Do Solitons Exchange Conserved Quantities During Collisions?

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The redistribution of conserved quantities among colliding solitons of the nonlinear Schrödinger equation is considered. An analogy with the theory of spatial solitons in nonlinear optics provides one way to calculate this redistribution. In this context, exchanges of conserved quantities among N colliding solitons can be completely described from a knowledge of the case for $N = 2$. It is shown that solitons generally exchange L^2 norm as they collide, with the fraction shared being small when the solitons differ significantly in velocity or amplitude. Exchanges of other conserved densities are also considered.

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It is well known that after a collision of N solitons making up a solution ψ of the focusing nonlinear Schrödinger equation (NLS), the individual solitons emerge intact with the same amplitude, wave shape, and velocity as before the collision, the only evidence of the interaction being a phase shift [1]. Suppose we choose arbitrarily one of the conserved densities of NLS, like the L^2 norm density $|\psi|^2$ or momentum density $\mathrm{Im}(\psi^* \partial_x \psi)$. That the integral of this density remains the same throughout the collision is indeed tautological, as is the fact that each soliton's contribution to the integral is the same before and after the interaction. However, if we permit ourselves to imagine that the conserved density represents a physical density of infinitesimal elements or particles that may be distinguished one from another, we may ask a more philosophical question. Are the particles carried by a given soliton prior to the collision redistributed during the interaction such that each soliton carries a fraction of them away afterwards? If so, is it possible to quantify exactly how the particles are shared?

As stated, the question is not well posed, because to answer it one needs some way to distinguish among all the particles in one of the solitons after collision those that came from a particular soliton before collision, and there may be several ways to do this, not all equivalent. Indeed, on the basis of NLS alone there is no way to label the particles, so one needs to make some additional assumptions. To be concrete, we choose here to distinguish the particles by a method borrowed from the theory of spatial solitons in nonlinear optics. The theory of "light-guiding light" [2–6] describes the use of strong self-focused beams to "write" waveguides in a medium with a nonlinear response—waveguides that may be subsequently used to guide weaker beams. If the waveguide written by the strong beam is somehow fixed in the material, say, chemically or by photolithography, then the physical process of the linear propagation of light through the waveguide becomes isolated from the physical process that created the waveguide. By self-consistency, one mode of this waveguide will be indistinguishable from the strong field that created the waveguide [7]. On the other hand, there will generally be additional bound modes. We take the point of view that these linear modes describe the flow of particles through the waveguide, *and hence through the nonlinear field that created it.* In the specific case when the strong beam that makes the waveguide can be described by an N-soliton solution of NLS, there will be bound states of the waveguide that vanish in all but one arm of the guide before collision. These bound states are particular solutions of the linear Schrödinger equation with the potential function given by the square modulus of the N-soliton solution of NLS. Because for such a mode all particles will be unambiguously confined to just one soliton before the collision, the mode structure beyond the collision will describe their redistribution.

In this Letter, we use the optical analogy to give one possible theoretical answer to the question posed in our title. We will arrive at the perhaps counterintuitive result that solitons do indeed exchange particles when they collide. The number exchanged will depend upon which kind of particles (that is, which conserved density of NLS) we consider. We will find that a consistent description of the redistribution is possible in the case that the particles are infinitesimal elements of the L^2 norm, leading to the conclusion that colliding solitons always exchange some L^2 norm, with the fraction shared during the collision being small when the solitons differ greatly in velocity or amplitude. On the other hand, some problems appear when one applies these same methods that work so well for the L^2 norm to other conserved quantities of NLS. Let us now proceed to calculate the redistribution of conserved quantities of NLS by evaluating them on special solutions of the linear Schrödinger equation that are bound to the potential wells of a time-dependent potential created by the collision of N solitons of NLS.

Calculating the Redistribution of Particles.—Let ψ be an N-soliton solution of the focusing NLS equation

$$i\partial_t\psi + \frac{1}{2}\partial_x^2\psi + |\psi|^2\psi = 0\,. \tag{1}$$

The solution ψ is specified by choosing N complex numbers $\lambda_j = a_j + ib_j$ which contain the velocity and amplitude of each soliton and N complex numbers γ_j which

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contain phase information. As $t \to \pm\infty$, ψ decouples into a superposition of N solitons of the form

$$\psi_j^{\pm}(x,t) = 2b_j \operatorname{sech}(2b_j x + 4a_j b_j t - \delta_j^{\pm}) \times \exp\{-i[2a_j x + 2(a_j^2 - b_j^2)t - \theta_j^{\pm}]\}, \quad (2)$$

where the phases $\delta_j^{\pm}$ and $\theta_j^{\pm}$ are determined from γ_j.

