Transfer matrices for multiport devices made from solitons

P. D. Miller and N. N. Akhmediev
Optical Sciences Centre, Research School of Physical Sciences and Engineering, Australian National University, Canberra, Australian Capitol Territory 0200, Australia
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The linear Schrödinger equation is explicitly solved for the case of a refractive index profile created by the interaction of N bright solitons in a Kerr medium. The bound states give exact solutions to the problem of determining how a beam of light is split as it passes through the collision of N solitons; the main result of this work is an analytical formula for the power transmission matrix of this linear problem. As specific examples, the cases of the two-soliton and three-soliton linear couplers are considered, and the power transmission characteristics for bound states are explicitly calculated in both cases. It is further shown that the behavior of linear waves propagating through an arbitrary collision of N solitons can be completely understood in terms of the scattering of linear waves from a soliton X junction. The unbound states of an N-soliton waveguide are also explicitly computable and describe the evolution of radiation modes.

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I. INTRODUCTION

Light-guiding-light effects [1] are promising for all-optical information processing. One such effect is the spatial soliton created by a strong beam, which can act as a linear waveguide, carrying a weak probe beam [2]. This guiding can be accomplished using either bright [3] or dark [4] spatial solitons.

Further, colliding solitons can serve as X junctions [5,6], and more generally as $2N$-port devices. Several of these devices have been analyzed to date [6,7]. Using a probe beam of the same frequency as the solitons is preferable because losses to radiation modes in the impact area of two solitons are avoided [6]. Moreover, such a probe beam sent into one of the channels is not reflected back into any of the input channels, but passes through the impact area of the solitons and is distributed among the output channels—an ideal feature of a multiport switch. However, in practice probe beams of the soliton frequency cannot be effectively separated from the pump, and it is more convenient to use probe beams at another frequency [7,8], which can be resolved by spectral filtering. The cost of this convenience is the partial reflection of the probe back into some of the input channels and some loss to radiation in the soliton impact area [7].

We can keep all the advantages of a purely solitonic device by supposing either that the shift in refractive index induced by the pump is initially fixed in the medium by some process so that a strong beam need not be simultaneously present with the probe at all, or that the pump and probe beams are orthogonally polarized. We study the case when this device arises from the collision of N bright Kerr solitons, use this $2N$-port device as an $N\times N$ switch (see Fig. 1 for an example of a refractive index profile corresponding to $N=3$), and investigate its transmission properties for linear waves of the same frequency as the pump field (which may or may not actually be present). These properties are encoded in the amplitude and power transfer matrices, which we will calculate exactly.

We begin in Sec. II by casting the problem of linear wave propagation in a multisoliton waveguide in a concrete mathematical form. Section III contains a detailed description of the method we will use, which is based upon the integrability of the nonlinear Schrödinger equation. In this section we will also isolate the bound and unbound linear waves that propagate through the multisoliton waveguide. Section IV is devoted to the asymptotic analysis of the bound states of the waveguide, and it is here that we obtain our main result—a precise analytical characterization of the wave transmission properties

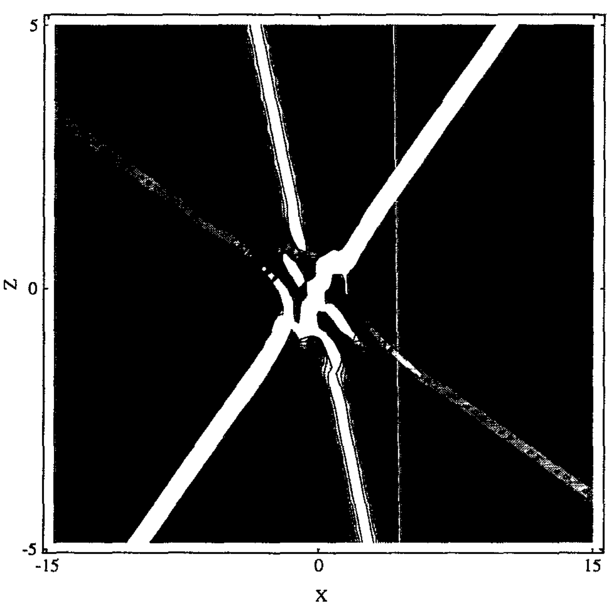

FIG. 1. The refractive index profile of a three-soliton collision. This profile may be considered as a six-port linear device, with three inputs at the bottom of the figure and three outputs at the top.

of a waveguide made from the arbitrary collision of N solitons in a Kerr medium. We will find that the transmission characteristics are independent of the relative optical phases of the interacting solitons and also of any detail of the geometry of the interaction. Only the soliton amplitudes and propagation angles are involved in any of the final formulas. This fact will permit us to interpret an N-soliton waveguide as a network of soliton X junctions. We will then demonstrate the simplicity of the formulas by specific calculations for the cases of $N=2$ and 3. Finally, in Sec. V, we will point out some related problems that we plan to address, and discuss the generality N-soliton waveguide as a $2N$-port linear device. In all that follows, the complex conjugate of a number λ is written as $\bar{\lambda}$.

