mathbf{U}(r,\theta) = \mathbf{H}^n_{m,\epsilon}(r,\theta)$ of $\overline{\partial}$ Problem 3.21 satisfies the integral equation

$$\mathbf{U}(r,\theta) = \mathbb{I} - \frac{1}{\pi}\iint \frac{\mathbf{U}(r',\theta')\mathbf{W}^n_{m,\epsilon}(r',\theta')}{z'-z}\,dA', \tag{3.60}$$

where $z = re^{i\theta}$, $z' = r'e^{i\theta'}$, and dA' is a positive area element $dA' = r'dr'd\theta'$. The integral is taken over the entire plane. □

Proof. Recall that the Cauchy kernel is a fundamental solution for the $\overline{\partial}$ operator. In the relation (3.54), we may replace $\overline{\partial}\mathbf{H}^n_{m,\epsilon}(r,\theta)$ by $\overline{\partial}[\mathbf{H}^n_{m,\epsilon}(r,\theta) - \mathbb{I}]$; multiplying by the Cauchy kernel and integrating over the whole plane gives the identity

$$-\frac{1}{\pi}\iint \frac{\overline{\partial}\left[\mathbf{H}^n_{m,\epsilon}(r',\theta') - \mathbb{I}\right]}{z'-z}\,dA' = -\frac{1}{\pi}\iint \frac{\mathbf{H}^n_{m,\epsilon}(r',\theta')\mathbf{W}^n_{m,\epsilon}(r',\theta')}{z'-z}\,dA'. \tag{3.61}$$

On the left-hand, side the $\overline{\partial}$ operator differentiates with respect to the primed variables. Since the Cauchy kernel is absolutely integrable, we may evaluate the integral on the left-hand side by replacing the domain of integration by the region $|z' - z| \geq \delta > 0$ and subsequently taking the limit $\delta \to 0$. For each positive δ, we may apply Stokes' theorem and use the facts that $\overline{\partial}[(z' - z)^{-1}] = 0$ for $|z' - z| \geq \delta$ and that $\mathbf{H}^n_{m,\epsilon}(r, \theta) - \mathbb{I}$ tends to zero as $r \to \infty$ to evaluate the integral over the region $|z' - z| \geq \delta$ in terms of a line integral over the boundary. Thus we have

$$\lim_{\delta \downarrow 0} \frac{1}{2\pi\delta} \int_{|z'-z|=\delta} \left(\mathbf{H}^n_{m,\epsilon}(r', \theta') - \mathbb{I}\right) d\ell' = -\frac{1}{\pi} \iint \frac{\mathbf{H}^n_{m,\epsilon}(r', \theta')\mathbf{W}^n_{m,\epsilon}(r', \theta')}{z' - z} dA', \tag{3.62}$$

where $d\ell'$ is an arc-length element. From the continuity of $\mathbf{H}^n_{m,\epsilon}(r, \theta)$, the integral equation (3.60) with $\mathbf{U}(r, \theta) = \mathbf{H}^n_{m,\epsilon}(r, \theta)$ follows. ■

3.2.2 Asymptotic solution of the integral equation. Estimates of $\mathbf{H}^n_{m,\epsilon}(r, \theta)$ and its derivatives for large n. As the knowledge of the matrix $\mathbf{H}^n_{m,\epsilon}(r, \theta)$ is equivalent to knowledge of $\mathbf{M}^n(z)$ and hence of the polynomial of degree n in the system of polynomials orthogonal on the circle with respect to ϕ, we would like to use the integral equation (3.60) to characterize $\mathbf{H}^n_{m,\epsilon}(r, \theta)$. There is a difficulty in that while existence of solutions for (3.60) is not an issue, one does not automatically have uniqueness. However, it turns out that if the parameter n is sufficiently large, then the integral equation (3.60) defines a contraction mapping and thus may be solved by iteration yielding a unique solution in the form of a Neumann series. In this connection, we can also obtain from (3.60) asymptotic information about the matrix $\mathbf{H}^n_{m,\epsilon}(r, \theta)$, and consequently of the orthogonal polynomial $\pi_n(z)$, in the limit $n \to \infty$.

In order to study (3.60), it is useful to characterize the family of matrix functions $\mathbf{W}^n_{m,\epsilon}(r, \theta)$ more concretely.

Proposition 3.24. Suppose that $V : S^1 \to \mathbb{R}$ is a real function of class $C^{k-1,1}(S^1)$ for some $k \geq 1$, that m is an integer satisfying $1 \leq m \leq k$, and that $\epsilon > 0$ is fixed. Let the integer D be defined as $D := \min(k - m, m - 1)$. Then, the matrix function $\mathbf{W}^n_{m,\epsilon}$ is of class $C_0^{D-1,1}(\mathbb{R}^2 \setminus \{0\})$ if $D > 0$, and of class $L_0^\infty(\mathbb{R}^2 \setminus \{0\})$ if $D = 0$. Moreover, if α and β are nonnegative integers such that $\alpha + \beta \leq D$, then there is a constant $C^{(\alpha,\beta)}_{m,\epsilon} > 0$ such that for all n, the estimate

$$\left\| \frac{\partial^{\alpha+\beta}}{\partial r^\alpha \partial \theta^\beta} \mathbf{W}^n_{m,\epsilon}(r, \theta) \right\| \leq C^{(\alpha,\beta)}_{m,\epsilon} n^\beta e^{-n|\log(r)|} \left|\log(r)\right|^{m-1-\alpha} \sum_{p=0}^{\alpha} n^p \left|\log(r)\right|^p \tag{3.63}$$

holds throughout the region $|\log(r)| \le \epsilon\log(2)$ containing the essential support of $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$. $\square$

Proof. $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$ vanishes identically outside of the annulus $|\log(r)| \le \epsilon\log(2)$. In the disjoint regions $r < 1$ and $r > 1$, the matrix function $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$ is infinitely differentiable with respect to r, and the issue is the continuity of these derivatives at $r = 1$. The relation (1.7) implies that for each fixed θ, $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$ is proportional to $(\log(r))^{m-1}$ near $r = 1$ (where $B(\log(r)/\epsilon) \equiv 1$ holds), and thus all derivatives of $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$ with respect to r through order $m-2$ are Lipschitz continuous at $r = 1$, and the derivative $\partial^{m-1}\boldsymbol{W}^n_{m,\epsilon}(r,\theta)/\partial r^{m-1}$ remains bounded as $r \to 1$, but experiences a jump discontinuity at $r = 1$.

On the other hand, if $r \neq 1$ is fixed, then from (1.5) and (1.7), the matrix $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$ depends analytically on derivatives $V^{(j)}(\theta)$ for $0 \le j \le m$. Since V is of class $C^{k-1,1}(S^1)$, all derivatives of $\boldsymbol{W}^n_{m,\epsilon}$ with respect to θ through order $k-1-m$ will be Lipschitz continuous, while the derivative $\partial^{k-m}\boldsymbol{W}^n_{m,\epsilon}(r,\theta)/\partial\theta^{k-m}$ will be defined for almost all θ and will be uniformly bounded.

To have all mixed partial derivatives of total order at most $D-1$ to be Lipschitz continuous, it is sufficient to have both $D \le m-1$ and $D \le k-m$. If for $1 \le m \le k$ these inequalities force $D = 0$, then no derivatives of $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$ may be taken at all, but $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$ is uniformly bounded and compactly supported in the annulus $|\log(r)| \le \epsilon\log(2)$, that is, $\boldsymbol{W}^n_{m,\epsilon} \in L^\infty_0(\mathbb{R}^2\setminus\{0\})$. If $D = \min(k-m, m-1) > 0$, then we learn that $\boldsymbol{W}^n_{m,\epsilon} \in C^{D-1,1}_0(\mathbb{R}^2\setminus\{0\})$.

Now there are absolute constants $C_\pm > 0$ such that

$$\left\|\frac{\partial^{\alpha+\beta}}{\partial r^\alpha\partial\theta^\beta}\boldsymbol{W}^n_{m,\epsilon}(r,\theta)\right\| = \begin{cases} C_+\left|\displaystyle\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} W_{m,\epsilon,\alpha'',\beta''}(r,\theta)\frac{\partial^{\alpha'+\beta'}}{\partial r^{\alpha'}\partial\theta^{\beta'}}z^n S_\phi(z)^2\right|, & r<1,\\[2ex] C_-\left|\displaystyle\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} W_{m,\epsilon,\alpha'',\beta''}(r,\theta)\frac{\partial^{\alpha'+\beta'}}{\partial r^{\alpha'}\partial\theta^{\beta'}}z^{-n} S_\phi(z)^{-2}\right|, & r>1,\end{cases} \tag{3.64}$$

where

$$W_{m,\epsilon,\alpha'',\beta''}(r,\theta) := \frac{\partial^{\alpha''+\beta''}}{\partial r^{\alpha''}\partial\theta^{\beta''}}\overline{\partial}\left[B\big(\log(r)/\epsilon\big)e^{E_m V(r,\theta)}\right]. \tag{3.65}$$

Generally, the derivatives indexed by α' and β' are uniformly bounded throughout the regions $2^{-\epsilon} < r < 1$ and $1 < r < 2^{\epsilon}$ by a constant multiple of $n^{\alpha'+\beta'}e^{-n|\log(r)|}$. Furthermore, using (1.7), the derivative $W_{m,\epsilon,\alpha'',\beta''}(r,\theta)$ is uniformly bounded by a constant multiple of $|\log(r)|^{m-1-\alpha''}$ throughout the region $|\log(r)| \leq \epsilon\log(2)$. Therefore setting $\alpha'' = \alpha - \alpha'$, an inequality of the form

$$\left\|\frac{\partial^{\alpha+\beta}}{\partial r^{\alpha}\partial\theta^{\beta}}\mathbf{W}^{n}_{m,\epsilon}(r,\theta)\right\| \leq \widetilde{C}^{(\alpha,\beta)}_{m,\epsilon}e^{-n|\log(r)|}\sum_{\alpha'=0}^{\alpha}\sum_{\beta'=0}^{\beta}n^{\beta'}\left|\log(r)\right|^{m-1-\alpha}n^{\alpha'}\left|\log(r)\right|^{\alpha'} \tag{3.66}$$

holds in the region $|\log(r)| \leq \epsilon\log(2)$, where $\widetilde{C}^{(\alpha,\beta)}_{m,\epsilon} > 0$ is a constant.

Finally, since $n^{\beta'} \leq n^{\beta}$, the inequality (3.63) follows, where $C^{(\alpha,\beta)}_{m,\epsilon} = (\beta+1)\widetilde{C}^{(\alpha,\beta)}_{m,\epsilon}$. ■

An important part of our analysis will be the estimation of certain two-dimensional Laplace-type integrals with Cauchy kernels. The main workhorse in this connection is the following lemma.

Lemma 3.25. Let $\epsilon > 0$ and $\nu \geq 1$ be fixed constants. Then there exists a corresponding constant $K_{\epsilon,\nu} > 0$ such that the estimate (note that $z' = r'e^{i\theta'}$)

$$\sup_{z\in\mathbb{C}}\int_{2^{-\epsilon}}^{2^{\epsilon}} r'dr'e^{-n|\log(r')|}\left|\log(r')\right|^{\nu-1}\int_{-\pi}^{\pi}\frac{d\theta'}{|z'-z|} \leq K_{\epsilon,\nu}\frac{\log(n)}{n^{\nu}} \tag{3.67}$$

holds for sufficiently large n. Moreover, for each $\rho > 2^{\epsilon}$, there exists a constant $K_{\epsilon,\nu,\rho} > 0$ such that the estimate

$$\sup_{|\log(|z|)|\geq\log(\rho)}\int_{2^{-\epsilon}}^{2^{\epsilon}} r'dr'e^{-n|\log(r')|}\left|\log(r')\right|^{\nu-1}\int_{-\pi}^{\pi}\frac{d\theta'}{|z'-z|} \leq \frac{K_{\epsilon,\nu,\rho}}{n^{\nu}} \tag{3.68}$$

holds for sufficiently large n. □

Proof. As θ' varies over S^1, the minimum value of $|z'-z|$ is achieved at $\theta' = \theta$, and we thus have $|z'-z| \geq |r'-r|$, which implies the inequality

$$\int_{-\pi}^{\pi}\frac{d\theta'}{|z'-z|} \leq \frac{2\pi}{|r'-r|}. \tag{3.69}$$

Let μ be a positive constant. Clearly, there is another positive constant C_1 depending on μ but not on r or r' such that

$$\left|\log(r') - \log(r)\right| > \mu \Longrightarrow \int_{-\pi}^{\pi}\frac{d\theta'}{|z'-z|} \leq C_1 \tag{3.70}$$

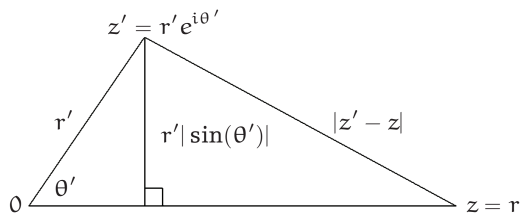

Figure 3.7 The estimate $|z'-z| \geq r'|\sin(\theta')|$. The right-hand side can be replaced by $r'|\theta'|/2$ for $|\theta'| < \pi/2$.

because the condition $|\log(r') - \log(r)| > \mu$ also bounds $|r'-r|$ away from zero. Furthermore, if μ is sufficiently small, there is a positive constant C_2 depending on μ but not on r or r' such that

$$\begin{aligned} &\left|\log(r') - \log(r)\right| \leq \mu, \qquad \left|\log(r')\right| \leq \epsilon \log(2) \\ &\quad \Longrightarrow \int_{-\pi}^{\pi} \frac{d\theta'}{|z'-z|} \leq C_2 \log\left(n + \frac{1}{\left|\log(r') - \log(r)\right|}\right). \end{aligned} \tag{3.71}$$

To establish (3.71), note that for μ sufficiently small, the condition $|\log(r') - \log(r)| \leq \mu$ implies that $|r'-r| < \pi/2$. Assuming without loss of generality that $\theta = 0$, we use the following estimates of the integrand. For $|\theta'| \leq |r'-r|$, we use the estimate $|z'-z| \geq |r'-r|$ which follows from the triangle inequality applied to the identity $z + (z'-z) = z'$. For $|r'-r| < |\theta'| \leq \pi/2$, we use the estimate $|z'-z| \geq r'|\sin(\theta')| \geq r|\theta'|/2$ which follows from the diagram shown in Figure 10. Finally, for $\pi/2 < |\theta'| \leq \pi$, we use the estimate $|z'-z| \geq r$ which follows from the law of cosines because $\cos(\theta') < 0 \Rightarrow |z'-z|^2 = r^2 + (r')^2 - 2rr'\cos(\theta') \geq r^2 + (r')^2 \geq r^2$ (again, see Figure 10). Combining these estimates, we have

$$\int_{-\pi}^{\pi} \frac{d\theta'}{|z'-z|} \leq 2 + \frac{4}{r'} \int_{|r'-r|}^{\pi/2} \frac{d\theta'}{\theta'} + \frac{\pi}{r} \leq 2 + 2^{2+\epsilon} \int_{|r'-r|}^{\pi/2} \frac{d\theta'}{\theta'} + 2^{\epsilon}\pi e^{\mu}. \tag{3.72}$$

This is clearly bounded above by a constant multiple of $\log(|r'-r|^{-1})$ for μ sufficiently small, and from this (3.71) follows as well (adding a constant inside the logarithm keeps the bound positive away from the singularity as long as $n \geq 1$, and for later purposes, it is convenient to take the additive constant to be n).