Now, consider the time-dependent potential function $V(x,t) = |\psi(x,t)|^2$. We want to try to deduce exchanges of particles among the N solitons making up $\psi(x,t)$ by studying the bound states of the *linear* Schrödinger equation for a function $\phi(x,t)$,

$$i\partial_t \phi + \tfrac{1}{2}\partial_x^2 \phi + V(x,t)\phi = 0. \quad (3)$$

For $|t|$ large, the potential breaks up into N distinct wells, each of which can carry one bound state, locally proportional to $\psi(x,t)$. Suppose that for large negative t the solution $\phi(x,t)$ of (3) has the form

$$\phi(x,t) \sim \sum_{j=1}^{N} \alpha_j \psi_j^{-}(x,t), \quad (4)$$

for given complex constants α_j. For large positive t the solution $\phi(x,t)$ will also be confined to the potential wells of the individual solitons. In this limit, the solution has the form

$$\phi(x,t) \sim \sum_{j=1}^{N}\left[\sum_{k=1}^{N} T_{jk}\alpha_k\right]\psi_j^{+}(x,t). \quad (5)$$

The complex $N \times N$ matrix $\mathbf{T}$ defined in this way is called the *complex amplitude transfer matrix.* This matrix contains all the information (both amplitude and phase) about the asymptotic behavior of bound states of (3).

In Ref. [8] an explicit algebraic algorithm is given for computing complex amplitude transfer matrices for time-dependent potentials $V = |\psi|^2$ obtained from arbitrary collisions ψ of N solitons of the NLS equation (1). For our immediate purposes, the most important property of the matrix $\mathbf{T}$ is that it is independent of the numbers γ_j that determine the specific geometry of the interaction of the N solitons. Only the soliton eigenvalues λ_j, which determine the asymptotic amplitude and velocity of each soliton, are involved in the formulas at all. Thus, the amplitude transfer matrix $\mathbf{T}$ for a time-dependent potential $V(x,t)$ in which the N solitons interact more or less at the same time is exactly the same as the amplitude transfer matrix for a potential $V(x,t)$ in which the solitons only interact pairwise. This means that it is possible to build the $N \times N$ amplitude transfer matrix out of the 2×2 matrix, and the asymptotic behavior of solutions of (3) for an N-soliton potential can be deduced from the behavior of solutions in the case $N = 2$. The 2×2 complex amplitude transfer matrix calculated in [8] is

$$\mathbf{T}(\lambda_1,\lambda_2) = \frac{1}{\lambda_1^* - \lambda_2}\begin{bmatrix} \lambda_1^* - \lambda_2^* & \lambda_2^* - \lambda_2 \\ \lambda_1^* - \lambda_1 & \lambda_1 - \lambda_2 \end{bmatrix}, \quad (6)$$

where it is assumed that $a_1 < a_2$. Reference [8] contains a description of how to use this simple matrix to calculate the complex amplitude transfer matrix for a potential created by an arbitrary collision of N solitons.

With this tool, we can now calculate how infinitesimal elements or particles of some conserved quantity of NLS, initially trapped in the potential well of one of the solitons as $t \to -\infty$, are redistributed among all the potential wells as $t \to +\infty$. Let $q[\psi](x,t)$ be one of the conserved densities of NLS, so that

$$\partial_t Q[\psi](t) \doteq \partial_t \int_{-\infty}^{+\infty} q[\psi](x,t)\,dx = 0, \quad (7)$$

as long as $\psi(x,t)$ solves (1) and vanishes along with its derivatives sufficiently rapidly as $x \to \pm\infty$ to allow the integral to converge and boundary terms to vanish. Consider the bound state solution $\phi(x,t)$ of (3) that behaves as (4) with $\alpha_j = I\delta_{jk}$ for some k and some complex number I, so that as $t \to -\infty$ the solution is approximated by $\phi(x,t) \sim I\psi_k^{-}(x,t)$. We define the initial number of particles in the kth potential well to be $Q_k^- = Q[I\psi_k^-]$. Then, according to (5), the solution ϕ is approximated as $t \to +\infty$ in the potential well of soliton j by $IT_{jk}\psi_j^+(x,t)$. The number of particles that are captured by the jth potential well after collision will then be $Q_j^+ = Q[IT_{jk}\psi_j^+]$. So, from the complex amplitude transfer matrix, and the form of (2), we can calculate the elements of the *transfer matrix for the conserved quantity* Q

$$S_{jk} = Q_j^+ / Q_k^-. \quad (8)$$

The elements of this matrix describe the redistribution of the particles of the conserved quantity $Q[\psi]$ that were unambiguously isolated in a particular potential well prior to the interaction of the wells. In general, the elements of the matrix will depend on the numbers a_j and b_j, as well as the parameter I.