II. STATEMENT OF THE PROBLEM

In this short section, we present the problem we are going to solve in precise mathematical form, in order to fix our notation. Consider a spatially modulated monochromatic plane light beam propagating in the z direction in a two-dimensional slab Kerr medium. This beam can be described by an electric field of the form

$$\mathbf{E}(x,z,t)=\mathbf{e}\psi(X,Z)\exp[i(\beta z-\omega t)]+\text{c.c.} , \quad (1)$$

where $X=\varepsilon x$, $Z=\varepsilon z$, ε is the small ratio between the wavelength of the carrier wave and the characteristic length scale of the modulation, and $\mathbf{e}$ is a unit vector indicating the polarization direction. In the case when the Kerr nonlinearity acts against diffraction, the complex envelope $\psi(X,Z)$ describing the spatial modulation will satisfy the focusing cubic nonlinear Schrödinger equation

$$i\partial_Z\psi+\tfrac{1}{2}\partial_X^2\psi+|\psi|^2\psi=0 , \quad (2)$$

which we have written in dimensionless variables. A solution $\psi(X,Z)$ of (2) induces an effective refractive index in the material proportional to $|\psi(X,Z)|^2$.

We want to consider the linear propagation of spatially modulated phase waves propagating in the z direction in materials with a spatially dependent refractive index given by $|\psi(X,Z)|^2$, where ψ is a solution of (2). The corresponding theory that we develop below will apply to problems in which the nonlinear induced refractive index is fixed by a photolithographic process, or maintained after the pump has been switched off in a nonlinear medium with memory (e.g., a photorefractive material [9,10]), or when weak beams of frequency ω orthogonally polarized with respect to the pump interact with the material in the presence of the (strong) field (1). In any of these cases, the weak field envelope $\phi(X,Z)$ solves the linear Schrödinger equation

$$i\partial_Z\phi+\tfrac{1}{2}\partial_X^2\phi+V(X,Z)\phi=0 , \quad (3)$$

with the potential function determined from the solution of (2) as

$$V(X,Z)=|\psi(X,Z)|^2 . \quad (4)$$

Of course, we always know one solution of this linear equation, namely $\phi=\psi$. In the case when ψ is a one-soliton solution of (2), this special solution $\phi=\psi$ is the only bound (finite power) state, up to scaling. On the other hand, if ψ is a multisoliton solution of (2), there will be more bound states. Below, we will obtain a linear space of solutions of (3) in the case where the solution $\psi(X,Z)$ of (2) corresponds to an arbitrary interaction of N bright solitons, analyzing the asymptotic behavior of bound states and scattering states. We believe that this space of solutions is complete, so that it contains the solution of (3) for arbitrary $L^2(\mathbb{R})$ initial data at $Z=0$. Some aspects of this complicated problem have been considered previously by Dubrovin *et al.* [11]

III. GENERAL METHOD

Our method, which is developed in greater detail in [12], is based upon the integrability of the nonlinear equation (2), which means that (2) is the consistency condition for the following two linear problems for a two-component vector $\mathbf{u}=(u_1,u_2)^T$:

$$\partial_X\mathbf{u}=\mathbf{L}\mathbf{u}=\begin{bmatrix}-i\lambda & \psi\\ -\bar{\psi} & i\lambda\end{bmatrix}\mathbf{u} , \quad (5)$$

$$i\partial_Z\mathbf{u}=\mathbf{B}\mathbf{u}=\begin{bmatrix}\lambda^2-\frac{1}{2}|\psi|^2 & i\lambda\psi-\frac{1}{2}\partial_X\psi\\ -i\lambda\bar{\psi}-\frac{1}{2}\partial_X\bar{\psi} & -\lambda^2+\frac{1}{2}|\psi|^2\end{bmatrix}\mathbf{u} , \quad (6)$$

where λ is an arbitrary complex parameter [13]. In the problem (5), λ plays the role of an eigenvalue. Each bound state eigenvalue λ_j indicates a soliton component in the field ψ. The compatibility condition for these two ordinary differential equations with nonconstant coefficients depending on $\psi(X,Z)$ is exactly

$$i\partial_Z\mathbf{L}-\partial_X\mathbf{B}+\mathbf{L}\mathbf{B}-\mathbf{B}\mathbf{L}=\mathbf{0} , \quad (7)$$

a matrix partial differential equation that is satisfied for all λ if and only if $\psi(X,Z)$ solves (2). For each solution $\psi(X,Z)$ of (2), there is then a basis of two linearly independent simultaneous solutions to (5) and (6) parametrized by λ. The link between these solutions $\mathbf{u}(X,Z;\lambda)$ and the solutions ϕ of (3) is given by

Theorem 1. Suppose that $\psi(X,Z)$ solves the nonlinear Schrödinger equation (2). Let $\mathbf{u}(X,Z;\lambda)$ be any simultaneous solution of the linear problems (5) and (6), for any complex λ. Then the function

$$\phi(X,Z)=u_1(X,Z;\lambda)\exp[-i(\lambda X+\lambda^2Z)] \quad (8)$$

is a solution of the linear Schrödinger equation (3).

The theorem is proved simply by substituting (8) into the linear Schrödinger equation (3) and eliminating derivatives with respect to X and Z by the linear problems (5) and (6). The *motivation* for transformation (8) is more subtle. One embeds the two problems (2) and (3) into the integrable Manakov (or coupled nonlinear Schrödinger) system [14], from which (8) arises from infinitesimal Bäcklund transformations. See [12] for details. At face value, theorem 1 merely connects the solution of the linear problem (3) with the solution of the other linear problems (5) and (6) that also involve the given function $\psi(X,Z)$ as nonconstant coefficients. However, many exact methods for finding functions $\psi(X,Z)$ that solve the nonlinear Schrödinger equation (2) also produce a basis of simultaneous solutions to (5) and (6) as a byproduct, and thus for functions ψ constructed by such methods, theorem 1 gives us many [recall that in the transformation formula (8) there is an arbitrary complex parameter λ] exact solutions to the problem of interest, (3).