Now we estimate the integral over r'. Using $r' \le 2^{\epsilon}$, along with (3.70) and (3.71), and changing the integration variable from r' to $s = n\log(r')$, we get

$$\begin{aligned}
&\int_{2^{-\epsilon}}^{2^{\epsilon}} r'dr' e^{-n|\log(r')|}\left|\log(r')\right|^{\nu-1}\int_{-\pi}^{\pi}\frac{d\theta'}{|z'-z|} \\
&\quad \le \frac{2^{2\epsilon}C_1}{n^{\nu}}\int_{\mathcal{A}} ds\, e^{-|s|}|s|^{\nu-1} + \frac{2^{2\epsilon}C_2}{n^{\nu}}\int_{\mathcal{B}} ds\, e^{-|s|}|s|^{\nu-1}\log\left(n + \frac{n}{\left|s - n\log(r)\right|}\right),
\end{aligned} \tag{3.73}$$

where

$$\begin{aligned}
\mathcal{A} &:= \left\{s \text{ such that } |s| \le n\epsilon\log(2) \text{ and } \left|s - n\log(r)\right| > n\mu\right\}, \\
\mathcal{B} &:= \left\{s \text{ such that } |s| \le n\epsilon\log(2) \text{ and } \left|s - n\log(r)\right| \le n\mu\right\}.
\end{aligned} \tag{3.74}$$

Finally, we have

$$\int_{\mathcal{A}} ds\, e^{-|s|}|s|^{\nu-1} \le \int_{-\infty}^{\infty} ds\, e^{-|s|}|s|^{\nu-1}, \tag{3.75}$$

which is finite and independent of n and r because $\nu \ge 1$, while

$$\begin{aligned}
&\int_{\mathcal{B}} ds\, e^{-|s|}|s|^{\nu-1}\left(n + \frac{n}{\left|s - n\log(r)\right|}\right) \\
&\quad \le \int_{-\infty}^{\infty} ds\, e^{-|s|}|s|^{\nu-1}\log\left(n + \frac{n}{\left|s - n\log(r)\right|}\right) \\
&\quad = \log(n)\int_{-\infty}^{\infty} ds\, e^{-|s|}|s|^{\nu-1} + \int_{-\infty}^{\infty} ds\, e^{-|s|}|s|^{\nu-1}\log\left(1 + \frac{1}{\left|s - n\log(r)\right|}\right),
\end{aligned} \tag{3.76}$$

and the last integral is bounded independently of n and r, since by Cauchy and Schwarz,

$$\begin{aligned}
&\int_{-\infty}^{\infty} ds\, e^{-|s|}|s|^{\nu-1}\log\left(1 + \frac{1}{\left|s - n\log(r)\right|}\right) \\
&\quad \le \left[\int_{-\infty}^{\infty} ds\, e^{-2|s|}|s|^{2\nu-2}\right]^{1/2}\left[\int_{-\infty}^{\infty} ds\left(\log\left(1 + \frac{1}{|s|}\right)\right)^2\right]^{1/2}
\end{aligned} \tag{3.77}$$

with both factors being finite. Thus, an upper bound for the integral of interest is proportional to $\log(n)/n^{\nu}$ in general, which proves (3.67). If $\rho > 2^{\epsilon}$ and $|\log(|z|)| \ge \log(\rho)$, then it is not necessary to divide the integration into sets $\mathcal{A}$ and $\mathcal{B}$, and the bound (3.70) can be used over the whole range of integration in which case the upper bound is then proportional to $1/n^{\nu}$, which proves (3.68). ■

With these results in hand, we can formulate and prove the following proposition.

Proposition 3.26. Suppose that $\phi = e^{-V}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k = 1, 2, 3, \ldots$. Let the integer m lie in the range $1 \le m \le k$ and fix $\epsilon > 0$. Define the integer $D := \min(k - m, m - 1) \ge 0$. Then, for all $n \ge 0$, the matrix $\mathbf{W}^n_{m,\epsilon}(r, \theta)$ is well defined almost everywhere by (3.55), and for all n sufficiently large, $\mathbf{H}^n_{m,\epsilon}(r, \theta)$ is given by a Neumann series

$$\mathbf{H}^n_{m,\epsilon}(r,\theta) = \mathbb{I} + \big(\mathcal{W}^n_{m,\epsilon}\mathbb{I}\big)(r,\theta) + \big(\mathcal{W}^n_{m,\epsilon} \circ \mathcal{W}^n_{m,\epsilon}\mathbb{I}\big)(r,\theta) + \cdots \tag{3.78}$$

which converges in the norm $|||\cdot|||_D$, where the double-integral operator $\mathcal{W}^n_{m,\epsilon}$ is defined by

$$\big(\mathcal{W}^n_{m,\epsilon}\mathbf{F}\big)(r,\theta) := -\frac{1}{\pi}\iint \frac{\mathbf{F}(r',\theta')\mathbf{W}^n_{m,\epsilon}(r',\theta')}{z'-z}\, dA'. \tag{3.79}$$

In particular, if $D = 0$, then $\mathbf{H}^n_{m,\epsilon}$ lies in the space $L^\infty(\mathbb{R}^2)$, and if $D > 0$ then $\mathbf{H}^n_{m,\epsilon}$ lies in the space $C^{D-1,1}(\mathbb{R}^2)$ and $|||\mathbf{H}^n_{m,\epsilon}|||_D$ is finite. For all integer p in the range $0 \le p \le D$, the following estimates hold for sufficiently large n:

$$\Big|\Big|\Big|\mathbf{H}^n_{m,\epsilon} - \mathbb{I}\Big|\Big|\Big|_p \le C^{(p)}_{m,\epsilon}\frac{\log(n)}{n^{m-p}}, \tag{3.80}$$

$$\Big|\Big|\Big|\mathbf{H}^n_{m,\epsilon} - \mathbb{I} - \mathcal{W}^n_{m,\epsilon}\mathbb{I}\Big|\Big|\Big|_p \le \left(C^{(p)}_{m,\epsilon}\frac{\log(n)}{n^{m-p}}\right)^2, \tag{3.81}$$

where $C^{(p)}_{m,\epsilon} > 0$ is a constant. Furthermore, for each $\rho > 2^\epsilon$ and for all integer p in the range $0 \le p \le D$, the following estimates hold for sufficiently large n:

$$\sum_{\alpha+\beta\le p} \sup_{\substack{-\pi<\theta<\pi \\ |\log(r)|\ge\log(\rho)}} \left\| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\left[\mathbf{H}^n_{m,\epsilon}(r,\theta) - \mathbb{I}\right]\right\| \le \widetilde{C}^{(p)}_{m,\rho}\frac{1}{n^{m-p}}, \tag{3.82}$$

$$\sum_{\alpha+\beta\le p} \sup_{\substack{-\pi<\theta<\pi \\ |\log(r)|\ge\log(\rho)}} \left\| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\left[\mathbf{H}^n_{m,\epsilon}(r,\theta) - \mathbb{I} - \big(\mathcal{W}^n_{m,\epsilon}\mathbb{I}\big)(r,\theta)\right]\right\| \le \widetilde{C}^{(p)2}_{m,\rho}\frac{\log(n)}{n^{2m-2p}}, \tag{3.83}$$

where $\widetilde{C}^{(p)}_{m,\rho} > 0$ is a constant. □

Proof. Fix $k \ge 1$ and m in the range $1 \le m \le k$, and set $D = \min(k - m, m - 1)$. Let p be a nonnegative integer satisfying $p \le D$. If $p > 0$, suppose that $F(r, \theta)$ is matrix function of class $C^{p-1,1}(\mathbb{R}^2)$ with all derivatives of total order no greater than p uniformly bounded in the whole plane, and if $p = 0$, suppose that $F(r,\theta)$ is of class $L^\infty(\mathbb{R}^2)$. For such $F(r,\theta)$, we recall the norm (1.15), where the Cartesian coordinates x and y are connected to the polar coordinates r and θ in the usual way: $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Because for $z' \neq z$,

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\frac{1}{z'-z} = (-1)^{\alpha+\beta}\frac{\partial^{\alpha+\beta}}{\partial x'^\alpha \partial y'^\beta}\frac{1}{z'-z}, \tag{3.84}$$

for $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha+\beta \leq p$, we have

$$\begin{aligned}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\left(\mathcal{W}^n_{m,\epsilon}F\right)(r,\theta) &= -\frac{(-1)^{\alpha+\beta}}{\pi}\iint F(r',\theta')\boldsymbol{W}^n_{m,\epsilon}(r',\theta')\frac{\partial^{\alpha+\beta}}{\partial x'^\alpha \partial y'^\beta}\left[\frac{1}{z'-z}\right]dA' \\ &= -\frac{1}{\pi}\iint \frac{\partial^{\alpha+\beta}}{\partial x'^\alpha \partial y'^\beta}\left[F(r',\theta')\boldsymbol{W}^n_{m,\epsilon}(r',\theta')\right]\frac{dA'}{z'-z}.\end{aligned} \tag{3.85}$$

Note that in order to integrate by parts in (3.85) for all α and β of interest, we must have $p \leq k - m$. Now, since $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$ is compactly supported in the annulus $|\log(r)| \leq \epsilon\log(2)$,

$$\begin{aligned}&\left\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\left[F(r,\theta)\boldsymbol{W}^n_{m,\epsilon}(r,\theta)\right]\right\| \\ &\quad\leq \sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}}\left[\sup_{\mathbb{R}^2}\left\|\frac{\partial^{\alpha'+\beta'}}{\partial x^{\alpha'}\partial y^{\beta'}}F(r,\theta)\right\|\right]\cdot\left\|\frac{\partial^{\alpha''+\beta''}}{\partial x^{\alpha''}\partial y^{\alpha''}}\boldsymbol{W}^n_{m,\epsilon}(r,\theta)\right\| \\ &\quad\leq |||F|||_p\cdot\sum_{\alpha''=0}^{\alpha}\sum_{\beta''=0}^{\beta}\left\|\frac{\partial^{\alpha''+\beta''}}{\partial x^{\alpha''}\partial y^{\beta''}}\boldsymbol{W}^n_{m,\epsilon}(r,\theta)\right\| \\ &\quad\leq |||F|||_p\sum_{\alpha'+\beta'\leq\alpha+\beta}K^{(\alpha',\beta')}_{m,\epsilon}\left\|\frac{\partial^{\alpha'+\beta'}}{\partial r^{\alpha'}\partial\theta^{\beta'}}\boldsymbol{W}^n_{m,\epsilon}(r,\theta)\right\|,\end{aligned} \tag{3.86}$$

where $K^{(\alpha',\beta')}_{m,\epsilon}$ are some positive constants. Using Proposition 3.24, we then find that

$$\begin{aligned}&\left\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\left[F(r,\theta)\boldsymbol{W}^n_{m,\epsilon}(r,\theta)\right]\right\| \\ &\quad\leq K^{(\alpha+\beta)}_{m,\epsilon}\cdot|||F|||_p e^{-n|\log(r)|}\left|\log(r)\right|^{m-1-(\alpha+\beta)}\sum_{j=0}^{\alpha+\beta}n^j\left|\log(r)\right|^j,\end{aligned} \tag{3.87}$$

where $K^{(l)}_{m,\epsilon} > 0$ is some constant. Finally, we arrive at the estimate

$$\left|\left|\left|\mathcal{W}^n_{m,\epsilon}\mathbf{F}\right|\right|\right|_p \leq \frac{\widetilde{K}^{(p)}_{m,\epsilon}}{\pi}\cdot|||\mathbf{F}|||_p\cdot\sum_{j=0}^{p}n^j\int_{2^{-\epsilon}}^{2^{\epsilon}}r'dr'e^{-n|\log(r')|}\left|\log(r')\right|^{m+j-1-p}\int_{-\pi}^{\pi}\frac{d\theta'}{|z'-z|}, \tag{3.88}$$

where $\widetilde{K}^{(p)}_{m,\epsilon} > 0$ is another constant.

Set $\nu := m + j - p$, and note that $\nu \geq 1$ since as j ranges from 0 to p, ν ranges from $m - p$ to m, and we have $p \leq D \leq m - 1$. Lemma 3.25 may thus be applied to each integral on the right-hand side of (3.88), with the result that

$$\left|\left|\left|\mathcal{W}^n_{m,\epsilon}\mathbf{F}\right|\right|\right|_p \leq \frac{C^{(p)}_{m,\epsilon}\log(n)}{n^{m-p}}|||\mathbf{F}|||_p, \tag{3.89}$$

for some constant $C^{(p)}_{m,\epsilon} > 0$ and n sufficiently large. We note in passing that in order for (3.89) to provide control of the operator $\mathcal{W}^n_{m,\epsilon}$, we need to have $p < m$. The two restrictions in force on p, namely $p < m$ and $p \leq k - m$, have been expressed in the statement of the proposition as the inequality $p \leq D$. If one restricts attention to those $z = re^{i\theta}$ for which $|\log(r)| \geq \log(\rho) > \epsilon\log(2)$, then Lemma 3.25 implies the inequality

$$\sum_{\alpha+\beta\leq p}\sup_{\substack{-\pi<\theta<\pi\\|\log(r)|\geq\log(\rho)}}\left\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial y^\beta}\left(\mathcal{W}^n_{m,\epsilon}\mathbf{F}\right)(r,\theta)\right\| \leq \frac{\widetilde{C}^{(p)}_{m,\rho}}{n^{m-p}}|||\mathbf{F}|||_p, \tag{3.90}$$

where $\widetilde{C}^{(p)}_{m,\rho} > 0$ is a constant and n is sufficiently large.

From (3.89), it is clear that if n is sufficiently large, the double-integral operator $\mathcal{W}^n_{m,\epsilon}$ defined by the formula (3.79) and acting in the integral equation (3.60) thus defines a contraction mapping in the space $C^{p-1,1}(\mathbb{R}^2)$ equipped with the norm $|||\cdot|||_p$, or in the space $L^\infty(\mathbb{R}^2)$ if $p = 0$. This implies that there is a unique solution of (3.60) in this space that may be found by iteration resulting in the $|||\cdot|||_p$-convergent Neumann series

$$\mathbf{U}(r,\theta) := \mathbb{I} + \left(\mathcal{W}^n_{m,\epsilon}\mathbb{I}\right)(r,\theta) + \left(\mathcal{W}^n_{m,\epsilon}\circ\mathcal{W}^n_{m,\epsilon}\mathbb{I}\right)(r,\theta) + \cdots. \tag{3.91}$$

In particular, choosing $p = 0$, one sees that the Neumann series (3.91) furnishes a unique solution of the integral equation (3.60) in the space $L^\infty(\mathbb{R}^2)$. Since Proposition 3.23 guarantees that the matrix $\mathbf{H}^n_{m,\epsilon}(r,\theta)$ is a known solution of the integral equation (3.60) that is (in particular) uniformly bounded in the plane, we may identify it with the Neumann series (3.91) for n sufficiently large. From this point forward in our proof, we assume that n is indeed large enough for this to be the case. Since the same Neumann series (3.91) also converges in the norm $|||\cdot|||_p$ where p may be taken to be as large as D, we also learn that if $D > 0$, then $\mathbf{H}^n_{m,\epsilon}(r,\theta)$ lies in the space $C^{D-1,1}(\mathbb{R}^2)$ and that $|||\mathbf{H}^n_{m,\epsilon}|||_D$ is finite.

From (3.91), taking $|||\cdot|||_p$ norms and using (3.89), we see that

$$|||\mathbf{H}^n_{m,\epsilon} - \mathbb{I}|||_p \leq \|\mathbb{I}\| \sum_{j=1}^{\infty} \left(\frac{C^{(p)}_{m,\epsilon} \log(n)}{n^{m-p}} \right)^j \leq 2\|\mathbb{I}\| \frac{C^{(p)}_{m,\epsilon} \log(n)}{n^{m-p}} \tag{3.92}$$

if n is large enough that n^{m-p} exceeds $2C^{(p)}_{m,\epsilon}\log(n)$. Since according to the integral equation (3.60) satisfied by $\mathbf{H}^n_{m,\epsilon}(r,\theta)$ we have

$$\mathbf{H}^n_{m,\epsilon}(r,\theta) - \mathbb{I} - \left(\mathcal{W}^n_{m,\epsilon}\mathbb{I}\right)(r,\theta) = \left(\mathcal{W}^n_{m,\epsilon}\left(\mathbf{H}^n_{m,\epsilon} - \mathbb{I}\right)\right)(r,\theta), \tag{3.93}$$

we may take norms and use (3.89) and (3.92) to learn that

$$|||\mathbf{H}^n_{m,\epsilon} - \mathbb{I} - \mathcal{W}^n_{m,\epsilon}\mathbb{I}|||_p \leq 2\|\mathbb{I}\| \frac{C^{(p)2}_{m,\epsilon} \log(n)^2}{n^{2m-2p}} \tag{3.94}$$

holds for sufficiently large n. The proof of the estimates (3.80) and (3.81) is complete upon appropriate redefinition of the constant $C^{(p)}_{m,\epsilon}$.