Specific examples.—Let us calculate the transfer matrices for the first few conserved quantities of NLS, and describe their properties. First, we consider the L^2 norm N, for which we have the density

$$q_N[\psi](x,t) = |\psi(x,t)|^2. \quad (9)$$

The elements of the corresponding transfer matrix are obtained as $S_{Njk} = b_j b_k^{-1}|T_{jk}|^2$, giving the *norm transfer matrix*

$$\mathbf{S}_N(\lambda_1,\lambda_2) = \frac{1}{\Delta_{21}^2 + (b_2 + b_1)^2}\begin{bmatrix} \Delta_{21}^2 + (b_2 - b_1)^2 & 4b_1 b_2 \\ 4b_1 b_2 & \Delta_{21}^2 + (b_2 - b_1)^2 \end{bmatrix}, \quad (10)$$

where we have introduced the notation Δ_{ij} for the relative velocity $a_i - a_j$. This matrix has a number of desirable properties. First, there is no dependence on the parameter I, which controls the total number of particles present as $t \to -\infty$. More important to our aim of describing the redistribution of particles is the fact that the norm of the linear waves is conserved through the collision, so that for each j we have $S_{N1k} + S_{N2k} = 1$. So, the total norm is the same before and after collision, and when all of the particles are initially trapped in the potential well of soliton k as $t \to -\infty$, the fraction trapped in the potential well of soliton j as $t \to +\infty$ is exactly S_{Njk}. Observe that some particles are always captured by the colliding potential well, but that the fraction captured vanishes in the limit of extreme relative velocity of the two wells ($\Delta_{21}^2 \to \infty$) and also in the limit of extreme difference in depth of the wells [$\ln(b_2/b_1) \to \infty$]. The norm transfer matrix thus encodes the results indicated at the beginning of this Letter.

Some of the nice properties of the norm transfer matrix disappear when we consider the momentum P, for which the density is

$$q_P[\psi](x,t) = \frac{1}{2i}[\psi^*(x,t)\partial_x\psi(x,t) - \psi(x,t)\partial_x\psi^*(x,t)]. \tag{11}$$

In this case, the elements of the transfer matrix are given by $S_{Pjk} = a_j b_j a_k^{-1} b_k^{-1} |T_{jk}|^2$, which gives the *momentum transfer matrix*

$$\mathbf{S}_P(\lambda_1, \lambda_2) = \frac{1}{\Delta_{21}^2 + (b_2 + b_1)^2} \begin{bmatrix} \Delta_{21}^2 + (b_2 - b_1)^2 & 4b_1b_2a_1/a_2 \\ 4b_1b_2a_2/a_1 & \Delta_{21}^2 + (b_2 - b_1)^2 \end{bmatrix}. \tag{12}$$

Again, there is no dependence on the initial total momentum (proportional to $|I|^2$). However, we now see that the momentum for linear waves is not generally conserved during the interaction of the two potential wells. The deviation from conservation of momentum is measured by the relative velocity Δ_{21}. Conservation of momentum fails because the colliding potential well applies some force to a body initially trapped in a stationary well, thus changing its momentum. The total momentum is only conserved when the linear solution $\phi(x,t)$ is proportional to the N-soliton collision $\psi(x,t)$ that created the potential.

The interpretation of the transfer matrix is even more difficult for the Hamiltonian H, which has the conserved density

$$q_H[\psi](x,t) = |\partial_x\psi(x,t)|^2 - |\psi(x,t)|^4. \tag{13}$$

Because the density is not homogeneous in ψ, the elements of the transfer matrix will now depend on the parameter I, and hence on the total Hamiltonian present as $t \to -\infty$. The elements of the *Hamiltonian transfer matrix* are

$$S_{Hjk} = |T_{jk}|^2 \frac{a^2 + b^2 - 2b^2(1 + |I|^2|T_{jk}|^2)/3}{a^2 + b^2 - 2b^2(1 + |I|^2)/3}. \tag{14}$$

Furthermore, as was the case with the momentum, the Hamiltonian is not conserved by the linear waves during the interaction of the potential wells.

We believe that the approach presented above gives a concrete consistent answer to the question of how particles of the L^2 norm, associated unambiguously with a given isolated soliton potential well as $t \to -\infty$, are distributed among the potential wells of all the solitons after their collision. This answer has two features that are desirable for a description of scattering of indestructible particles by a linear process: (i) The total number of particles is preserved by the process. (ii) The scattering properties are independent of the total number of particles involved. When the scattering of particles of other conserved quantities of NLS is considered, these two features are generally no longer present. In general, the scattering properties of particles of any conserved quantity of (1) whose density is of homogeneous degree in ψ will be independent of the total number of particles present as $t \to -\infty$. This was the case, for example, with the scattering of momentum particles. However, in that case we found that the total number of particles is not preserved by the scattering, with the total momentum being altered by the motion of the time-dependent potential. A general conserved density of (1) will not be homogeneous in ψ, and for particles of such conserved quantities both of the properties of norm transfer mentioned above will be absent.