Let us present the construction of the N-soliton solution of (2) and the corresponding basis of simultaneous solutions to (5) and (6), using a method due to Krichever [15] and Manin [16], and elaborated by Date [17]. Choose an integer N, the number of solitons. Define

$$\mathbf{u}^{+}(X,Z;\lambda)=\begin{bmatrix}\sum_{n=0}^{N-1} A_n(X,Z)\lambda^n \\ \lambda^N+\sum_{n=0}^{N-1} B_n(X,Z)\lambda^n\end{bmatrix}\times\exp[i(\lambda X+\lambda^2 Z)]\ , \tag{9}$$

$$\mathbf{u}^{-}(X,Z;\lambda)=\begin{bmatrix}\lambda^N+\sum_{n=0}^{N-1} C_n(X,Z)\lambda^n \\ \sum_{n=0}^{N-1} D_n(X,Z)\lambda^n\end{bmatrix}\times\exp[-i(\lambda X+\lambda^2 Z)]\ . \tag{10}$$

Choose N complex numbers λ_j, and N complex numbers γ_j, $j=1,\ldots,N$. Then determine the $4N$ coefficient functions A_n, B_n, C_n, and D_n by imposing the linear relations

$$\mathbf{u}^{+}(X,Z;\lambda_j)=\gamma_j\mathbf{u}^{-}(X,Z;\lambda_j)\ , \tag{11}$$

$$-\bar{\gamma}_j\mathbf{u}^{+}(X,Z;\bar{\lambda}_j)=\mathbf{u}^{-}(X,Z;\bar{\lambda}_j)\ . \tag{12}$$

If the numbers λ_j are distinct and nonreal [note that without loss of generality we can assume $\mathrm{Im}(\lambda_j)>0$ for all j], then these relations are independent linear conditions on the coefficient functions, and thus the vector functions given in (9) and (10) are uniquely defined. Now define

$$\psi(X,Z)=2iA_{N-1}(X,Z)\ . \tag{13}$$

Then, $\psi(X,Z)$ solves the nonlinear Schrödinger equation (2), and the vector functions $\mathbf{u}^{+}(X,Z;\lambda)$ and $\mathbf{u}^{-}(X,Z;\lambda)$ form a basis of the simultaneous solutions of the Lax pair (5) and (6) for all complex λ (except at the points λ_j and $\bar{\lambda}_j$, where the basis degenerates, forming a bound state). The solutions of (2) so constructed correspond to the interaction of N bright solitons of the form

$$\psi_j(X,Z)=2b_j\,\mathrm{sech}(2b_jX+4a_jb_jZ-\delta_j)\times\exp\{-i[2a_jX+2(a_j^2-b_j^2)Z-\theta_j]\}\ , \tag{14}$$

where a_j, b_j, δ_j, and θ_j are all real, and $\lambda_j=a_j+ib_j$ and $\gamma_j=-\exp(i\theta_j-\delta_j)$. In all that follows, we assume without loss of generality that $a_j>a_k$ whenever $j>k$, and that $b_j>0$ for all j.

We now use this basis of simultaneous solutions to (5) and (6) and appeal to theorem 1 to give two families of solutions, each parametrized by the arbitrary complex parameter λ, to the linear Schrödinger equation (3). Define

$$\phi^{+}(X,Z;\lambda)=u_1^{+}(X,Z;\lambda)\exp[-i(\lambda X+\lambda^2 Z)]=\sum_{n=0}^{N-1}A_n(X,Z)\lambda^n\ , \tag{15}$$

$$\phi^{-}(X,Z;\lambda)=u_1^{-}(X,Z;\lambda)\exp[-i(\lambda X+\lambda^2 Z)]=\left[\lambda^N+\sum_{n=0}^{N-1}C_n(X,Z)\lambda^n\right]\times\exp[-2i(\lambda X+\lambda^2 Z)]\ . \tag{16}$$

The function $\phi^{+}(X,Z;\lambda)$ sweeps out an N-dimensional vector space of solutions to (3), as the parameter λ varies. We will see that these solutions correspond to the bound states. On the other hand, the function $\phi^{-}(X,Z;\lambda)$ sweeps out an infinite-dimensional linear space of solutions to (3). For real values of λ, these solutions are complex exponentials in parts of the (X,Z) plane that are distant from the centers of mass of any of the solitons. As λ range over all real values, the function $\phi^{-}(X,Z;\lambda)$ sweeps out the linear space of scattering states of (3).

A. Bound states

Of particular interest in the theory of guided waves are the bound states of the linear Schrödinger equation (3). We will see in this section that in the case when the ψ is an N-soliton solution of the nonlinear Schrödinger equation (2), then the bound states of (3) are all contained in the family of solutions $\phi^{+}(X,Z;\lambda)$. This family is N-dimensional, as can be seen from formula (15). First, we claim that formula (15) represents a family of bound state solutions to (3).

Theorem 2. For each complex λ, the function $\phi^{+}(X,Z;\lambda)$ is a bound state of the linear Schrödinger equation (3).