Note that (3.90) implies that the upper bounds in (3.92) and (3.94) can be reduced by a factor of $\log(n)$ if in each case the supremum on the left-hand side is taken over only those values of r satisfying $|\log(r)| \geq \log(\rho)$ for a fixed $\rho > 2^{\epsilon}$. This completes the proof of the estimates (3.82) and (3.83) upon appropriate redefinition of the constant $\widetilde{C}^{(p)}_{m,\rho}$. ■

Remark 3.27. To uniformly control p derivatives of $\mathbf{H}^n_{m,\epsilon}(r,\theta)$, Proposition 3.26 requires that m should lie in the range $1+p \leq m \leq k-p$, and therefore to guarantee the existence of suitable values of m, $V : S^1 \to \mathbb{R}$ should be of class $C^{k-1,1}(S^1)$ for some $k \geq 2p+1$. Also, note that the utility of the estimates (3.81) and (3.83) is that the matrix $\mathbf{W}^n_{m,\epsilon}(r,\theta)$ is off-diagonal, so the diagonal matrix elements of $\mathbf{H}^n_{m,\epsilon}(r,\theta) - \mathbb{I}$ experience more rapid decay than do the off-diagonal elements.

3.3 Proofs of theorems stated in Section 3.1

If we solve (3.52) for $\mathbf{M}^n(z)$ in terms of $\mathbf{H}^n_{m,\epsilon}(r,\theta)$, then since $M^n_{11}(z) = \pi_n(z)$, the monic polynomial of degree n in the system of polynomials orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_\phi$ defined by (1.9), we can easily obtain from Proposition 3.26 asymptotic formulae for $\pi_n(z)$ and its derivatives, valid for large n, with uniform error estimates.

3.3.1 Asymptotic behavior of $\pi_n(z)$ *for* $|z| > 1$

Proof of Theorem 3.1. In the region $r > 1$, we have for each $\epsilon > 0$ and for each $m = 1, \ldots, k$ the exact representation

$$\pi_n(z) = H^n_{m,\epsilon,11}(r,\theta) z^n e^{N(z)} - H^n_{m,\epsilon,12}(r,\theta) B\left(\frac{\log(r)}{\epsilon}\right) e^{E_m V(r,\theta) - N(z)}. \tag{3.95}$$

Here we have used (3.5) to write $S_\phi(z) = e^{N(z)}$ for $|z| \geq 1$. Equivalently,

$$\begin{aligned} &\pi_n(z) z^{-n} e^{-N(z)} - 1 \\ &\quad = \left[H^n_{m,\epsilon,11}(r,\theta) - 1\right] - z^{-n} H^n_{m,\epsilon,12}(r,\theta) B\left(\frac{\log(r)}{\epsilon}\right) e^{E_m V(r,\theta) - 2N(z)}, \quad r \geq 1. \end{aligned} \tag{3.96}$$

If $\rho > 1$ is fixed, then we may choose $\epsilon > 0$ small enough that $B(\log(r)/\epsilon) \equiv 0$ whenever $|z| \geq \rho$. In this case, we have simply

$$\pi_n(z) z^{-n} e^{-N(z)} - 1 = H^n_{m,\epsilon,11}(r,\theta) - 1, \quad |z| \geq \rho > 1. \tag{3.97}$$

The best decay estimate comes from taking $m = k$. In this case, using Proposition 3.26 (specifically recalling the estimate (3.83) and the fact that $\mathbf{W}^n_{m,\epsilon}(r,\theta)$ is an off-diagonal matrix), we see that for some constant $K_\rho > 0$,

$$\sup_{\substack{-\pi < \theta < \pi \\ r \geq \rho > 1}} \left|H^n_{m,\epsilon,11}(r,\theta) - 1\right| \leq K_\rho \frac{\log(n)}{n^{2k}}. \tag{3.98}$$

Now as the combination $\pi_n(z) z^{-n} e^{-N(z)} - 1$ is a function of z that is analytic in the region $|z| > 1$ and decaying to zero as $z \to \infty$, we may express arbitrary derivatives of it as Cauchy integrals:

$$\frac{d^p}{dz^p}\left[\pi_n(z) z^{-n} e^{-N(z)} - 1\right] = -\frac{p!}{2\pi i} \oint_{|s|=\rho} \frac{\pi_n(s) s^{-n} e^{-N(s)} - 1}{(s-z)^p} ds, \tag{3.99}$$

where the contour of integration is oriented in the counterclockwise direction, and $|z| > \rho$. Using (3.98) to bound the integrand then gives a uniform bound of the same order of magnitude for derivatives over regions bounded away from the circle $|z| = \rho$, which can be taken arbitrarily close to the unit circle. This proves (3.7), and completes the proof of Theorem 3.1.

3.3.2 *Asymptotic behavior of* $\pi_n(z)$ *for* $|z| = 1$

Proof of Theorem 3.2. Theorem 3.2 follows from the estimate (3.9) by noting that the error is an analytic function of z in the exterior domain $|z| > 1$ that decays as $z \to \infty$, and therefore (3.9) implies the more general result stated in Theorem 3.2 via the maximum modulus principle. To prove (3.9), we fix any positive value of ϵ and consider $1 \leq r \leq 2^{\epsilon/2}$ in which case $B(\log(r)/\epsilon) \equiv 1$, and therefore (3.96) implies the following formula:

$$\frac{d^p}{dz^p}\left[\pi_n(z)z^{-n}e^{-N(z)} - 1\right] = \partial^p\left[H^n_{m,\epsilon,11}(r,\theta) - 1\right] - \partial^p\left[z^{-n}H^n_{m,\epsilon,12}(r,\theta)e^{E_m V(r,\theta)-2N(z)}\right], \quad 1 \leq r \leq 2^{\epsilon/2}. \tag{3.100}$$

We remind the reader that estimates on derivatives like (3.100) are valid for $1 + p \leq m \leq k - p$ (see the remark at the end of Section 3.2). Using the estimate (3.81) from Proposition 3.26 and noting that $\boldsymbol{W}^n_{m,\epsilon}(r,\theta)$ is an off-diagonal matrix, we see that for some constant $K_p > 0$,

$$\sup_{\mathbb{R}^2}\left|\partial^p\left[H^n_{m,\epsilon,11}(r,\theta) - 1\right]\right| \leq K_p\frac{\log(n)^2}{n^{2m-2p}}, \tag{3.101}$$

if $m \leq k - p$. On the other hand, the dominant contributions actually come from those terms in the second member of the right-hand side of (3.100) in which none of the p derivatives fall on the exponential factor $e^{E_m V(r,\theta)-2N(z)}$ (which has $k - m + 1$ uniformly bounded derivatives). Since $|z| \geq 1$, it suffices to estimate $n^j\partial^{p-j}H^n_{m,\epsilon,12}(r,\theta)$ with the use of the inequality (3.80) in Proposition 3.26. Taking $m = k - p$ for the best possible decay estimate then gives

$$\sup_{\substack{-\pi<\theta<\pi \\ 1\leq r\leq 2^{\epsilon/2}}}\left|\frac{d^p}{dz^p}\left[\pi_n(z)z^{-n}e^{-N(z)} - 1\right]\right| \leq K_p\frac{\log(n)}{n^{k-2p}}, \tag{3.102}$$

where $K_p > 0$ is a constant. This proves (3.9), upon taking the limit $r \downarrow 1$ and writing the z derivatives in terms of θ (differentiation commutes with the limit process).

3.3.3 *Asymptotic behavior of* $\pi_n(z)$ *for* $|z| < 1$ *and of* γ_n.

Proof of Theorems 3.6, 3.9, and 3.11. Using (3.52) and the fact (see (3.5)) that $S_\phi(0) = e^{-V_0}$, we have the exact representation:

$$\gamma^2_{n-1} = -M^n_{21}(0) = H^n_{m,\epsilon,22}(0,\theta)e^{V_0}, \tag{3.103}$$

and, whenever $|z| < 1$,

$$\begin{aligned}\pi_n(z) = M_{11}^n(z) = z^n e^{-V_0 - \overline{N(1/\overline{z})}} B\left(\frac{\log(r)}{\epsilon}\right) e^{E_m V(r,\theta)} H_{m,\epsilon,11}^n(r,\theta) \\ - e^{V_0 + \overline{N(1/\overline{z})}} H_{m,\epsilon,12}^n(r,\theta),\end{aligned} \tag{3.104}$$

where (3.5) has been used, and $z = re^{i\theta}$.

To prove Theorem 3.9, we simply apply (3.83) from Proposition 3.26 in the case $p = 0$ and $m = k$ to the identity (3.103). This immediately yields (3.17) and completes the proof.

The proof of Theorem 3.6 is based on a similar analysis of (3.104). Recalling that $\rho \in (0,1)$, one can choose $\epsilon > -2\log(\rho)/\log(2)$ and then $B(\log(r)/\epsilon) \equiv 1$ in (3.104) for $\rho < |z| < 1$. The estimate (3.13) then follows by taking $m = k$, and using (3.81) from Proposition 3.26 in the case $p = 0$. Similarly, choosing $\epsilon < -\log(\rho)/\log(2)$, we have $B(\log(r)/\epsilon) \equiv 0$ in (3.104) for $|z| < \rho$. Again taking $m = k$, one obtains (3.14) by using (3.83) from Proposition 3.26 with $p = 0$. This completes the proof of Theorem 3.6.

The rest of this section will be devoted to the proof of Theorem 3.11. We begin with (3.104) for $m = k$, a formula that is valid for all z with $|z| < 1$. Using (3.81) from Proposition 3.26, and keeping the term corresponding to $\mathcal{W}_{k,\epsilon}^n \mathbb{I}$, we arrive at the formula

$$\begin{aligned}\pi_n(z) = z^n e^{-V_0 - \overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} B\left(\frac{\log(r)}{\epsilon}\right) \\ + \frac{e^{V_0 + \overline{N(1/\overline{z})}}}{\pi} \iint_{2^{-\epsilon} < r' < 1} \frac{W_{k,\epsilon,12}^n(r',\theta')}{z' - z} dA' + O\left(\frac{\log(n)^2}{n^{2k}}\right),\end{aligned} \tag{3.105}$$

as $n \to \infty$, where the error term is uniform for $|z| < 1$. Here, $dA' = r'dr'd\theta'$ is an area element, and in the integral θ' varies over S^1 (the support of $W_{k,\epsilon,12}^n(r,\theta)$ is the annulus given by the inequalities $2^{-\epsilon} \le r \le 1$).

In the annulus of support of $W_{k,\epsilon,12}^n(r,\theta)$, we have from (3.55) that

$$W_{k,\epsilon,12}^n(r,\theta) = \overline{\partial}\left[z^n e^{-2V_0 - 2\overline{N(1/\overline{z})}} B\left(\frac{\log(r)}{\epsilon}\right) e^{E_k V(r,\theta)}\right]. \tag{3.106}$$

Remark 3.28. The fact that $W_{k,\epsilon,12}^n(r,\theta)$ is in the range of $\overline{\partial}$ means that the double integral in (3.105) can be reduced without approximation to the sum of an explicit contribution and a contour integral. Indeed, by the inversion of the $\overline{\partial}$ operator,

$$\begin{aligned}-\frac{1}{\pi} \iint_{2^{-\epsilon} < r' < 1} \frac{W_{k,\epsilon,12}^n(r',\theta')}{z' - z} dA' \\ = F(z) + z^n e^{-2V_0 - 2\overline{N(1/\overline{z})}} B\left(\frac{\log(r)}{\epsilon}\right) e^{E_k V(r,\theta)} \chi_{(2^{-\epsilon},1)}(r),\end{aligned} \tag{3.107}$$

where χ_I denotes the characteristic function of an interval I, and where F(z) is function analytic except on the circles $|z| = 2^{-\epsilon}$ and $|z| = 1$ bounding the support of $W^n_{k,\epsilon,12}(r,\theta)$ that is chosen to make the right-hand side continuous and decaying as $z \to \infty$. These latter properties uniquely identify F(z) with the Cauchy integral

$$F(z) = -\frac{1}{2\pi i}\oint_{|s|=1} \frac{s^n e^{-V_0 + i\Omega(\arg(s))}}{s - z} ds, \tag{3.108}$$

where the contour of integration is oriented counterclockwise. Note that it is the presence of the bump function $B(\log(r)/\epsilon)$ that makes F(z) continuous at $|z| = 2^{-\epsilon}$. Unfortunately, this interesting formula, while apparently simpler than a double integral, is not as useful for asymptotic analysis as the alternative approach we now follow.

Continuing our analysis, we carry out the differentiation in (3.106) in the region $|z| < 1$ with the use of (1.7):

$$\begin{aligned} &W^n_{k,\epsilon,12}(r,\theta) \\ &= z^n e^{-2V_0 - 2\overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} \left[\overline{\partial} B\left(\frac{\log(r)}{\epsilon}\right) + B\left(\frac{\log(r)}{\epsilon}\right)\overline{\partial} E_k V(r,\theta)\right] \\ &= z^n e^{-2V_0 - 2\overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} \left[\overline{\partial} B\left(\frac{\log(r)}{\epsilon}\right) + \frac{ie^{i\theta}}{2r}\frac{V^{(k)}(\theta)}{(k-1)!}(-i\log(r))^{k-1} B\left(\frac{\log(r)}{\epsilon}\right)\right], \end{aligned} \tag{3.109}$$

and in the special case that $r = |z| > 2^{-\epsilon/2}$, we have $B(\log(r)/\epsilon) \equiv 1$ and $\overline{\partial} B(\log(r)/\epsilon) \equiv 0$, so

$$W^n_{k,\epsilon,12}(r,\theta) = z^n e^{-2V_0 - 2\overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} \cdot \frac{ie^{i\theta}}{2r}\frac{V^{(k)}(\theta)}{(k-1)!}(-i\log(r))^{k-1}, \quad \text{for } 2^{-\epsilon/2} < r < 1. \tag{3.110}$$

The presence of the z^n factor together with the absolute integrability of the Cauchy kernel in two dimensions means that

$$-\frac{1}{\pi}\iint_{2^{-\epsilon}<r'<1} \frac{W^n_{k,\epsilon,12}(r',\theta')}{z' - z} dA' = -\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1} \frac{W^n_{k,\epsilon,12}(r',\theta')}{z' - z} dA' + O(2^{-\epsilon n/2}) \tag{3.111}$$

holds as $n \to \infty$ uniformly for all z with $|z| < 1$. Therefore using the simpler formula (3.110) in the integrand and integrating over the smaller annulus $2^{-\epsilon/2} < r' < 1$ introduces an error that is uniformly exponentially small for $|z| < 1$.