When the total number of particles of a given conserved quantity of (1) is not conserved, it is an indication that delicate interference effects among particles simultaneously trapped in the individual wells of the potential as $t \to -\infty$ are required in order to conserve the total number. In other words, the linear solution $\phi(x,t)$ must be everywhere proportional to the N-soliton collision solution $\psi(x,t)$ of (1) to conserve the particles. Of course, this is precisely the solution we would like to study, but for such a bound state it is not possible to distinguish among the particles present at $t = +\infty$ those that were trapped in a particular potential well at $t = -\infty$. For the L^2 norm, the fact that the total number of particles is preserved even for bound states $\phi(x,t)$ of (3) that differ essentially from the N-soliton collision $\psi(x,t)$ is what enables us to distinguish among the particles present as $t \to +\infty$.

As we mentioned above, there is no way on the basis of a given soliton equation alone to assign identity to infinitesimal elements comprising a wave field that represents the collision of N solitons. It is necessary either to have some *a priori* information not contained in the soliton equation, or to make additional assumptions. However, in any real problem a soliton equation arises as an approximation to a complicated physical process, and it may happen that this underlying process contains information relating to the identity of particles—information that gets neglected in the derivation of the soliton equation as a model. As an example, we might return to the theory of planar optical waveguides and consider the propagation of spatially modulated stationary monochromatic beams of two different specially chosen orthogonal polarizations denoted by unit vectors $\mathbf{e}_{\pm}$ for which the electric field can be written as

$$\mathbf{E}(x,z,t) = [\mathbf{e}_{+}\psi_{+}(X,Z) + \mathbf{e}_{-}\psi_{-}(X,Z)] \times \exp[i(\beta z - \omega t)] + \text{c.c.}, \quad (15)$$

where $X = \epsilon x$, $Z = \epsilon z$, and ϵ is the small ratio between the optical wavelength and the characteristic scale of the spatial modulation. We imagine a medium having an exotic nonlinearity that responds to the two chosen polarizations by producing a refractive index profile $n(X,Z) = |\psi_{+}(X,Z) + \psi_{-}(X,Z)|^2$ *seen by both polarizations.* We see no fundamental physical reason why such a medium cannot exist. In dimensionless form, the equations for the complex envelopes in such a medium would then be

$$i\partial_Z\psi_{\pm} + \tfrac{1}{2}\partial_X^2\psi_{\pm} + |\psi_{+} + \psi_{-}|^2\psi_{\pm} = 0. \quad (16)$$

Together, these equations imply the NLS equation for the sum of the envelopes $\psi = \psi_{+} + \psi_{-}$

$$i\partial_Z\psi + \tfrac{1}{2}\partial_X^2\psi + |\psi|^2\psi = 0, \quad (17)$$

and, using ϕ to represent either ψ_{+} or ψ_{-}, we obtain

$$i\partial_Z\phi + \tfrac{1}{2}\partial_X^2\phi + |\psi|^2\phi = 0. \quad (18)$$

These are the same equations we have analyzed above. One can consider a two soliton collision of the nonlinear equation (17) for the sum of the envelopes, but for which prior to the collision one of the solitons contains only photons polarized in the $\mathbf{e}_{+}$ direction, and the other contains only photons polarized in the $\mathbf{e}_{-}$ direction. After the collision, each soliton of the envelope sum will carry the same number of photons as before the collision. However, the output beams will each now contain some mixture of photons of each polarization. In a case like this, the physical sharing of particles can be obtained by enlarging the scope and working with the model (16), which incorporates the neglected physical mechanism (here, polarization) that allows the particles to be distinguished. This approach is in some ways more attractive than the one we have described at the beginning of this Letter, since now the physical process used to analyze the redistribution of the particles is the same as the physical process that leads to the propagation of the solitons; we can thus consider the redistribution of particles without fixing the refractive index first. Moreover, in principle, experiments can be done with a medium of the type described here to confirm that the exchanges of photons are indeed described by the norm transfer (interpreted in this case as photon transfer) matrix presented in this Letter. It is interesting to consider the degree to which the properties of exchange of particles as dictated by some underlying physical process like (16) leading to a given soliton equation like (17) might be universal and thus independent of the details of the underlying process.

Certainly, other methods of distinguishing among infinitesimal elements participating in a multisoliton collision can also be suggested, including analogies with quantum mechanics and self-consistent field theory. We invite debate on this subject in the future. In particular, we would like to see some discussion of other possible ways of distinguishing among particles involved in soliton collisions—ways that might be useful both in considering conserved densities other than the L^2 norm and also in treating soliton interactions in physical systems modeled at some level of approximation by integrable equations other than NLS.

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