Proof: It is a consequence of theorem 1 that for each complex λ, $\phi^{+}(X,Z;\lambda)$ solves (3). We now analyze $\phi^{+}(X,Z;\lambda)$ along straight lines through the origin in the (X,Z) plane. Choose a real slope v and introduce the variables $\chi=X+2vZ$ and $\zeta=Z$. We want to examine the behavior of ϕ^{+} in the limits $\zeta\to\pm\infty$ for fixed χ. Conditions (11) and (12) determine ϕ^{+} and ϕ^{-} by giving the coefficients $A_n(X,Z)$ and $C_n(X,Z)$ as solutions of the linear system

$$\sum_{n=0}^{N-1}A_n(X,Z)\lambda_j^n=\gamma_j\left[\lambda_j^N+\sum_{n=0}^{N-1}C_n(X,Z)\lambda_j^n\right]\times\exp[-2i(\lambda_jX+\lambda_j^2Z)]\ , \tag{17}$$

$$\sum_{n=0}^{N-1}A_n(X,Z)\bar{\lambda}_j^n=-\frac{1}{\bar{\gamma}_j}\left[\bar{\lambda}_j^N+\sum_{n=0}^{N-1}C_n(X,Z)\bar{\lambda}_j^n\right]\times\exp[-2i(\bar{\lambda}_jX+\bar{\lambda}_j^2Z)]\ , \tag{18}$$

where $j=1,\ldots,N$. Taking $v>a_j$ for all j puts us in the moving frame of an observer moving far to the right (left) of the solitons for large negative (positive) ζ. Then, using $b_j>0$ for all j, in the limit of $|\zeta|\to\infty$ one obtains the relations

$$\begin{aligned}&\sum_{n=0}^{N-1} A_n(\chi-2v\zeta,\zeta)\lambda_j^n=0, \quad \zeta\to+\infty, \quad j=1,\ldots,N\ ,\\&\sum_{n=0}^{N-1} A_n(\chi-2v\zeta,\zeta)\bar{\lambda}_j^n=0, \quad \zeta\to-\infty, \quad j=1,\ldots,N\ .\end{aligned} \tag{19}$$

Likewise, taking $v<a_j$ for all j yields

$$\begin{aligned}&\sum_{n=0}^{N-1} A_n(\chi-2v\zeta,\zeta)\bar{\lambda}_j^n=0, \quad \zeta\to+\infty, \quad j=1,\ldots,N\ ,\\&\sum_{n=0}^{N-1} A_n(\chi-2v\zeta,\zeta)\lambda_j^n=0, \quad \zeta\to-\infty, \quad j=1,\ldots,N\ .\end{aligned} \tag{20}$$

The $N\times N$ matrix $\mathbf{V}(\lambda_1,\ldots,\lambda_N)=\{V_{jk}=\lambda_j^{k-1}\}$ is a Vandermonde matrix which is invertible because the λ_j are distinct [the same is true of $\mathbf{V}(\bar{\lambda}_1,\ldots,\bar{\lambda}_N)$]. Thus the limiting values of A_n are identically zero for all $n=0,\ldots,N-1$. Inserting these limiting values into formula (15) we see that $\phi^+(\chi-2v\zeta,\zeta;\lambda)$ vanishes far to the right and left of all the solitons for all values of the arbitrary parameter λ.

It is easy to extend the arguments in the above proof to show that as long as $v\neq a_j$ for $j=1,\ldots,N$ one has $\phi^+\to 0$ for all λ as $\zeta\to\pm\infty$. Thus the solutions (15) of (3) are asymptotically confined to the individual soliton waveguides. These solutions are essentially characterized by the power confined in each guide as $\zeta\to\pm\infty$. We will calculate this power shortly. But first, we consider the class of solutions of (3) described by the function $\phi^-(X,Z;\lambda)$.

B. Scattering states

Here we begin to investigate the solutions of the linear Schrödinger equation (3) that are built from the function $\phi^-(X,Z;\lambda)$ which was defined by (16). As λ ranges over all real numbers, the function $\phi^-(X,Z;\lambda)$ sweeps out an infinite-dimensional space of unbound radiation modes of (3). We state this as follows.

Theorem 3. Let λ be real. Then, the function $\phi^-(X,Z;\lambda)$ is a solution of (3) that for large Z behaves as a constant times the bounded exponential $\exp[-2i(\lambda x+\lambda^2 t)]$ in any constant velocity moving frame that is not the frame of any of the solitons. Moreover, if $\lambda\neq\mu$ then $\phi^-(X,Z;\lambda)$ and $\phi^-(X,Z;\mu)$ are linearly independent.

Proof: As with ϕ^+, theorem 1 guarantees that $\phi^-(X,Z;\lambda)$ solves (3) for all complex λ, and in particular for all real λ. Now, arguing as in the proof of theorem 2, we choose a moving frame described by the change of coordinates $\chi=X+2vZ$ and $\zeta=Z$ and analyze the linear system (17) and (18) as $|\zeta|\to\infty$. Suppose that $a_k<v<a_{k+1}$. Then, as $\zeta\to+\infty$, one obtains

$$\begin{bmatrix} C_0\\ \vdots \\ C_{N-1}\end{bmatrix} = -\mathbf{V}(\bar{\lambda}_1,\ldots,\bar{\lambda}_k,\lambda_{k+1},\ldots,\lambda_N)^{-1}\begin{bmatrix}\bar{\lambda}_1^N\\ \vdots\\ \bar{\lambda}_k^N\\ \lambda_{k+1}^N\\ \vdots\\ \lambda_N^N\end{bmatrix}, \tag{21}$$

and, as $\zeta\to-\infty$,

$$\begin{bmatrix} C_0\\ \vdots \\ C_{N-1}\end{bmatrix} = -\mathbf{V}(\lambda_1,\ldots,\lambda_k,\bar{\lambda}_{k+1},\ldots,\bar{\lambda}_N)^{-1}\begin{bmatrix}\lambda_1^N\\ \vdots\\ \lambda_k^N\\ \bar{\lambda}_{k+1}^N\\ \vdots\\ \bar{\lambda}_N^N\end{bmatrix}. \tag{22}$$

Again, the Vandermonde matrices are invertible because the soliton eigenvalues are distinct, and thus the functions $C_n(X,Z)$ have asymptotically constant values along the lines of constant χ. The only dependence on X and Z that remains in $\phi^-(X,Z;\lambda)$ in these limits is thus in the exponential factor, which is bounded because λ is real. The linear independence of these functions for different real values of λ then follows from the linear independence of the exponentials.