Using the simple identity

$$\bar{\partial}\big(-\log(r)\big)^k = -\frac{kz}{2r^2}\big(-\log(r)\big)^{k-1}, \tag{3.112}$$

we may rewrite (3.110) in the form

$$W^n_{k,\epsilon,12}(r,\theta) = -\frac{i^k}{k!}z^n e^{-2V_0-2\overline{N(1/\bar{z})}}e^{E_k V(r,\theta)}V^{(k)}(\theta)\bar{\partial}\big(-\log(r)\big)^k, \quad \text{for } 2^{-\epsilon/2}<r<1. \tag{3.113}$$

Our subsequent analysis will be specialized to the case where $V^{(k)}(\theta)$ is piecewise continuous, with jump discontinuities at $\ell < \infty$ angles $-\pi < \theta_1 < \cdots < \theta_\ell < \pi$, and is (at first) only Lipschitz between the points of discontinuity. Then, between the points of discontinuity, $V^{(k+1)}(\theta)$ exists almost everywhere and may be identified with a bounded function. Under these circumstances, we may "integrate by parts" (i.e., apply Stokes' theorem) with the following formula:

$$\begin{aligned} &-\frac{1}{\pi}\iint \frac{f(r',\theta')\bar{\partial}g(r',\theta')}{z'-z}dA' \\ &\quad = -\frac{1}{\pi}\iint \frac{\bar{\partial}\big[f(r',\theta')g(r',\theta')\big]}{z'-z}dA' + \frac{1}{\pi}\iint \frac{g(r',\theta')\bar{\partial}f(r',\theta')}{z'-z}dA', \end{aligned} \tag{3.114}$$

and the first integral on the right-hand side may be exchanged for a sum of explicit terms and a contour integral as described in the above remark. In (3.114) and in the rest of the proof, whenever the operator $\bar{\partial}$ appears in the integrand, it acts on the primed variables.

To prepare to use this technique, we begin with

$$\begin{aligned} &-\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1} \frac{W^n_{k,\epsilon,12}(r',\theta')}{z'-z}dA' \\ &\quad = -\frac{i^k}{k!}\sum_{j=1}^{\ell}\left[-\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1,\,\theta'\in I_j} \frac{(z')^n h(r',\theta')V^{(k)}(\theta')\bar{\partial}\big(-\log(r')\big)^k}{z'-z}dA'\right], \end{aligned} \tag{3.115}$$

where $h(r,\theta)$ is shorthand for the following terms:

$$h(r,\theta) := e^{-2V_0-2\overline{N(1/\bar{z})}}e^{E_k V(r,\theta)}, \tag{3.116}$$

and I_j refers to the interval in S^1 of initial angle θ_j and final angle equal to the point of next jump discontinuity as the circle is traversed in the counterclockwise direction.

Using (3.114), we therefore find

$$-\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1}\frac{W^n_{k,\epsilon,12}(r',\theta')}{z'-z}dA' = -J_k(r,\theta)+K_k(r,\theta), \tag{3.117}$$

where

$$\begin{aligned}J_k(r,\theta) &:= \frac{i^k}{k!}\sum_{j=1}^{\ell}\left[-\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1,\,\theta'\in I_j}\frac{(z')^n\overline{\partial}\left[h(r',\theta')V^{(k)}(\theta')\right]\left(-\log(r')\right)^k}{z'-z}dA'\right],\\ K_k(r,\theta) &:= \frac{i^k}{k!}\sum_{j=1}^{\ell}\left[-\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1,\,\theta'\in I_j}\frac{\overline{\partial}\left[(z')^n h(r',\theta')V^{(k)}(\theta')\left(-\log(r')\right)^k\right]}{z'-z}dA'\right].\end{aligned} \tag{3.118}$$

The double-integral expression $K_k(r,\theta)$ may be reduced to contour integrals as follows:

$$K_k(r,\theta) = \frac{i^k}{k!}z^n h(r,\theta)V^{(k)}(\theta)\left(-\log(r)\right)^k\chi_{(2^{-\epsilon/2},1)}(r)+G_k(z), \tag{3.119}$$

where

$$\begin{aligned}G_k(z) := &-\frac{i^k}{k!}\cdot\frac{1}{2\pi i}\oint_{|s|=2^{-\epsilon/2}}\frac{s^n h(|s|,\arg(s))V^{(k)}\left(\arg(s)\right)\left(-\log\left(|s|\right)\right)^k}{s-z}ds\\ &+\frac{i^k}{k!}\sum_{j=1}^{\ell}\frac{\Delta_j^{(k)}}{2\pi i}\int_{2^{-\epsilon/2}e^{i\theta_j}}^{e^{i\theta_j}}\frac{s^n h(|s|,\theta_j)\left(-\log\left(|s|\right)\right)^k}{s-z}ds,\end{aligned} \tag{3.120}$$

and $\Delta_j^{(k)} := V^{(k)}(\theta_j+)-V^{(k)}(\theta_j-)$. Since the $|||\cdot|||_{\circ,1}$ norm (see (1.16)) of the numerator in the first Cauchy integral is proportional to $2^{-\epsilon n/2}$, we may also write $G_k(z)$ in the form

$$G_k(z) = \frac{i^k}{k!}\sum_{j=1}^{\ell}\frac{\Delta_j^{(k)}}{2\pi i}\int_{2^{-\epsilon/2}e^{i\theta_j}}^{e^{i\theta_j}}\frac{s^n h(|s|,\theta_j)\left(-\log\left(|s|\right)\right)^k}{s-z}ds+O\left(2^{-\epsilon n/2}\right) \tag{3.121}$$

as $n\to\infty$, where the exponentially small error term is uniform for $|z|<1$ (right up to the contour of integration; this is a consequence of the Plemelj-Privalov theorem [22]).

Let us now consider $J_k(r,\theta)$. Note that

$$\begin{aligned}\overline{\partial}\left[h(r,\theta)V^{(k)}(\theta)\right] &= \left[V^{(k)}(\theta)\overline{\partial}E_k V(r,\theta)+\frac{iz}{2r^2}V^{(k+1)}(\theta)\right]h(r,\theta)\\ &= \frac{iz}{2r^2}\left[\frac{V^{(k)}(\theta)^2}{(k-1)!}\left(-i\log(r)\right)^{k-1}+V^{(k+1)}(\theta)\right]h(r,\theta).\end{aligned} \tag{3.122}$$

Therefore, inserting this into the integrand for $J_k(r,\theta)$ and applying Lemma 3.25 with $\nu = 2k$ to the integrals resulting from the first term above, we find

$$J_k(r,\theta) = -\frac{i^{k+1}}{(k+1)!}\sum_{j=1}^{\ell}\left[-\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1,\,\theta'\in I_j}\left[-\frac{(k+1)z'}{2(r')^2}\big(-\log(r')\big)^k\right]\right.$$
$$\left.\times\frac{(z')^n h(r',\theta')V^{(k+1)}(\theta')}{z'-z}dA'\right] + O\left(\frac{\log(n)}{n^{2k}}\right), \tag{3.123}$$

where the error is uniformly small as $n \to \infty$ for all $|z| < 1$. With $V^{(k+1)}(\theta)$ bounded in each I_j, we could in principle apply Lemma 3.25 to the remaining integrals. However, this would only give a bound of order $\log(n)/n^{k+1}$, and as $G_k(z)$ will turn out to be (for most z) of size $1/n^{k+1}$, and we will want to consider $G_k(z)$ to provide the dominant term, we need to impose additional conditions on $V^{(k+1)}$ in each I_j to see that $J_k(r,\theta)$ is indeed subdominant. Therefore, we first use (3.112) to write $J_k(r,\theta)$ in the form

$$J_k(r,\theta) = -\frac{i^{k+1}}{(k+1)!}$$
$$\times\sum_{j=1}^{\ell}\left[-\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1,\,\theta'\in I_j}\frac{(z')^n h(r',\theta')V^{(k+1)}(\theta')\overline{\partial}\big(\log(r')\big)^{k+1}}{z'-z}dA'\right]$$
$$+\,O\left(\frac{\log(n)}{n^{2k}}\right). \tag{3.124}$$

The double integral above is of the same form as the original double integral we are trying to compute (see (3.115)), but with k replaced by $k+1$ (everywhere it appears explicitly; in the function $h(r,\theta)$, k remains k).

To continue the analysis of $J_k(r,\theta)$, we therefore use the further assumption that in each of the intervals I_j, $V^{(k+1)}(\theta')$ is a Lipschitz continuous function so that $V^{(k+2)}(\theta')$ exists almost everywhere in I_j and can be identified there with a bounded function. Repeating the above steps, we find that

$$J_k(r,\theta) = -J_{k+1}(r,\theta) + K_{k+1}(r,\theta), \tag{3.125}$$

and with the help of the identity

$$\overline{\partial}\big[h(r,\theta)V^{(k+1)}(\theta)\big] = \frac{iz}{2r^2}\left[\frac{V^{(k)}(\theta)V^{(k+1)}}{(k-1)!}\big(-i\log(r)\big)^{k-1} + V^{(k+2)}(\theta)\right]h(r,\theta), \tag{3.126}$$

we may apply Lemma 3.25 to the integrals that result from substituting the above into $J_{k+1}(r,\theta)$ with $\nu = 2k+1$ (from the first term above) and with $\nu = k+2$ (from the second term above). Consequently,

$$J_{k+1}(r,\theta) = O\left(\frac{\log(n)}{n^{k+2}}\right), \tag{3.127}$$

where the error is uniformly small as $n \to \infty$ for all $r < 1$. As before, $K_{k+1}(r,\theta)$ is given by (3.119) with $G_{k+1}(z)$ given by (3.121) (note that in substituting $k+1$ for k in these formulae, one leaves $h(r,\theta)$ alone).

Combining these results, we have shown that

$$\begin{aligned} -\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1} & \frac{W^n_{k,\epsilon,12}(r',\theta')}{z'-z}\,dA' \\ &= \frac{i^k}{k!}\sum_{j=1}^{\ell}\frac{\Delta_j^{(k)}}{2\pi i}\int_{2^{-\epsilon/2}e^{i\theta_j}}^{e^{i\theta_j}} \frac{s^n h(|s|,\theta_j)\big(-\log(|s|)\big)^k}{s-z}\,ds \\ &\quad - \frac{i^{k+1}}{(k+1)!}\sum_{j=1}^{l}\frac{\Delta_j^{(k+1)}}{2\pi i}\int_{2^{-\epsilon/2}e^{i\theta_j}}^{e^{i\theta_j}} \frac{s^n h(|s|,\theta_j)\big(-\log(|s|)\big)^{k+1}}{s-z}\,ds \\ &\quad + O\left(\frac{\log(n)}{n^{k+2}}\right) + O\Big(r^n\big|\log(r)\big|^k \chi_{(2^{-\epsilon/2},1)}(r)\Big) \end{aligned} \tag{3.128}$$

holds uniformly for $r = |z| < 1$ under the assumptions in force on $V(\theta)$.

Now in addition to $|z| < 1$, we choose arbitrarily a constant $\sigma > 0$ and consider those z for which $\log(|z|) \le -(k-\sigma)\log(n)/n$ (this is the interesting case, since according to Corollary 3.7 we are excluding a zero-free annulus near the unit circle whenever $\sigma > \delta$, where δ is the arbitrary positive parameter in the statement of Corollary 3.7; note that both δ and σ may be taken to be arbitrarily small). Since the function $r^n|\log(r)|^k$ achieves its maximum value when $|\log(r)|$ is proportional to $1/n$, we have

$$\begin{aligned} \max_{\log(r)\le-(k-\sigma)\log(n)/n} r^n\big|\log(r)\big|^k &= r^n\big|\log(r)\big|^k\Big|_{\log(r)=-(k-\sigma)\log(n)/n} \\ &= (k-\sigma)^k n^{\sigma-2k}\big(\log(n)\big)^k. \end{aligned} \tag{3.129}$$

Since we are assuming here that $k \ge 2$, $\sigma > 0$ may be chosen small enough that both error terms in (3.128) may be replaced by $o(n^{-(k+1)})$ as $n \to \infty$ uniformly for $\log(|z|) \le -(k-\sigma)\log(n)/n$.

It remains to evaluate the explicit integrals in (3.128) by Laplace's method. Letting $a = k$ or $a = k+1$, we consider

$$\begin{aligned} Y_{a,j}(z) &:= \int_{2^{-\epsilon/2}e^{i\theta_j}}^{e^{i\theta_j}} \frac{s^n h(|s|,\theta_j)\big(-\log(|s|)\big)^a}{s-z}\,ds \\ &= e^{in\theta_j}\int_{2^{-\epsilon/2}}^{1} \frac{x^n h(x,\theta_j)\big(-\log(x)\big)^a}{x-ze^{-i\theta_j}}\,dx. \end{aligned} \tag{3.130}$$

Expecting the dominant contribution to come from the neighborhood of $x = 1$, we write

$$\begin{aligned} Y_{a,j}(z) &= \frac{e^{in\theta_j}h(1,\theta_j)}{1-ze^{-i\theta_j}}\int_{2^{-\epsilon/2}}^{1} x^n\big(-\log(x)\big)^a dx \\ &\quad + e^{in\theta_j}\int_{2^{-\epsilon/2}}^{1}\left[\frac{h(x,\theta_j)}{x-ze^{-i\theta_j}} - \frac{h(1,\theta_j)}{1-ze^{-i\theta_j}}\right]x^n\big(-\log(x)\big)^a dx. \end{aligned} \tag{3.131}$$

Finding a common denominator and extracting from $Y_{a,j}(z)$ a factor of $e^{in\theta_j}(1-ze^{-i\theta_j})^{-1}$, we see that

$$\begin{aligned} Y_{a,j}(z) = \frac{e^{in\theta_j}}{1-ze^{-i\theta_j}}\Bigg[& h(1,\theta_j)\int_{2^{-\epsilon/2}}^{1} x^n\big(-\log(x)\big)^a dx \\ & + \int_{2^{-\epsilon/2}}^{1}\frac{ze^{-i\theta_j}\big(h(1,\theta_j)-h(x,\theta_j)\big)+h(x,\theta_j)-xh(1,\theta_j)}{x-ze^{-i\theta_j}} \\ & \times x^n\big(-\log(x)\big)^a dx\Bigg]. \end{aligned} \tag{3.132}$$

If z does not approach the point $e^{i\theta_j}$, then the fraction in the integrand of the second integral is easily seen to be bounded by a multiple of $-\log(x)$ that is independent of n. More generally, if z is allowed to approach the point $e^{i\theta_j}$ as $n \to \infty$, then we note that the integrand is analytic in x, and the path of integration may be deformed in such a way that throughout the path of integration, $|x - ze^{-i\theta_j}|$ is bounded away from zero by a quantity that is proportional to $\log(n)/n$ because $\log(|z|) \le -(k-\sigma)\log(n)/n$, and therefore the fraction in the integrand of the second integral is bounded by a multiple of $-\log(x)$ that is proportional to $n/\log(n)$. Therefore,

$$Y_{a,j}(z) = \frac{e^{in\theta_j}h(1,\theta_j)}{1-ze^{-i\theta_j}}\frac{a!}{n^{a+1}}\left(1+O\left(\frac{1}{\log(n)}\right)\right), \tag{3.133}$$

where the error is uniformly small for $\log(|z|) \leq -(k-\sigma)\log(n)/n$. Since $E_k V(1,\theta) = V(\theta)$, we have $h(1,\theta) = e^{-V_0 + i\Omega(\theta)}$. It follows that

$$\begin{aligned} &-\frac{1}{\pi}\iint_{2^{-\epsilon/2}<r'<1} \frac{W^n_{k,\epsilon,12}(r',\theta')}{z'-z}\,dA' \\ &\quad = \frac{i^{k-1}e^{-V_0}}{2\pi n^{k+1}}\sum_{j=1}^{\ell}\Delta_j^{(k)} e^{i\Omega(\theta_j)}\frac{e^{i(n+1)\theta_j}}{e^{i\theta_j}-z} + o\left(n^{-(k+1)}\right) \end{aligned} \tag{3.134}$$

holds uniformly in the region $\log(|z|) < -(k-\sigma)\log(n)/n$, and therefore by (3.105) so does

$$\begin{aligned} \pi_n(z) &= z^n e^{-V_0 - \overline{N(1/\overline{z})}} e^{E_k V(r,\theta)} B\left(\frac{\log(r)}{\epsilon}\right) \\ &\quad + \frac{i^{k+1} e^{\overline{N(1/\overline{z})}}}{2\pi n^{k+1}} \sum_{j=1}^{\ell} \Delta_j^{(k)} e^{i\Omega(\theta_j)} \frac{e^{i(n+1)\theta_j}}{e^{i\theta_j}-z} + o\left(n^{-(k+1)}\right). \end{aligned} \tag{3.135}$$

This concludes the proof of Theorem 3.11.