From formulas (21) and (22) it is possible to normalize the functions $\phi^-(X,Z;\lambda)$ appropriately and calculate the transmission and reflection due to each arm of the multisoliton waveguide in turn. We leave this calculation for a future publication.

IV. TRANSFER MATRICES FOR BOUND STATES

We now return to the calculation of the asymptotic confinement properties of bound states of the N-soliton waveguide. Let $v=a_k$ for some $k=1,\ldots,N$ describe the frame of reference in which the soliton ψ_k defined by (14) (modulo phase shifts) is stationary. Then Eqs. (17) and (18) imply that, as $\zeta\to+\infty$,

$$\begin{aligned}&\sum_{n=0}^{N-1} A_n(\chi-2v\zeta,\zeta)\lambda_j^n=0, \quad j<k\ ,\\&\sum_{n=0}^{N-1} A_n(\chi-2v\zeta,\zeta)\bar{\lambda}_j^n=0, \quad j>k\ ,\end{aligned} \tag{23}$$

and, as $\zeta\to-\infty$,

$$\begin{aligned}&\sum_{n=0}^{N-1} A_n(\chi-2v\zeta,\zeta)\bar{\lambda}_j^n=0, \quad j<k\ ,\\&\sum_{n=0}^{N-1} A_n(\chi-2v\zeta,\zeta)\lambda_j^n=0, \quad j>k\ .\end{aligned} \tag{24}$$

In each of the two limits $\zeta\to\pm\infty$, we thus have $N-2$ equations among the asymptotic values of A_0 through A_{N-1}. Recall that $A_{N-1}(X,Z)$ is distinguished by (13) as being proportional to the N-soliton collision, whose asymptotic behavior is known. Thus, we can use (23) and (24) to eliminate A_n, $n=0,\ldots,N-2$ in favor of A_{N-1}. We find as $\zeta\to+\infty$ that

$$\begin{bmatrix} A_0 \\ \vdots \\ A_{N-2} \end{bmatrix} = A_{N-1}\mathbf{V}(\lambda_1,\ldots,\lambda_{k-1},\bar{\lambda}_{k+1},\ldots,\bar{\lambda}_N)^{-1}\begin{bmatrix} \lambda_1^{N-1} \\ \vdots \\ \lambda_{k-1}^{N-1} \\ \bar{\lambda}_{k+1}^{N-1} \\ \vdots \\ \bar{\lambda}_N^{N-1} \end{bmatrix}, \tag{25}$$

and, as $\zeta\to-\infty$, that

$$\begin{bmatrix} A_0 \\ \vdots \\ A_{N-2} \end{bmatrix} = A_{N-1}\mathbf{V}(\bar{\lambda}_1,\ldots,\bar{\lambda}_{k-1},\lambda_{k+1},\ldots,\lambda_N)^{-1}\begin{bmatrix} \bar{\lambda}_1^{N-1} \\ \vdots \\ \bar{\lambda}_{k-1}^{N-1} \\ \lambda_{k+1}^{N-1} \\ \vdots \\ \lambda_N^{N-1} \end{bmatrix}. \tag{26}$$

Again, we have the guarantee that the Vandermonde matrices are invertible. This might be a good place to give the formula for the inverse of a Vandermonde matrix. The matrix elements of $\mathbf{V}(a_1,\ldots,a_N)^{-1}$ are given by

$$V_{ij}^{-1}=(-1)^{N-i}\frac{\sum_{\mathbf{k}}\prod_{m=1}^{N-i} a_{k_m}}{\prod_{k=1,k\neq i}^{N-1}(a_i-a_k)}, \tag{27}$$

where the sum is taken over all vectors $\mathbf{k}=(k_1,\ldots,k_{N-i})^T$, where $k_m<k_n$ whenever $m<n$ and $k_m\neq j$ for all m. This means that there are constants A_n^k, found by solving these linear systems, such that as $\zeta\to+\infty$,

$$A_n(\chi-2v\zeta,\zeta)=A_n^k A_{N-1}(\chi-2v\zeta,\zeta), \quad n=0,\ldots,N-2, \tag{28}$$

and as $\zeta\to-\infty$,

$$A_n(\chi-2v\zeta,\zeta)=\overline{A_n^k} A_{N-1}(\chi-2v\zeta,\zeta), \quad n=0,\ldots,N-2. \tag{29}$$

We now can calculate the behavior of $\phi^+(X,Z;\lambda)$ as $Z\to\pm\infty$ in any of the waveguide arms. To proceed, we take as a convenient basis of the vector space of all bound state solutions to (3) the collection of N functions $\phi_k^+(X,Z)=\phi^+(X,Z;\lambda_k)$. By taking linear combinations of these basis elements, we want to find a one-dimensional space of bound state solutions of (3) that vanish in all but one waveguide at $\zeta=-\infty$, say, the waveguide indexed by the soliton eigenvalue λ_j. Let us call an element of this latter space $\Phi_j(X,Z)$. Expanding in our basis with coefficients f_{jk} gives the following representation for $\Phi_j(X,Z)$:

$$\Phi_j(X,Z)=\sum_{k=1}^{N} f_{jk}\phi_k^+(X,Z) = \sum_{k=1}^{N} f_{jk}\left[\sum_{n=0}^{N-1} A_n(X,Z)\lambda_k^n\right]. \tag{30}$$

The coefficients f_{jk} are determined by imposing the conditions that Φ_j vanish as $\zeta\to-\infty$ in all waveguides m except for $m=j$, in which we will normalize by taking Φ_j to be asymptotically equal to the N-soliton collision $\psi(X,Z)$. As $\zeta\to-\infty$, we have, in waveguide m,

$$\Phi_j(X,Z)=\left[\sum_{k=1}^{N} f_{jk}\left[\lambda_k^{N-1}+\sum_{n=0}^{N-2}\overline{A_n^m}\lambda_k^n\right]\right]\times A_{N-1}(X,Z) = \frac{1}{2i}\psi(X,Z)\sum_{k=1}^{N} f_{jk}g_{km}^-, \tag{31}$$

where we have defined

$$g_{km}^-=\lambda_k^{N-1}+\sum_{n=0}^{N-2}\overline{A_n^m}\lambda_k^n. \tag{32}$$

Thus we determine the elements f_{jk} of the matrix $\mathbf{F}$ in terms of the elements g_{km}^- of the matrix $\mathbf{G}_-$ by

$$\mathbf{F}=2i\mathbf{G}_{-}^{-1} . \quad (33)$$

Now, as $\zeta \to +\infty$, in waveguide m we have

$$\Phi_j(X,Z)=T_{mj}\psi(X,Z) , \quad (34)$$

where we have defined the *complex amplitude transfer matrix* by

$$\mathbf{T}=\frac{1}{2i}[\mathbf{F}\mathbf{G}_{+}]^{T} = [\mathbf{G}_{-}^{-1}\mathbf{G}_{+}]^{T} , \quad (35)$$

and the elements g_{km}^{+} of the matrix $\mathbf{G}_{+}$ are given by

$$g_{km}^{+}=\lambda_k^{N-1}+\sum_{n=0}^{N-2}\lambda_k^n A_n^m . \quad (36)$$

Thus an arbitrary bound beam, which as $Z \to -\infty$ is of the form

$$\phi(X,Z)=\sum_{j=1}^{N}\alpha_j\psi_j(X,Z) , \quad (37)$$

will be scattered by the N-soliton refractive index $|\psi(X,Z)|^2$ to become

$$\phi(X,Z)=\sum_{m=1}^{N}\left[\sum_{j=1}^{N}T_{mj}\alpha_j\right]\psi_m(X,Z) , \quad (38)$$

as $Z \to +\infty$. A very important observation that can be made at this point is that the complex amplitude transfer matrix depends only on the soliton eigenvalues λ_i which contain the soliton amplitude and propagation angle information; *there is no dependence whatsoever on the relative phases involved in the N-soliton collision* which are contained in the parameters γ_i (see Fig. 2).

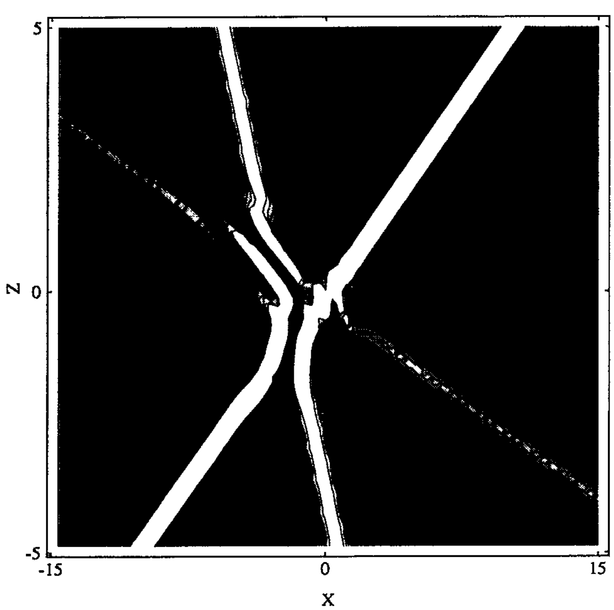

FIG. 2. The refractive index profile of a three-soliton collision. This waveguide has the same transmission properties as that shown in Fig. 1.

Finally, we turn our attention to the calculation of the power transfer. The power of the free soliton beam $\psi_j(X,Z)$ is

$$\int_{-\infty}^{\infty}|\psi_j(X,Z)|^2 dX=8b_j . \quad (39)$$

Then the *power transfer matrix* is defined in terms of the amplitude transfer matrix by

$$P_{mj}=\frac{b_m}{b_j}|T_{mj}|^2 . \quad (40)$$

The interpretation of this matrix is that p units of power input into waveguide j at $Z=-\infty$ (with no power input into any of the other waveguides) will be split among all the waveguides at $Z=+\infty$, with power pP_{mj} in waveguide m. If more than one waveguide is illuminated at $Z=-\infty$, then there will be interference effects that are taken into account by the complex amplitude transfer matrix —for which there is a superposition principle— but not by the simple definition of power transfer that we have given above. However, from the complex amplitude transfer matrix, the power transfer matrix inherits the important property that it does not depend on the exact geometry of the N-soliton collision that forms the waveguide, only on the sizes of the solitons and their angles of propagation.

The essential results of this section are summarized in the following.