4 Exponentially varying weights

4.1 Asymptotic formulae for $\pi_n(z)$ and γ_n in the varying-weights case

In this section, we consider weights of the form

$$\phi(\theta) = e^{-nV(\theta)}, \quad \forall \theta \in S^1, \tag{4.1}$$

where $V : S^1 \to \mathbb{R}$ is a given real-valued function of period 2π. The weight (4.1) varies exponentially according to a parameter n, and for each n, we may associate with ϕ the corresponding sequence of monic orthogonal polynomials $\pi_0(z), \pi_1(z), \pi_2(z), \ldots$ and normalization constants $\gamma_0, \gamma_1, \gamma_2, \ldots$. Properly speaking, these quantities depend on the parameter n, and we should invent notation to express this dependence, such as $\pi_j^{(n)}(z)$. We will not introduce this cumbersome notation. However, the reader should take note of this dependence. The limit of interest here is to study the behavior of the particular monic polynomial $\pi_n(z)$ of degree n in this system along with its normalization constant γ_n, in the limit $n \to \infty$. Thus the large parameter n enters simultaneously into the degree of the polynomial and also into the weight, and we are studying the asymptotic behavior along the diagonal of a doubly indexed sequence.

The asymptotic behavior in this limit is governed by the function V, along with some associated functions. First, recall the analytic function $N(z)$ defined by (1.20) for

$|z| \geq 1$, which is associated with the negative frequency component of the Fourier series of $V(\theta)$. Now define

$$\kappa(\theta) := \theta - i\left[N\left(e^{i\theta}\right) - \overline{N\left(e^{i\theta}\right)}\right] = \theta + \Omega(\theta), \tag{4.2}$$

where $\Omega(\theta)$ is the periodic function defined in (1.21). Both functions κ and Ω are as smooth as V is.

Our asymptotic results in this case are contained in the following theorem.

Theorem 4.1. Let $p \geq 0$ be a fixed integer. Suppose that $\phi(\theta) = e^{-nV(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k \geq 2p+2$. If $\kappa'(\theta)$ is strictly positive, then for each $\rho > 1$, there is a constant $K_{p,\rho} > 0$ such that

$$\sup_{|z|\geq\rho} \left| \frac{d^p}{dz^p}\left[\pi_n(z) z^{-n} e^{-nN(z)} - 1\right]\right| \leq K_{p,\rho} \frac{\log(n)}{n^{2(k-1)}} \tag{4.3}$$

holds for all sufficiently large n. □

Theorem 4.2. Let $p \geq 0$ be a fixed integer. Suppose that $\phi(\theta) = e^{-nV(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k \geq 2p+2$. If $\kappa'(\theta)$ is strictly positive, then there is a constant $K_p > 0$ such that

$$\sup_{|z|\geq 1} \left| \frac{d^p}{dz^p}\left[\pi_n(z) z^{-n} e^{-nN(z)} - 1\right]\right| \leq K_p \frac{\log(n)}{n^{k-2p-1}} \tag{4.4}$$

holds for all sufficiently large n. □

Remark 4.3. As in the fixed-weights case (see (3.9)), a special case of the estimate (4.4) is the following estimate (holding under the same conditions):

$$\sup_{-\pi<\theta<\pi} \left| \left(-ie^{-i\theta}\frac{d}{d\theta}\right)^p \left[\pi_n\left(e^{i\theta}\right) e^{-in\theta} e^{-nN\left(e^{i\theta}\right)} - 1\right]\right| \leq K_p \frac{\log(n)}{n^{k-2p-1}}, \tag{4.5}$$

asymptotically characterizing $\pi_n(z)$ on the unit circle.

Once again, the uniform nature of the convergence on the circle allows us to prove the following mean result (the proof is the same as for the analogous result in the fixed-weight case).

Corollary 4.4. Let $p \geq 0$ be a fixed integer. Suppose that $\phi(\theta) = e^{-nV(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 2p+2$. If $\kappa'(\theta)$ is strictly positive, then

$$\lim_{n\to\infty} \frac{1}{n^p} \cdot \frac{\|\pi_n^{(p)}\|_\phi}{\|\pi_n\|_\phi} = 1. \tag{4.6}$$

□

The next result concerns the asymptotic behavior of the polynomial $\pi_n(z)$ for z inside a closed annular region whose outer boundary is the unit circle.

Theorem 4.5. Suppose that $\phi(\theta) = e^{-nV(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k \geq 2$. If $\kappa'(\theta)$ is strictly positive, then for each ρ satisfying $0 < \rho < 1$, there are constants $K_\rho^\pm > 0$ such that the estimates

$$\sup_{\rho<|z|<1} \left|\pi_n(z)e^{-n\overline{N(1/\overline{z})}} - z^n e^{inE_k\Omega(r,\theta)}\right| \leq K_\rho^- \frac{\log(n)}{n^{k-1}}, \tag{4.7}$$

$$\sup_{|z|<\rho} \left|\pi_n(z)e^{-n\overline{N(1/\overline{z})}}\right| \leq \frac{K_\rho^+}{n^{k-1}} \tag{4.8}$$

hold for all n sufficiently large. □

An immediate corollary is that there exists an annulus inside the unit circle that asymptotically contains no zeros.

Corollary 4.6 (zero-free regions)**.** Suppose that $\phi(\theta) = e^{-nV(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k \geq 2$, and suppose that $\kappa'(\theta)$ is strictly positive. Let $\delta > 0$ be an arbitrarily small number. Then there are no zeros of $\pi_n(z)$ in the region

$$\left\{z = re^{i\theta} \mid \log(r) > -\left(\frac{k-1-\delta}{1+\Omega'(\theta)}\right)\frac{\log(n)}{n}\right\} \tag{4.9}$$

as long as n is sufficiently large. □

Remark 4.7. In the case of a fixed weight, Corollary 3.7 established the existence of a zero-free annulus with outer radius 1, and whose inner radius depended explicitly on k, the degree of smoothness. We subsequently considered a family of weights for which we could compute explicitly the behavior of the zeros, and showed that Corollary 3.7 is sharp. Indeed, for explicit families of weights with k degrees of smoothness, some zeros achieve a distance of $k\log(n)/n + \log(\log(n))/n$ from the unit circle. In addition, for these examples, a majority of the zeros approach a circle whose radius is $1 - (k+1)\log(n)/n$. For the case of varying weights, it is our belief that Corollary 4.6 is similarly sharp. Moreover, it is to be expected that for some canonical family of examples constructed such that $\Omega^{(k)}$ has jump discontinuities, it should be similarly possible to obtain a very detailed asymptotic description of the zeros.

Remark 4.8. The zero-free region is determined by observing that for z to be a zero of $\pi_n(z)$, $z^n e^{inE_k\Omega(r,\theta)}$ must be roughly the same size as $\log(n)/n^{k-1}$. This is clearly not true for z on the unit circle, and investigating the size of $z^n e^{inE_k\Omega(r,\theta)}$ relative to $\log(n)/n^{k-1}$ yields the result. For $|z| < 1$, $z^n e^{inE_k\Omega(r,\theta)}$ is small not only because z^n is small, but also because (as will be clear in Section 4.2) we constructed $e^{inE_k\Omega(r,\theta)}$ so that for $r < 1$ but $1 - r$ sufficiently small,

$$e^{inE_k\Omega(r,\theta)} = O\big(r^{n\Omega'(\theta)}\big). \tag{4.10}$$

Thus our extension of the function Ω plays a role in determining the zero-free region near the unit circle.

Remark 4.9. The quantity $1+\Omega'(\theta)$ appearing in Corollary 4.6 is strictly positive because $1 + \Omega'(\theta) = \kappa'(\theta)$, and $\kappa'(\theta)$ is strictly positive. It is interesting to note that this condition also guarantees that there are no gaps in the support of the equilibrium measure (see Appendix A). The occurrence of a gap in the support of the equilibrium measure is heralded by the development of a zero of the function $\kappa'(\theta)$. Although Corollary 4.6 would not apply if $\kappa'(\theta)$ vanished at some θ_0, an intuitive consideration of the set defined in (4.9) indicates that near θ_0, the zeros are pushed away, further from the unit circle. As a gap develops then, one might expect the zeros to accumulate on a contour approaching the unit circle, but with a gap aligned with the gap in the support of the equilibrium measure. We will not carry out such an analysis here, but clearly the methods outlined here can be adapted in this direction.

Just as Theorem 3.6 leads to Corollary 3.8, (4.8) from Theorem 4.5 yields the following result. (Recall from the definition (1.20) that $N(z) \to 0$ as $z \to \infty$, and so $\overline{N(1/\overline{z})} \to 0$ as $z \to 0$.)

Corollary 4.10 (varying recurrence coefficients)**.** Suppose that $\phi(\theta) = e^{-nV(\theta)}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ with $k \geq 2$. If $\kappa'(\theta)$ is strictly positive, then there is a constant $K > 0$ such that the estimate

$$|\alpha_n| \leq \frac{K}{n^{k-1}} \tag{4.11}$$

□

holds for sufficiently large n.

Finally, we have the following result concerning the asymptotic behavior of the normalization constant $\gamma_n > 0$, defined such that $\|\gamma_n \pi_n(z)\|_\phi = 1$, with $\phi(\theta)$ given in the varying-weights case by (4.1).

Theorem 4.11. Suppose that V is a real function of class $C^{k-1,1}(S^1)$ for some $k \geq 2$. If $\kappa'(\theta)$ is strictly positive, then there is a constant $K > 0$ such that

$$\left|\gamma_{n-1}^2 e^{-nV_0} - 1\right| \leq \frac{K \log(n)}{n^{2(k-1)}} \tag{4.12}$$

holds for all n sufficiently large. □

Remark 4.12. At first glance, the asymptotic formulae (4.12) and (4.11) may appear to be inconsistent with the identity (2.12). However, in (4.12), γ_{n-1} is the norming coefficient of the $(n-1)$st degree polynomial orthogonal with respect to the n-dependent weight e^{-nV}. To verify (2.12) in the varying-weights case, one would have to compute the asymptotics for γ_n as well, rather than merely replacing n by $n+1$ in (4.12).

4.2 The $\overline{\partial}$ steepest descent method for exponentially varying weights

One of the main points of this paper is that while the limit appropriate for exponentially varying weights of the form (4.1) lies for the most part beyond the reach of classical techniques applicable for fixed weights (meaning primarily the approximation of $\phi(\theta)^{-1}$ by positive trigonometric polynomials), analysis of this limit by means of the $\overline{\partial}$ steepest descent method presents almost no further difficulty beyond the analysis carried out for fixed weights in Section 3. To illustrate the ease with which many of the techniques carry over from the fixed-weights case, we now outline the analogous calculations for the varying weight (4.1).

As in the case of fixed weights, we begin with the matrix $\mathbf{M}^n(z)$ solving Riemann-Hilbert Problem 2.1 and introduce a sequence of explicit transformations. In order that Riemann-Hilbert Problem 2.1 indeed characterizes the polynomial of degree n orthogonal to all lower-degree polynomials with respect to the weight (4.1), we make the following assumption.

Assumption 4.13. V is a real continuous function on the circle that, for some exponent $\nu \in (0, 1]$ and for some constant $K > 0$, satisfies a uniform Hölder continuity condition $|V(\theta_2) - V(\theta_1)| \leq K|\theta_2 - \theta_1|^\nu$.

4.2.1 Conversion to an equivalent $\overline{\partial}$ problem. Solution of the $\overline{\partial}$ problem in terms of integral equations. Now, define $\mathbf{S}^n(z)$ in terms of $\mathbf{M}^n(z)$ as follows:

$$\mathbf{S}^n(z) := \begin{cases} e^{nV_0\sigma_3/2}\mathbf{M}^n(z)e^{-nV_0\sigma_3/2}\begin{pmatrix} z^{-n}e^{-nN(z)} & 0 \\ 0 & z^n e^{nN(z)} \end{pmatrix}, & \text{for } |z| > 1, \\ e^{nV_0\sigma_3/2}\mathbf{M}^n(z)e^{-nV_0\sigma_3/2}\begin{pmatrix} e^{-n\overline{N(1/\overline{z})}} & 0 \\ 0 & e^{n\overline{N(1/\overline{z})}} \end{pmatrix}, & \text{for } |z| < 1. \end{cases} \tag{4.13}$$

Since $N(z) \to 0$ as $z \to \infty$, it is easy to see that as $z \to \infty$, $\mathbf{S}^n(z) \to \mathbb{I}$. Moreover, $\mathbf{S}^n(z)$ is analytic for $|z| \neq 1$, and its boundary values $\mathbf{S}^n_+(z)$ (resp., $\mathbf{S}^n_-(z)$) taken on the circle S^1 from the inside (resp., outside) satisfy

$$\mathbf{S}^n_+(e^{i\theta}) = \mathbf{S}^n_-(e^{i\theta})\begin{pmatrix} e^{in\kappa(\theta)} & 1 \\ 0 & e^{-in\kappa(\theta)} \end{pmatrix}. \tag{4.14}$$

Note that according to **Assumption 4.13**, $\kappa(\theta)$ also satisfies a uniform Hölder continuity condition with exponent ν.

As in **Section 3.2.1**, we have available an algebraic factorization of this jump condition:

$$\mathbf{S}^n_+(e^{i\theta}) = \mathbf{S}^n_-(e^{i\theta})\begin{pmatrix} 1 & 0 \\ e^{-in\kappa(\theta)} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e^{in\kappa(\theta)} & 1 \end{pmatrix}. \tag{4.15}$$

To take advantage of this factorization, we need to extend the functions $e^{\pm in\kappa(\theta)}$ from S^1 to an annulus containing S^1. To have continuity of these extensions, we make the following assumption.

Assumption 4.14. The function V is of class $C^{k-1}(S^1)$ for some $k = 1, 2, 3, \ldots$.

With this assumption, which provides new information only if $k > 1$, we can extend $e^{\pm in\kappa(\theta)}$ as follows. Writing $e^{\pm in\kappa(\theta)} = e^{\pm in\theta}e^{\pm in\Omega(\theta)}$, we extend the factor $e^{\pm in\theta}$ analytically as $z^{\pm n}$. On the other hand, $\Omega : S^1 \to \mathbb{R}$ is a well-defined function on the circle, and therefore we may apply the extension operator E_m defined in (1.5) to Ω, resulting in an extension of $e^{\pm in\Omega(\theta)}$ to the domain $\mathbb{R}^2 \setminus \{0\}$ as a continuous function $e^{\pm inE_m\Omega(r,\theta)}$ for any m in the range $1 \le m \le k$.

Let $\epsilon > 0$ be the radius parameter of the annular domains $A_\pm$ (see Figure 3.2.1). Recall the "bump" function B defined in Section 1.5. We define a new unknown matrix $\mathbf{T}^n_{m,\epsilon}(r,\theta)$ by setting

$$\mathbf{T}^n_{m,\epsilon}(r,\theta) := \begin{cases} \mathbf{S}^n(z)\begin{pmatrix} 1 & 0 \\ B\left(\dfrac{\log(r)}{\epsilon}\right) r^{-n}e^{-in\theta}e^{-inE_m\Omega(r,\theta)} & 1 \end{pmatrix}, & \text{for } z = re^{i\theta} \in A_-, \\ \mathbf{S}^n(z)\begin{pmatrix} 1 & 0 \\ -B\left(\dfrac{\log(r)}{\epsilon}\right) r^{n}e^{in\theta}e^{inE_m\Omega(r,\theta)} & 1 \end{pmatrix}, & \text{for } z = re^{i\theta} \in A_+, \\ \mathbf{S}^n(z), & \text{for } z = re^{i\theta} \notin \overline{A_+ \cup A_-}. \end{cases} \tag{4.16}$$

Note that the presence of the factor $B(\log(r)/\epsilon)$ ensures that $\mathbf{T}^n_{m,\epsilon}(r,\theta)$ may be continuously extended to the outer boundary of A_- and the inner boundary of A_+, that is, $\mathbf{T}^n_{m,\epsilon}(r,\theta)$ is a continuous function for $r \neq 1$.