Theorem 4 (transfer characteristics of N-soliton waveguides). The complete transfer characteristics for bound states of an N-soliton junction waveguide are given in terms of the N soliton eigenvalues λ_j by the following simple algebraic algorithm:

(1) For $1 \le k \le N$ and $0 \le n \le N-2$, calculate the numbers A_n^k by solving the linear equations

$$\sum_{n=1}^{N-1}\lambda_j^{n-1}A_{n-1}^k=\lambda_j^{N-1}, \quad 1 \le j \le k-1 ,$$

$$\sum_{n=1}^{N-1}\bar{\lambda}_j^{n-1}A_{n-1}^k=\bar{\lambda}_j^{N-1}, \quad k+1 \le j \le N .$$

(2) Then construct the $N \times N$ matrices $\mathbf{G}_{+}$ and $\mathbf{G}_{-}$ according to

$$g_{km}^{+}=\lambda_k^{N-1}+\sum_{n=0}^{N-2}A_n^m\lambda_k^n, \quad g_{km}^{-}=\lambda_k^{N-1}+\sum_{n=0}^{N-2}\overline{A_n^m}\lambda_k^n .$$

(3) The complex amplitude transfer matrix is then given explicitly as

$$\mathbf{T}=[\mathbf{G}_{-}^{-1}\mathbf{G}_{+}]^{T} .$$

(4) The elements of the $N \times N$ power transfer matrix are then

$$P_{mj}=\frac{b_m}{b_j}|T_{mj}|^2 .$$

The fact that the transfer characteristics are independent of the phase shifts or optical phases of the N individual soliton beams making up the waveguide indicates that a remarkably simple and beautiful idea lies behind the algorithm given in theorem 4. Suppose one wanted to calculate the complex amplitude transfer matrix for a given N-soliton waveguide. Armed with the knowledge that phase shifts cannot affect the final asymptotic properties, one can spatially translate the individual solitons participating in the collision until the $N\times N$ collision is well resolved into $(N^2-N)/2$ isolated 2×2 soliton collisions. *The guidance properties of an N-soliton waveguide can be completely deduced from the guidance properties of the two-soliton waveguide. In this context, an N-soliton collision is nothing more than the sum total of the pairwise collisions among the N solitons.* This property is clearly analogous to the well known fact that the phase shift that a soliton accrues as it simultaneously collides with $N-1$ others is exactly equal to the sum of the pairwise phase shifts, as if the collision were in fact with one soliton at a time. We now consider some concrete examples of the algorithm described in theorem 4.

A. Fundamental example: The general soliton X junction

As an illustration, let us calculate the transfer matrices for the special case of a soliton X junction, for which $N=2$. First, we calculate A_0^m, and obtain the simple result that

$$A_0^1=\bar{\lambda}_2, \quad A_0^2=\lambda_1 \ . \tag{41}$$

The complex amplitude transfer matrix is given by

$$\mathbf{T}=\begin{bmatrix}\lambda_1+\bar{\lambda}_2 & \lambda_2+\bar{\lambda}_2\\ 2\lambda_1 & \lambda_2+\lambda_1\end{bmatrix}\begin{bmatrix}\lambda_1+\lambda_2 & 2\lambda_2\\ \lambda_1+\bar{\lambda}_1 & \lambda_2+\bar{\lambda}_1\end{bmatrix}^{-1}$$
$$=\frac{1}{\bar{\lambda}_1-\lambda_2}\begin{bmatrix}\bar{\lambda}_1-\bar{\lambda}_2 & \bar{\lambda}_2-\lambda_2\\ \bar{\lambda}_1-\lambda_1 & \lambda_1-\lambda_2\end{bmatrix}. \tag{42}$$

As we have noted, this 2×2 matrix is of fundamental importance, since all $N\times N$ complex amplitude transfer matrices can be built out of it. We will see how shortly. Finally, using (40), we obtain the power transfer matrix for the soliton X junction:

$$\mathbf{P}=\frac{1}{\Delta_{21}^2+(b_2+b_1)^2}\times\begin{bmatrix}\Delta_{21}^2+(b_2-b_1)^2 & 4b_1b_2\\ 4b_1b_2 & \Delta_{21}^2+(b_2-b_1)^2\end{bmatrix}, \tag{43}$$

where we have introduced the notation $\Delta_{ij}=a_i-a_j$ for the angle difference between the soliton waveguides. Thus we observe that, in addition to the power transfer not depending on any phase information about the interaction of the waveguides, there is also no dependence on the overall angle of the guides in the medium—only the angle difference is involved. It is easy to check that, for each j,

$$\sum_{k=1}^{N} P_{kj}=1 \ , \tag{44}$$

which expresses the conservation of norm in Eq. (3). Another interesting property of the transmission matrix (43) is its symmetry. The matrix is symmetric even if the amplitudes of the two colliding solitons are unequal. When the soliton amplitudes coincide, matrix (43) describes transmission properties obtained numerically in [6]. Also, observe that if the colliding solitons have unequal amplitudes the diagonal elements of the matrix are nonzero even when two solitons are almost parallel ($\Delta_{21}\to 0$).

One of the obvious applications of our results is to the design of efficient signal routing devices like the optical X junction, a device meant to take two optical signals as inputs, redistributing the power in a predictable way among two output ports. The X junctions typically fabricated today guide signals imperfectly through the device, in the sense that some power is lost to radiation in the interaction region [18]. However, it is now clear that if a device is designed with the index profile of the junction given by the geometry of a two Kerr soliton collision, these losses will be reduced to zero. Moreover, the size of a device made in this way can in principle be comparable to the operating wavelength, in contrast to existing devices whose lengths are comparable to the beat length of their linear modes—much greater than a wavelength. These features suggest strongly that the soliton based X junctions considered here can be efficiently used in integrated optical circuits.

Moreover, since their transmission behavior is now established, such devices are easy to design to specification. As a concrete example, we show how to use the power transfer matrix for the X junction to design a signal splitter. Choose a splitting ratio $0<r<1$, and imagine building the index profile of the device from two solitons of the same amplitude $b=1$. Then, to have the signal-splitter power transfer matrix

$$\mathbf{P}=\begin{bmatrix}1-r & r\\ 1 & 1-r\end{bmatrix}, \tag{45}$$

one must only choose the angle between the solitons according to $\Delta_{21}^2=4(r^{-1}-1)$.