Remark 4.15. Here our analysis rests upon extending $\Omega(\theta)$ from the circle into domains $A_\pm$. By contrast in our treatment of the fixed-weights case in Section 3.2 it was more convenient to extend the function $V(\theta)$, which is related to $\Omega(\theta)$ by a Cauchy transform.

If $m = 1$, then it is easy to see that $E_m\Omega(r,\theta) = E_1\Omega(r,\theta) = \Omega(\theta)$ for all $r > 0$, so the off-diagonal matrix elements in $\mathbf{S}^n(re^{i\theta})^{-1}\mathbf{T}^n_{1,\epsilon}(r,\theta)$ are bounded in magnitude by $e^{-n|\log(r)|}$. Therefore, in this case (4.16) represents an exponentially near-identity transformation when n is large and $|\log(r)|$ is not too small. We want to ensure that a similar situation prevails when $m \ge 2$ as well. Since $1 \le m \le k$, the consideration of $m \ge 2$ requires that $k \ge 2$. In this case, Assumption 4.14 implies that $\kappa(\theta)$ is continuously differentiable, and hence so is $\Omega(\theta)$. The crucial conditions for (4.16) to be a near-identity transformation when n is large are that $\Omega'(\theta) + 1$ is strictly positive and that the parameter ϵ is chosen small enough, as the following lemma shows.

Lemma 4.16. Suppose that $\Omega(\theta)$ is a real function of class $C^{m-1}(S^1)$ for some $m \ge 2$, such that $\Omega'(\theta) + 1$ is strictly positive. Then there exist constants $\epsilon_0 > 0$ and $\mu > 0$ such that whenever $0 < \epsilon < \epsilon_0$,

$$\begin{aligned} \left|e^{-iE_m\Omega(r,\theta)}\right| &\leq r^{1-\mu}, \quad \forall\,\theta \text{ and for } 1 \leq r \leq 2^{\epsilon}, \\ \left|e^{iE_m\Omega(r,\theta)}\right| &\leq r^{\mu-1}, \quad \forall\,\theta \text{ and for } 2^{-\epsilon} \leq r \leq 1. \end{aligned} \tag{4.17}$$

□

Proof. By hypothesis, we have

$$\inf_{-\pi<\theta<\pi} \Omega'(\theta) = \tau - 1, \quad \tau > 0. \tag{4.18}$$

Note that $\log|e^{\mp iE_m\Omega(r,\theta)}| = \pm\Im(E_m\Omega(r,\theta))$. From (1.5), we have

$$\begin{aligned} \Im\big(E_m\Omega(r,\theta)\big) &= \sum_{p=0}^{P(m)} (-1)^{p+1} \frac{\Omega^{(2p+1)}(\theta)}{(2p+1)!} \log(r)^{2p+1} \\ &= -\log(r)\left[\Omega'(\theta) + \sum_{p=1}^{P(m)} (-1)^p \frac{\Omega^{(2p+1)}(\theta)}{(2p+1)!} \log(r)^{2p}\right], \end{aligned} \tag{4.19}$$

where $P(m) = (m-2)/2$ if m is even and $P(m) = (m-3)/2$ if m is odd. Since Ω has $m-1$ continuous derivatives, we have

$$\left|\sum_{p=1}^{P(m)} (-1)^p \frac{\Omega^{(2p+1)}(\theta)}{(2p+1)!} \log(r)^{2p}\right| \leq \sum_{p=1}^{P(m)} \frac{\sup_{-\pi<\theta<\pi} \left|\Omega^{(2p+1)}(\theta)\right|}{(2p+1)!} \big(\epsilon \log(2)\big)^{2p} \tag{4.20}$$

for all r satisfying $|\log(r)| \leq \epsilon\log(2)$, where $\epsilon > 0$. Therefore, $\epsilon_0 > 0$ may be chosen sufficiently small that the inequality $0 < \epsilon < \epsilon_0$ implies that

$$\Omega'(\theta) + \sum_{p=1}^{P(m)} (-1)^p \frac{\Omega^{(2p+1)}(\theta)}{(2p+1)!} \log(r)^{2p} \geq \frac{\tau}{2} - 1 \tag{4.21}$$

holds whenever $|\log(r)| \leq \epsilon\log(2)$. Then, we have

$$\log\left|e^{-iE_m\Omega(r,\theta)}\right| \leq -\log(r)\left[\frac{\tau}{2} - 1\right], \quad \text{holding for } 1 \leq r \leq 2^{\epsilon},$$

$$\log\left|e^{iE_m\Omega(r,\theta)}\right| \le \log(r)\left[\frac{\tau}{2}-1\right], \quad \text{holding for } 2^{-\epsilon} \le r \le 1. \tag{4.22}$$

The estimates (4.17) thus both hold with the choice $\mu := \tau/2 > 0$. ∎

Thus, we are led to propose the following assumption.

Assumption 4.17. If V is of class $C^1(S^1)$, then $\kappa'(\theta)$ defined by (4.2) is strictly positive.

This assumption is not an explicit assumption on $V(\theta)$, however it is possible to give conditions on $V(\theta)$ that are sufficient to make Assumption 4.17 hold. For example, since for $V \in C^1(S^1)$, we have

$$\kappa'(\theta) = 1 - 2\sum_{j=1}^{\infty} j\left|V_j\right| \cos\left(j\theta + \arg\left(V_j\right)\right), \tag{4.23}$$

where we recall the Fourier coefficients of V defined by (1.19), the condition

$$\sum_{j=-\infty}^{\infty} |j|\left|V_j\right| < 1 \tag{4.24}$$

is sufficient to guarantee that Assumption 4.17 holds. If in fact $V \in C^2(S^1)$, then the convexity condition

$$\frac{d^2V}{d\theta^2}(\theta) > -\frac{1}{2} \tag{4.25}$$

also guarantees that $\kappa'(\theta)$ is strictly positive. Our proof that the condition (4.25) implies that $\kappa'(\theta)$ is strictly positive makes use of certain aspects of logarithmic potential theory and is given in Appendix A. In Appendix B, we show that neither are the sufficient conditions (4.24) and (4.25) equivalent, nor does either condition imply the other. Rather, the two conditions are independent and thus complement each other.

Remark 4.18. In the approach taken in [3], where a particular case of a varying weight in which the function $V(\theta)$, and hence $\kappa(\theta)$, is an analytic function, the analytic extension $E_\infty\Omega(r,\theta)$ is used, and the "bump" function factor $B(\log(r)/\epsilon)$ is omitted. The latter omission has the effect of introducing exponentially small jump discontinuities in $T^n_{m,\epsilon}(r,\theta)$ across the circles $\Sigma_\pm$.

Since for $m = 1, \dots, k$ we have $E_m\Omega(1,\theta) = \Omega(\theta)$, the matrix $\mathbf{T}_m^n(r,\theta)$ satisfies the jump condition

$$\lim_{r\uparrow 1} \mathbf{T}_{m,\epsilon}^n(r,\theta) = \lim_{r\downarrow 1} \mathbf{T}_{m,\epsilon}^n(r,\theta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.26}$$

To remove this jump discontinuity, introduce one further change of variables:

$$\mathbf{J}_{m,\epsilon}^n(r,\theta) := \begin{cases} \mathbf{T}_{m,\epsilon}^n(r,\theta), & \text{for } r > 1, \\ \mathbf{T}_{m,\epsilon}^n(r,\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{for } r < 1. \end{cases} \tag{4.27}$$

At this point, we can relate $\mathbf{J}_{m,\epsilon}^n(r,\theta)$ directly to $\mathbf{M}^n(z)$. Combining (4.13), (4.16), and (4.27), we have by definition

$$\mathbf{J}_{m,\epsilon}^n(r,\theta)$$
$$:= \begin{cases} e^{nV_0\sigma_3/2}\mathbf{M}^n(z)e^{-nV_0\sigma_3/2} \begin{pmatrix} 0 & -e^{-n\overline{N(1/\overline{z})}} \\ e^{n\overline{N(1/\overline{z})}} & e^{n\overline{N(1/\overline{z})}} B\left(\dfrac{\log(r)}{\epsilon}\right) r^n e^{in\theta} e^{inE_m\Omega(r,\theta)} \end{pmatrix}, \\ \qquad \text{for } 0 \le r < 1, \\ e^{nV_0\sigma_3/2}\mathbf{M}^n(z)e^{-nV_0\sigma_3/2} \begin{pmatrix} e^{-nN(z)} r^{-n} e^{-in\theta} & 0 \\ e^{nN(z)} B\left(\dfrac{\log(r)}{\epsilon}\right) e^{-inE_m\Omega(r,\theta)} & e^{nN(z)} r^n e^{in\theta} \end{pmatrix}, \\ \qquad \text{for } r > 1. \end{cases} \tag{4.28}$$

Given the assumptions in force, this matrix function is clearly continuous throughout the plane. To determine its deviation from being an analytic function in the regions $r < 1$ and $r > 1$, we need to control a derivative, and consequently we make the following assumption.

Assumption 4.19. The function V is of class $C^{k-1,1}(S^1)$ for some $k \ge 1$. That is, $V^{(k-1)}(\theta)$ is Lipschitz continuous.

Then, for $1 \le m \le k$, we see by direct calculation that

$$\overline{\partial}\mathbf{J}^{n}_{m,\epsilon}(r,\theta) = \begin{cases} e^{nV_0\sigma_3/2}\mathbf{M}^n(z)e^{-nV_0\sigma_3/2}\begin{pmatrix} 0 & 0 \\ 0 & e^{n\overline{N(1/\overline{z})}}r^n e^{in\theta}\overline{\partial}\left[B\left(\dfrac{\log(r)}{\epsilon}\right)e^{inE_m\Omega(r,\theta)}\right]\end{pmatrix}, \\ \qquad\qquad \text{for } 0 \le r < 1, \\ e^{nV_0\sigma_3/2}\mathbf{M}^n(z)e^{-nV_0\sigma_3/2}\begin{pmatrix} 0 & 0 \\ e^{nN(z)}\overline{\partial}\left[B\left(\dfrac{\log(r)}{\epsilon}\right)e^{-inE_m\Omega(r,\theta)}\right] & 0\end{pmatrix}, \\ \qquad\qquad \text{for } r > 1. \end{cases} \tag{4.29}$$

Eliminating $\mathbf{M}^n(z)$ in favor of $\mathbf{J}^n_{m,\epsilon}(r,\theta)$ using (4.28) again yields

$$\overline{\partial}\mathbf{J}^{n}_{m,\epsilon}(r,\theta) = \mathbf{J}^{n}_{m,\epsilon}(r,\theta)\mathbf{X}^{n}_{m,\epsilon}(r,\theta), \quad \text{for } r \neq 1 \text{ and almost all } \theta \in S^1, \tag{4.30}$$

where

$$\mathbf{X}^{n}_{m,\epsilon}(r,\theta) := \begin{cases} \begin{pmatrix} 0 & r^n e^{in\theta}\overline{\partial}\left[B\left(\dfrac{\log(r)}{\epsilon}\right)e^{inE_m\Omega(r,\theta)}\right] \\ 0 & 0\end{pmatrix}, & \text{for } 0 \le r < 1, \\ \begin{pmatrix} 0 & 0 \\ r^{-n}e^{-in\theta}\overline{\partial}\left[B\left(\dfrac{\log(r)}{\epsilon}\right)e^{-inE_m\Omega(r,\theta)}\right] & 0\end{pmatrix}, & \text{for } r > 1. \end{cases} \tag{4.31}$$

Then it is easy to see that $\mathbf{J}^n_{m,\epsilon}(r,\theta)$ satisfies the following $\overline{\partial}$ problem.

$\overline{\partial}$ Problem 4.20. Find a 2×2 matrix $\mathbf{U}(r,\theta)$ with the following properties.

Smoothness. $\mathbf{U}(r,\theta)$ is a Lipschitz continuous function throughout $\mathbb{R}^2$.

Deviation from analyticity. The relation

$$\overline{\partial}\mathbf{U}(r,\theta) = \mathbf{U}(r,\theta)\mathbf{X}^{n}_{m,\epsilon}(r,\theta) \tag{4.32}$$

holds for all points in $\mathbb{R}^2$ with the exception of a set of measure zero. The matrix $\mathbf{X}^n_{m,\epsilon}(r,\theta)$ is defined almost everywhere by (4.31) and is essentially compactly supported.

Normalization. The matrix $\mathbf{U}(r,\theta)$ is normalized at $r=\infty$ as follows:

$$\lim_{r\to\infty}\mathbf{U}(r,\theta)=\mathbb{I}. \tag{4.33}$$

In exactly the same way that Proposition 3.22 was proved, we have the following proposition.

Proposition 4.21. Suppose that $\phi = e^{-nV}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k = 1,2,3,\ldots$. Then for all $n = 0,1,2,3,\ldots$, for $m = 1,2,\ldots,k$, and for all $\epsilon > 0$, the matrix $\mathbf{X}^n_{m,\epsilon}(r,\theta)$ is well defined almost everywhere by (4.31) and $\overline{\partial}$ Problem 4.20 has a unique solution, namely $\mathbf{U}(r,\theta) = \mathbf{J}^n_{m,\epsilon}(r,\theta)$. □

Also, the proof of Proposition 3.23 carries over to the context of the varying weight (4.1) in the following form.

Proposition 4.22. Suppose that $\phi = e^{-nV}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k = 1,2,3,\ldots$. Then for all $n = 0,1,2,3,\ldots$, for $m = 1,2,\ldots,k$, and for all $\epsilon > 0$, the matrix $\mathbf{X}^n_{m,\epsilon}(r,\theta)$ is well defined almost everywhere by (4.31) and the corresponding solution $\mathbf{U}(r,\theta) = \mathbf{J}^n_{m,\epsilon}(r,\theta)$ of $\overline{\partial}$ Problem 4.20 satisfies the integral equation

$$\mathbf{U}(r,\theta)=\mathbb{I}-\frac{1}{\pi}\iint\frac{\mathbf{U}(r',\theta')\mathbf{X}^n_{m,\epsilon}(r',\theta')}{z'-z}\,dA', \tag{4.34}$$

where $z = re^{i\theta}$, $z' = r'e^{i\theta'}$, and dA' is a positive area element $dA' = r'dr'd\theta'$. The integral is taken over the entire plane. □

4.2.2 Asymptotic solution of the integral equation. Estimates of $\mathbf{J}^n_{m,\epsilon}(r,\theta)$ *and its derivatives for large* n. As in Section 3.2.2, it is possible to characterize $\mathbf{J}^n_{m,\epsilon}(r,\theta)$ by analyzing the integral equation (4.34) as long as n is large enough. However, in the context of the varying weight (4.1), we will require the monotonicity condition expressed in Assumption 4.17, and that the radius parameter ϵ be taken sufficiently small for each admissible given V that the conclusion of Lemma 4.16 holds. Moreover, the exponential character of the varying weight (4.1) suggests that by comparison with the analysis presented in Section 3.2.2, this approach is only fruitful in giving the same degree of control on $\mathbf{J}^n_{m,\epsilon}(r,\theta)$ as was achieved for $\mathbf{H}^n_{m,\epsilon}(r,\theta)$ if $V(\theta)$ has more smoothness than was required in Proposition 3.26. In particular, we will require that the following assumption holds.

Assumption 4.23. The function V is of class $C^{k-1,1}(S^1)$ for some $k \geq 2$.

In other words, to achieve the same convergence rates as in the fixed-weight case, one more derivative of V will be required in the exponentially varying-weight case.

As in Section 3.2.2, we need some bounds for $\mathbf{X}^n_{m,\epsilon}(r,\theta)$ and its derivatives.

Proposition 4.24. Suppose that $V : S^1 \to \mathbb{R}$ is a real function of class $C^{k-1,1}(S^1)$ for some $k \geq 1$ for which $\Omega'(\theta) + 1$ is strictly positive, that m is an integer satisfying $1 \leq m \leq k$, and that $\epsilon > 0$ is sufficiently small. Let the integer D be defined as $D := \min(k-m, m-1)$. Then, the matrix function $\mathbf{X}^n_{m,\epsilon}$ is of class $C_0^{D-1,1}(\mathbb{R}^2\backslash\{0\})$ if $D > 0$, and of class $L_0^\infty(\mathbb{R}^2\backslash\{0\})$ if $D = 0$. Moreover, if α and β are nonnegative integers such that $\alpha + \beta \leq D$, then there are constants $\mu > 0$ and $C^{(\alpha,\beta)}_{m,\epsilon} > 0$ such that for all n, the estimate

$$\left\| \frac{\partial^{\alpha+\beta}}{\partial r^\alpha \partial \theta^\beta} \mathbf{X}^n_{m,\epsilon}(r,\theta) \right\| \leq C^{(\alpha,\beta)}_{m,\epsilon} n^{1+\beta} e^{-n\mu|\log(r)|} \left|\log(r)\right|^{m-1-\alpha} \sum_{p=0}^{\alpha} n^p \left|\log(r)\right|^p \tag{4.35}$$

holds throughout the region $|\log(r)| \leq \epsilon \log(2)$ containing the essential support of $\mathbf{X}^n_{m,\epsilon}(r,\theta)$. □

Proof. This proposition is proved in almost the same way as Proposition 3.24. In this case, it is essential to recall Lemma 4.16 which provides the constant $\mu > 0$ and thus the exponential decay of the term $e^{-n\mu|\log(r)|}$ for $\epsilon > 0$ sufficiently small. Also, the initial application of the $\overline{\partial}$ operator in the definition (4.31) yields a factor of n that does not appear in the proof of Proposition 3.24. ■

Remark 4.25. This result should be compared with Proposition 3.24. The only important difference between the estimates (4.35) and (3.63) is the presence of an additional factor of n.

The proof of Proposition 3.26, with references to Proposition 3.24 replaced by references to Proposition 4.24, applies to the asymptotic estimation of $\mathbf{J}^n_{m,\epsilon}(r,\theta)$, with the only important difference being an additional factor of n. This results in the following proposition.

Proposition 4.26. Suppose that $\phi = e^{-nV}$, where $V : S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k = 2, 3, 4, \ldots$. Let the integer m lie in the range $2 \leq m \leq k$ and fix $\epsilon > 0$ sufficiently small. Define the integer $\widetilde{D} := \min(k - m, m - 2) \geq 0$. Then, for all $n \geq 0$, the matrix $\mathbf{X}^n_{m,\epsilon}(r,\theta)$ is well defined almost everywhere by (4.31), and for all n sufficiently large, $\mathbf{J}^n_{m,\epsilon}(r,\theta)$ is given by a Neumann series

$$\mathbf{J}^n_{m,\epsilon}(r,\theta) = \mathbb{I} + \left(\mathcal{X}^n_{m,\epsilon}\mathbb{I}\right)(r,\theta) + \left(\mathcal{X}^n_{m,\epsilon} \circ \mathcal{X}^n_{m,\epsilon}\mathbb{I}\right)(r,\theta) + \cdots, \tag{4.36}$$

which converges in the norm $|||\cdot|||_{\widetilde{D}}$, where the double-integral operator $\mathcal{X}^n_{m,\epsilon}$ is defined by

$$(\mathcal{X}^n_{m,\epsilon}\mathbf{F})(r,\theta) := -\frac{1}{\pi}\iint \frac{\mathbf{F}(r',\theta')\mathbf{X}^n_{m,\epsilon}(r',\theta')}{z'-z}\, dA'. \tag{4.37}$$

In particular, if $\widetilde{D}=0$, then $\mathbf{J}^n_{m,\epsilon}$ lies in the space $L^\infty(\mathbb{R}^2)$, and if $\widetilde{D}>0$ then $\mathbf{J}^n_{m,\epsilon}$ lies in the space $C^{\widetilde{D}-1,1}(\mathbb{R}^2)$ and $|||\mathbf{J}^n_{m,\epsilon}|||_{\widetilde{D}}$ is finite. For all integer p in the range $0\le p\le \widetilde{D}$, the following estimates hold for sufficiently large n:

$$\left|\left|\left|\mathbf{J}^n_{m,\epsilon}-\mathbb{I}\right|\right|\right|_p \le C^{(p)}_{m,\epsilon}\frac{\log(n)}{n^{m-p-1}}, \tag{4.38}$$

$$\left|\left|\left|\mathbf{J}^n_{m,\epsilon}-\mathbb{I}-\mathcal{X}^n_{m,\epsilon}\mathbb{I}\right|\right|\right|_p \le C^{(p)2}_{m,\epsilon}\frac{\log(n)^2}{n^{2m-2p-2}}, \tag{4.39}$$

where $C^{(p)}_{m,\epsilon}$ is a positive constant. Furthermore, for each $\rho>2^\epsilon$ and for all integer p in the range $0\le p\le D$, the following estimates hold for sufficiently large n:

$$\sum_{\alpha+\beta\le p}\ \sup_{\substack{-\pi<\theta<\pi\\ |\log(r)|\ge\log(\rho)}}\left\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial y^\beta}\left[\mathbf{J}^n_{m,\epsilon}(r,\theta)-\mathbb{I}\right]\right\|\le \widetilde{C}^{(p)}_{m,\rho}\frac{1}{n^{m-p-1}}, \tag{4.40}$$

$$\sum_{\alpha+\beta\le p}\ \sup_{\substack{-\pi<\theta<\pi\\ |\log(r)|\ge\log(\rho)}}\left\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial y^\beta}\left[\mathbf{J}^n_{m,\epsilon}(r,\theta)-\mathbb{I}-\left(\mathcal{X}^n_{m,\epsilon}\mathbb{I}\right)(r,\theta)\right]\right\|\le \widetilde{C}^{(p)2}_{m,\rho}\frac{\log(n)}{n^{2m-2p-2}}, \tag{4.41}$$

where $\widetilde{C}^{(p)}_{m,\rho}$ is a positive constant. □

Remark 4.27. Note that in (4.38), due to the extra factor of n introduced into the estimates by **Proposition 4.24**, convergence of the Neumann series in the $|||\cdot|||_p$ norm follows provided that $m-p-1>0$, and so we may only consider $m\ge 2$. In order to uniformly control p derivatives of $\mathbf{J}^n_{m,\epsilon}(r,\theta)$, **Proposition 4.26** requires that m should lie in the range $2+p\le m\le k-p$, and therefore to guarantee the existence of suitable values of m, $V:S^1\to\mathbb{R}$ should be of class $C^{k-1,1}(S^1)$ for some $k\ge 2p+2$.

Remark 4.28. **Assumption 4.17**, that $V(\theta)$ is such that $\kappa(\theta)$ is a strictly increasing function of the angle θ, ties together two crucial aspects of our analysis. First of all, as can be seen from the matrix factors involved in the change of variables (4.16) between $\mathbf{S}^n(z)$ and $\mathbf{T}^n_{m,\epsilon}(r,\theta)$, the inequality $\kappa'(\theta)>0$ is precisely what makes this a near-identity change of variables in the regions $\Omega_\pm$. On the other hand, it would not suffice to replace $\kappa'(\theta)$ by another unrelated positive quantity, because in order to control the $\overline{\partial}$ problem (i.e., to sufficiently bound the operator $\mathcal{X}^n_{m,\epsilon}$), it is necessary that the extension of $\kappa(\theta)$ should provide some degree of vanishing of the $\overline{\partial}$ derivative at $|z| = 1$. This vanishing is built into the extension operators we have defined in (1.5) by the key property (1.7). Moreover, *any* smooth extension of $\kappa(\theta)$ that has a vanishing $\overline{\partial}$ derivative on the unit circle will behave similarly near the unit circle. So we conclude that while the analyticity of $\kappa(\theta)$ is not important, the monotonicity of this function is crucial. To make an analogy with the asymptotic analysis of oscillatory exponential integrals, the $\overline{\partial}$ steepest descent method is a closer relative of Kelvin's method of stationary phase than of the saddle-point method. On the other hand, it is important to note that the $\overline{\partial}$ steepest descent method remains fundamentally a method of deformation into the complex plane, and is not based on integration by parts,[1] the Riemann-Lebesgue lemma, or related arguments of harmonic analysis. For an approach based on the latter, see Varzugin [27].

4.3 Proofs of theorems stated in Section 4.1

The proofs are generally based on expressing $\pi_n(z) = M_{11}^n(z)$ in terms of explicit functions and $J_{m,\epsilon}^n(r,\theta)$ by means of (4.28). Then, one applies Proposition 4.26 to control $J_{m,\epsilon}^n(r,\theta) - \mathbb{I}$ and its derivatives.

4.3.1 *Asymptotic behavior of $\pi_n(z)$ for $|z| \geq 1$.*

Proof of Theorems 4.1 and 4.2. From (4.28), we have the following exact representation for $\pi_n(z)$ valid for $|z| > 1$:

$$\pi_n(z) = e^{nN(z)}\left[z^n J_{m,\epsilon,11}^n(r,\theta) - B\left(\frac{\log(r)}{\epsilon}\right)e^{-inE_m\Omega(r,\theta)}J_{m,\epsilon,12}^n(r,\theta)\right], \quad |z| > 1. \tag{4.42}$$

Here ϵ should be taken to be sufficiently small. It follows that

$$\begin{aligned}\pi_n(z)z^{-n}e^{-nN(z)} - 1 &= \left[J_{m,\epsilon,11}^n(r,\theta) - 1\right] \\ &\quad - B\left(\frac{\log(r)}{\epsilon}\right)z^{-n}e^{-inE_m\Omega(r,\theta)}J_{m,\epsilon,12}^n(r,\theta), \quad |z| > 1.\end{aligned} \tag{4.43}$$

If $\rho > 1$ is fixed, then perhaps by making $\epsilon > 0$ smaller yet, it can be arranged that $B(\log(r)/\epsilon) \equiv 0$ whenever $|z| \geq \rho$. In this case, we have

$$\frac{d^p}{dz^p}\left[\pi_n(z)z^{-n}e^{-nN(z)}\right] = \partial^p\left[J_{m,\epsilon,11}^n(r,\theta) - 1\right], \quad |z| \geq \rho > 1. \tag{4.44}$$

[1] Properly speaking, we do not rely on integration by parts (or more generally, Stokes' theorem for the $\overline{\partial}$ operator in the plane—in as much as this can be considered a generalization of the standard $\overline{\partial}$ inversion formula) to establish the existence of an asymptotic expansion. However, such methods are useful in the detailed analysis of individual terms in the expansion. This technique was used, for example, in the proof of Theorem 3.11.

Using the estimate (4.41) from Proposition 4.26 in the case $m = k - p$, and the fact that $\mathbf{X}^n_{m,\epsilon}(r,\theta)$ is off-diagonal, we see that for some constant $C > 0$,

$$\sup_{\substack{-\pi<\theta<\pi \\ r\geq\rho>1}} \left|\partial^p\left[J^n_{k-p,\epsilon,11}(r,\theta) - 1\right]\right| \leq C\frac{\log(n)}{n^{2(k-2p-1)}}. \tag{4.45}$$

Now since $J^n_{k-p,\epsilon,11}(z) - 1$ is analytic for $|z| > \rho$, and tends to zero like z^{-1} as $z \to \infty$, Cauchy's theorem for an exterior domain, together with (4.45) for $p = 0$, proves (4.3).

To prove (4.5), we fix ϵ sufficiently small and consider $1 \leq r \leq 2^{\epsilon/2}$ in which case $B(\log(r)/\epsilon) \equiv 1$ so that

$$\begin{aligned}&\frac{d^p}{dz^p}\left[\pi_n(z)z^{-n}e^{-nN(z)} - 1\right] \\ &\quad = \partial^p\left[J^n_{m,\epsilon,11}(r,\theta) - 1\right] - \partial^p\left[z^{-n}e^{-inE_m\Omega(r,\theta)}J^n_{m,\epsilon,12}(r,\theta)\right], \quad 1 \leq r \leq 2^{\epsilon/2}.\end{aligned} \tag{4.46}$$

Again, since $\mathbf{X}^n_{m,\epsilon}(r,\theta)$ is an off-diagonal matrix, we see from (4.39) that there is a constant $C > 0$ such that

$$\sup_{\mathbb{R}^2}\left|\partial^p\left[J^n_{k-p,\epsilon,11}(r,\theta) - 1\right]\right| \leq C\frac{\log(n)^2}{n^{2(k-2p-1)}}. \tag{4.47}$$

As in the fixed-weights case, the dominant contribution comes from the remaining terms on the right-hand side of (4.46). Using (4.38) from Proposition 4.26 to see that $n^j\partial^{p-j}J^n_{m,\epsilon,12}(r,\theta)$ is of order $\log(n)/n^{m-p-1}$ and taking the best case of $m = k - p$, we then find that

$$\sup_{\substack{-\pi<\theta<\pi \\ 1\leq r\leq 2^{\epsilon/2}}} \left|\frac{d^p}{dz^p}\left[\pi_n(z)z^{-n}e^{-nN(z)} - 1\right]\right| \leq K\frac{\log(n)}{n^{k-2p-1}}. \tag{4.48}$$

This proves (4.5). Now the maximum modulus principle applied to $\pi_n(z)z^{-n}e^{-nN(z)}$ implies (4.4).

4.3.2 Asymptotic behavior of $\pi_n(z)$ for $|z| < 1$ and of γ_{n-1}

Proof of Theorems 4.5 and 4.11. The proof of Theorem 4.11 is based on the identity $\gamma^2_{n-1} = -M^n_{21}(0)$. Using (4.28), we see that

$$\gamma^2_{n-1}e^{-nV_0} = J^n_{m,\epsilon,22}(0,\theta), \tag{4.49}$$

and thus Theorem 4.11 is proved by applying the estimate (4.41) from Proposition 4.26 in the case of $p = 0$ and $m = k$.

Proving Theorem 4.5 begins with the exact formula

$$\pi_n(z) = -e^{n\overline{N(1/\overline{z})}} J^n_{m,\epsilon,12}(r,\theta), \quad 0 \le r \le 2^{-\epsilon}, \tag{4.50}$$

which follows from (4.28) using $\pi_n(z) = M^n_{11}(z)$ and the fact that $B(\log(r)/\epsilon) \equiv 0$ for $r < 2^{-\epsilon}$. Using (4.40) from Proposition 4.26 with $p = 0$ and $m = k$ then completes the proof of (4.8).

For the proof of (4.7), one begins with the following formula for π_n:

$$\pi_n(z) = -e^{n\overline{N(1/\overline{z})}} J^n_{m,\epsilon,12}(r,\theta) + e^{n\overline{N(1/\overline{z})}} e^{-inE_k\Omega(r,\theta)} J^n_{m,\epsilon,11}(r,\theta), \tag{4.51}$$

which again follows from (4.28), and the fact that $B(\log(r)/\epsilon) \equiv 1$ for $r > 2^{-\epsilon/2}$. Using (4.38) from Proposition 4.26 with $p = 0$ and $m = k$ then completes the proof of (4.7), and this completes the proof of Theorem 4.5.

Appendices

A Logarithmic potential theory of orthogonal polynomials on the unit circle

A more general and systematic strategy for extracting asymptotics of $\mathbf{M}^n(z)$ in the exponentially varying-weight case when $\phi(\theta)$ is of the form (4.1) is the following. First introduce a function $g(z)$ (to be determined) satisfying $g(z) \sim \log(z)$ as $z \to \infty$ and that $e^{g(z)}$ is analytic for $|z| \neq 1$, taking continuous boundary values on the unit circle Σ. Then one converts Riemann-Hilbert Problem 2.1 for $\mathbf{M}^n(z)$ into one with identity asymptotics as $z \to \infty$ by the change of variables

$$\mathbf{M}^n(z) = \mathbf{Y}^n(z) e^{ng(z)\sigma_3}, \tag{A.1}$$

resulting in a new unknown matrix $\mathbf{Y}^n(z)$. In addition to the normalization condition $\lim_{z\to\infty} \mathbf{Y}^n(z) = \mathbb{I}$, $\mathbf{Y}^n(z)$ satisfies the jump condition

$$\mathbf{Y}^n_+(e^{i\theta}) = \mathbf{Y}^n_-(e^{i\theta}) \begin{pmatrix} e^{-n(g_+(e^{i\theta}) - g_-(e^{i\theta}))} & e^{n(-V(\theta) - i\theta + g_+(e^{i\theta}) + g_-(e^{i\theta}))} \\ 0 & e^{n(g_+(e^{i\theta}) - g_-(e^{i\theta}))} \end{pmatrix}, \quad \text{for } \theta \in S^1, \tag{A.2}$$

where the subscript "+" indicates the boundary value taken from within the circle, and "−" indicates the boundary value taken from outside. Without any loss of generality, we may represent $g(z)$ in the form of a complex logarithmic potential

$$g(z) = \int_{-\pi}^{\pi} L_{\theta'}(z)\psi(\theta')d\theta', \tag{A.3}$$

where for each $\theta \in S^1$, we consider the function $L_\theta(z) := \log(z - e^{i\theta})$ to be real for $z - e^{i\theta}$ sufficiently large and positive real, with the branch cut from the point $z = e^{i\theta}$ in the clockwise direction along the unit circle to the negative real axis, and then along the negative real axis to $z = -\infty$. We also require that

$$\int_{-\pi}^{\pi} \psi(\theta')d\theta' = 1 \tag{A.4}$$

in order to satisfy the required normalization condition $g(z) \sim \log(z)$ as $z \to \infty$.

Additional conditions may now be placed on g, or equivalently on ψ, in order to make the jump condition (A.2) for $\mathbf{Y}^n(z)$ asymptotically tractable. One key condition is that the function ψ should be real-valued. From this condition, it follows that $g_+(z) + g_-(z) - \log(z)$ has a constant imaginary part for $|z| = 1$, as the following argument shows. When $|z| = e^{i\theta}$, the identity $d/d\theta = izd/dz$ yields

$$\frac{d}{d\theta}\Im\big(g_+(z) + g_-(z) - \log(z)\big) = \Re\big(zg_+'(z) + zg_-'(z) - 1\big). \tag{A.5}$$

Using (A.4) and differentiating (A.3) under the integral with respect to z for $|z| \neq 1$, we obtain

$$\frac{d}{d\theta}\Im\big(g_+(z) + g_-(z) - \log(z)\big) = -\lim_{\epsilon\downarrow 0}\Re\left[\int_{-\pi}^{\pi} \frac{e^{2i\theta'} - (1+\epsilon^2)e^{2i\theta}}{\left(e^{i\theta'} - e^{i\theta}\right)^2 - \epsilon^2 e^{2i\theta}}\psi(\theta')d\theta'\right]. \tag{A.6}$$

Assuming reality of ψ, we can pass the real part under the integral and thus arrive at

$$\begin{aligned}\frac{d}{d\theta}&\Im\big(g_+(z) + g_-(z) - \log(z)\big) \\ &= -\lim_{\epsilon\downarrow 0}\int_{-\pi}^{\pi} \frac{\epsilon^4 + 2\epsilon^2 + 2\epsilon^2\cos(\zeta) - 4\epsilon^2\cos^2(\zeta)}{4 + \epsilon^4 + \left(4\epsilon^2 - 8\right)\cos(\zeta) + \left(4 - 4\epsilon^2\right)\cos^2(\zeta)}\psi(\theta')d\theta',\end{aligned} \tag{A.7}$$

where $\zeta = \theta' - \theta$. The numerator of the fraction in the integrand is clearly uniformly bounded by a quantity of order ϵ^2, and the minimum value of the denominator is achieved when $\cos(\zeta) = (2 - \epsilon^2)/(2 - 2\epsilon^2) \in (-1, 1)$ yielding a minimum value that has the asymptotic expansion $\epsilon^2 - \epsilon^4 + O(\epsilon^6)$ as $\epsilon \downarrow 0$. Consequently, the fraction is uniformly bounded independent of ϵ. Moreover, the fraction is easily seen to tend to zero pointwise in the limit $\epsilon \downarrow 0$ for $\zeta \neq 0$. A dominated convergence argument therefore shows that the limit on the right-hand side of (A.7) is zero; see also [1], where it is also shown that the constant value is exactly $\Im(g_+(z)+g_-(z)-\log(z)) = \pi$. Another consequence of assuming that ψ is real comes from noting that

$$g_+\left(e^{i\theta}\right) - g_-\left(e^{i\theta}\right) = 2\pi i \int_{\theta}^{\pi} \psi(\theta')d\theta'. \tag{A.8}$$

It therefore follows that $g_+(z) - g_-(z)$ is purely imaginary for $|z| = 1$.

Since

$$\Re\left(g_+\left(e^{i\theta}\right) + g_-\left(e^{i\theta}\right)\right) = 2\int_{-\pi}^{\pi} \log\left|e^{i\theta} - e^{i\theta'}\right|\psi(\theta')d\theta', \tag{A.9}$$

one is led to seek ψ so that the circle is split into intervals of two different types.

Bands. For θ in a band I, $\psi(\theta)$ is real and positive, and

$$2\int_{-\pi}^{\pi} \log\left|e^{i\theta} - e^{i\theta'}\right|\psi(\theta')d\theta' - V(\theta) = \ell, \quad \text{for } \theta \text{ in a band}, \tag{A.10}$$

where ℓ is a real constant (the same constant for all bands—in fact if there is only one band, it turns out that $\ell = -V_0$). Thus, in a band, the jump condition (A.2) for $\mathbf{Y}^n(z)$ takes the form

$$\mathbf{Y}^n_+(z) = \mathbf{Y}^n_-(z)\begin{pmatrix} e^{in(\kappa_I(\theta)-\pi)} & e^{n(\ell+i\pi)} \\ 0 & e^{-in(\kappa_I(\theta)-\pi)} \end{pmatrix}, \quad \text{for } \theta \text{ in a band I}, \tag{A.11}$$

where $\kappa_I(\theta)$ is a strictly increasing real function.

Gaps. For θ in a gap Γ, $\psi(\theta) \equiv 0$ and we have the strict inequality

$$2\int_{-\pi}^{\pi} \log\left|e^{i\theta} - e^{i\theta'}\right|\psi(\theta')d\theta' - V(\theta) < \ell, \quad \text{for } \theta \text{ in a gap}. \tag{A.12}$$

Thus, in a gap, the jump condition (A.2) for $\mathbf{Y}^n(z)$ takes the form

$$\mathbf{Y}^n_+(z) = \mathbf{Y}^n_-(z)\begin{pmatrix} e^{in(\kappa_\Gamma-\pi)} & \text{exponentially small } e^{n(\ell+i\pi)} \\ 0 & e^{-in(\kappa_\Gamma-\pi)} \end{pmatrix}, \quad \text{for } \theta \text{ in a gap } \Gamma, \tag{A.13}$$

where κ_Γ is a real constant.

The alternative conditions (A.10) and (A.12) are exactly the Euler-Lagrange conditions for the minimization of the weighted logarithmic energy

$$E[\psi] := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log\left(\frac{1}{|e^{i\theta} - e^{i\theta'}|}\right)\psi(\theta')\psi(\theta)d\theta' d\theta + \int_{-\pi}^{\pi} V(\theta')\psi(\theta')d\theta' \tag{A.14}$$

over all probability measures $\psi(\theta)d\theta > 0$ supported on the unit circle. The constant ℓ is the Lagrange multiplier introduced to enforce the constraint (A.4). The minimizing measure with density $\psi(\theta)$ is called the *equilibrium measure*. The connection of this extremal problem with the jump condition (A.2) through the Euler-Lagrange variational conditions suggests that logarithmic potential theory plays an important role in the asymptotic theory of orthogonal polynomials for general exponentially varying weights of the form (4.1). The use of equilibrium measures for the asymptotic analysis of the Riemann-Hilbert problem associated to orthogonal polynomials on $\mathbb{R}$ with analytic exponentially varying weights was carried out in [11, 10]. For orthogonal polynomials on the unit circle with analytic exponentially varying weights of a specific form, this was done in [1].

The transformation

$$\mathbf{Z}^n(z) = \begin{cases} -e^{-n\ell\sigma_3/2}\mathbf{Y}^n(z)e^{n\ell\sigma_3/2}, & |z| < 1, \\ e^{-n\ell\sigma_3/2}\mathbf{Y}^n(z)e^{n\ell\sigma_3/2}, & |z| > 1, \end{cases} \tag{A.15}$$

leads to jump conditions of the form

$$\mathbf{Z}^n_+(z) = \mathbf{Z}^n_-(z)\begin{pmatrix} e^{in\kappa_I(\theta)} & 1 \\ 0 & e^{-in\kappa_I(\theta)} \end{pmatrix}, \quad \text{for } z = e^{i\theta} \text{ in a band } I, \tag{A.16}$$

which is of exactly the same form as (4.14), and

$$\mathbf{Z}^n_+(z) = \mathbf{Z}^n_-(z)\begin{pmatrix} e^{in\kappa_\Gamma} & \text{exponentially small} \\ 0 & e^{-in\kappa_\Gamma} \end{pmatrix}, \quad \text{for } z = e^{i\theta} \text{ in a gap } \Gamma. \tag{A.17}$$

Clearly, we also have the normalization condition $\mathbf{Z}^n(z) \to \mathbb{I}$ as $z \to \infty$.

Now, if the function $V(\theta)$ is such that the whole unit circle consists of a single band, or equivalently there are no gaps in the support of the equilibrium measure, then in fact $\mathbf{Z}^n(z) = \mathbf{S}^n(z)$ as defined in Section 4.2.1 and the analysis proceeds as in the main body of this paper. However, in the more general context—when it is only true that the support of the equilibrium measure consists of a finite number of disjoint intervals on the unit circle—a modification of the $\overline{\partial}$ steepest descent method described in Section 4 is required. A more general method may still be based on the algebraic factorization (4.15) of the jump matrix in each band; however the annuli $A_\pm$ must be replaced with a system of lens-shaped regions $A_\pm^I$ adjacent to each band I. Since the variational inequalities become less effective near the band edges, a local analysis must be supplied to control the error. When such a method is developed, the condition (4.24) or (4.25) on the function $V(\theta)$ can be dropped as long as it is known that there are only a finite number of gaps in the support of the equilibrium measure. In [20], it is shown how to carry out such a program in the context of nonanalytic exponentially varying weights on the real line, where a convexity condition is known to guarantee the existence of a single isolated band $[\alpha, \beta] \subset \mathbb{R}$.

Let us show how the logarithmic potential theory described briefly above leads to the condition (4.25) guaranteeing that the support of the equilibrium measure is the entire unit circle. If V is of class $C^2(S^1)$, then for θ in any gap in the support of the equilibrium measure, the real function

$$\Phi(\theta) := 2\int_{-\pi}^{\pi} \log\left|e^{i\theta} - e^{i\theta'}\right|\psi(\theta')d\theta' - V(\theta) \tag{A.18}$$

is also twice differentiable. In fact, a calculation shows that

$$\Phi''(\theta) = -2\int_{-\pi}^{\pi} \frac{\psi(\theta')d\theta'}{\left|e^{i\theta} - e^{i\theta'}\right|^2} - V''(\theta). \tag{A.19}$$

Note that this is not a singular integral since $e^{i\theta}$ is assumed to lie outside the support of the equilibrium measure. Let us assume both the condition (4.25), and also the existence of a gap in the support of the equilibrium measure; we will then derive a contradiction. Now, $\Phi(\theta)$ is continuous for $-\pi < \theta \leq \pi$. Therefore, since $\Phi(\theta) = \ell$ at both endpoints of the gap (according to (A.10)), and $\Phi(\theta) < \ell$ strictly in the interior of the gap (according to (A.12)), there must be a point θ in the gap at which $\Phi''(\theta) > 0$. However, since $|e^{i\theta} - e^{i\theta'}| \leq 2$ for all angles θ', we see that

$$\Phi''(\theta) \leq -\frac{1}{2} - V''(\theta) \tag{A.20}$$

which is negative in view of the assumption (4.25), thus establishing the desired contradiction.

B Comparison of the conditions (4.24) and (4.25)

In this appendix, we illustrate by concrete examples how the two conditions (4.24) and (4.25), while both sufficient for the prevention of gaps in the support of the equilibrium measure, are completely independent. Thus neither condition implies the other.

First, consider the example $V(\theta) = A\cos(k\theta)$, where $k = 1, 2, 3, \dots$ and $A \in \mathbb{R}$ are parameters. A direct calculation shows that

$$\begin{aligned} &\text{condition (4.24)} \Longleftrightarrow |A| < \frac{1}{k}, \\ &\text{condition (4.25)} \Longleftrightarrow |A| < \frac{1}{2k^2}. \end{aligned} \tag{B.1}$$

Since $2k^2 > k$ for all $k = 1, 2, 3, \dots$, we see that if

$$\frac{1}{2k^2} < |A| < \frac{1}{k}, \tag{B.2}$$

then condition (4.24) is satisfied but (4.25) is not.

Next, consider for $M > 0$ and $\epsilon > 0$ the example

$$V(\theta) = \begin{cases} -\dfrac{M}{6}|\theta|^3 + \left(\dfrac{M\epsilon}{2} - \dfrac{M\epsilon^2}{4\pi}\right)\theta^2 + \dfrac{M\epsilon^4}{4\pi} - \dfrac{M\epsilon^3}{3}, & |\theta| \le \epsilon, \\ -\dfrac{M\epsilon^2}{4\pi}\theta^2 + \dfrac{M\epsilon^2}{2}|\theta| + \dfrac{M\epsilon^4}{4\pi} - \dfrac{M\epsilon^3}{2}, & \epsilon < |\theta| < \pi. \end{cases} \tag{B.3}$$

On one hand, we have

$$V''(\theta) = \begin{cases} M\big(\epsilon - |\theta|\big) - \dfrac{M\epsilon^2}{2\pi}, & |\theta| \le \epsilon, \\ -\dfrac{M\epsilon^2}{2\pi}, & \epsilon < |\theta| < \pi, \end{cases} \tag{B.4}$$

and so

$$\text{condition (4.25)} \Longleftrightarrow M < \frac{\pi}{\epsilon^2}. \tag{B.5}$$

On the other hand, direct calculation of the Fourier coefficients gives

$$V_k = \begin{cases} -\dfrac{2M}{\pi k^4}\sin^2\left(\dfrac{k\epsilon}{2}\right), & k \ne 0, \\ \dfrac{7M\epsilon^4}{48\pi} - \dfrac{M\epsilon^3}{4} + \dfrac{M\pi\epsilon^2}{12}, & k = 0. \end{cases} \tag{B.6}$$

Thus, we may estimate the sum in the condition (4.24) as follows:

$$\sum_{k=-\infty}^{\infty} |kV_k| = \sum_{k=1}^{\infty} \frac{4M}{\pi k^3} \sin^2\left(\frac{k\epsilon}{2}\right) \geq \sum_{k=1}^{\lceil \pi/\epsilon \rceil} \frac{4M}{\pi k^3} \sin^2\left(\frac{k\epsilon}{2}\right) \geq \sum_{k=1}^{\lceil \pi/\epsilon \rceil} \frac{M\epsilon^2}{4k} \tag{B.7}$$

since $|\sin(x)| \geq |x|/2$ for $|x| \leq \pi/2$. If the product $M\epsilon^2$ is held fixed, this lower bound can be made arbitrarily large (and in particular larger than one) simply by taking ϵ sufficiently small due to the divergence of the harmonic series. We therefore see that if $0 < c \leq M\epsilon^2 < \pi$ and if ϵ is sufficiently small, then condition (4.25) is satisfied but condition (4.24) is not.

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K. T.-R. McLaughlin: Department of Mathematics, The University of Arizona, Tucson, AZ 85721-0089, USA
E-mail address: mcl@math.arizona.edu

P. D. Miller: Department of Mathematics, The University of Michigan, East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, USA
E-mail address: millerpd@umich.edu