B. Another example: The general soliton six-port device

As a further demonstration, we consider the case of a three-armed waveguide constructed out of the collision of three soliton solutions of (2), that is we take $N=3$ (see Figs. 1 and 2 for example index profiles). Rather than calculating the amplitude transfer matrix elements from the algorithm in theorem 4, let us show how elements of the 3×3 complex amplitude transfer matrix can be calculated simply from knowledge of the fundamental 2×2 complex amplitude transfer matrix (42). Let us adopt the notation $\mathbf{T}^{(2)}(\lambda_j,\lambda_k)$ for the 2×2 amplitude transfer matrix associated with the collision of the solitons indexed by λ_j and λ_k with $k>j$. In Fig. 3 we diagram a collision among three solitons indexed by the eigenvalues λ_1, λ_2, and λ_3, where we have used the freedom of the phase shifts of the solitons to separate the impact into three isolated 2×2 impacts. There are several ways to separate

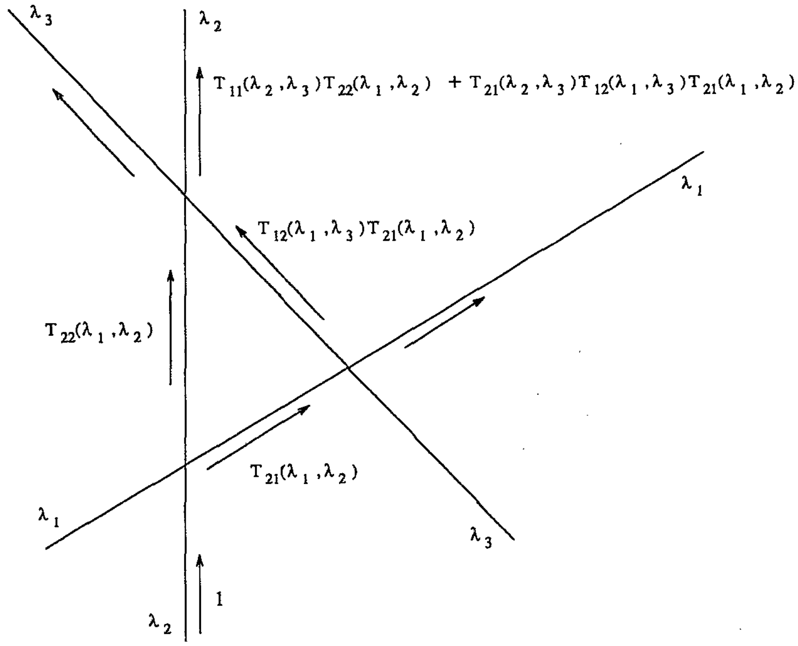

FIG. 3. Calculation of the $N=3$ matrix element T_{22} from the $N=2$ matrix elements.

the soliton collisions like this; however, the final expression for the amplitude transfer matrix is, of course, independent of any particular choice. Let us use the diagram in Fig. 3 to calculate the element T_{22} of the 3×3 amplitude transfer matrix. We assume that for large negative Z only the waveguide indexed by λ_2 is illuminated, and that the field envelope there agrees with the three-soliton interaction that created the waveguide. As this beam passes through the impact with the waveguide indexed by λ_1, the beam is split into two components. The field component that remains in the λ_2 waveguide picks up a complex factor of $T_{22}^{(2)}(\lambda_1,\lambda_2)$, and the component that is conducted into the other waveguide picks up a complex factor of $T_{21}^{(2)}(\lambda_1,\lambda_2)$. This other component propagates in the λ_1 waveguide until collision with the λ_3 waveguide, whereupon it is again split into two. We are not concerned with the part that remains in the λ_1 waveguide, since it cannot exit through the λ_2 guide; the other part picks up a complex factor of $T_{12}^{(2)}(\lambda_1,\lambda_3)$. Now this beam propagating in the λ_3 waveguide collides with the beam that emerged from the very first collision of the λ_2 waveguide, and the superposition is split among the output waveguides indexed by λ_2 and λ_3. We are only concerned with the beam that exists from the λ_2 waveguide; the overall factor output in this waveguide is

$$T_{22}=T_{11}^{(2)}(\lambda_2,\lambda_3)T_{22}^{(2)}(\lambda_1,\lambda_2)+T_{21}^{(2)}(\lambda_2,\lambda_3)T_{12}^{(2)}(\lambda_1,\lambda_3)T_{21}^{(2)}(\lambda_1,\lambda_2)\ . \qquad (46)$$

This procedure concretely establishes the link between the properties of an N-soliton waveguide and those of the soliton X junction. In practice, one may choose between calculating the amplitude transfer matrix either by this method or directly from the algorithm given in theorem 4. Of course the results are the same in both cases.

With the complex amplitude transfer matrix calculated by either one of these methods, we can obtain the elements of the 3×3 power transfer matrix for the general soliton six-port device. The power transfer matrix is symmetric, the independent elements being

$$\begin{aligned} P_{12}=P_{21}&=\frac{4b_1b_2[\Delta_{31}^2+(b_3-b_1)^2]}{[\Delta_{21}^2+(b_2+b_1)^2][\Delta_{31}^2+(b_3+b_1)^2]}\ ,\\ P_{13}=P_{31}&=\frac{4b_1b_3}{\Delta_{31}^2+(b_3+b_1)^2}\ ,\\ P_{23}=P_{32}&=\frac{4b_2b_3[\Delta_{31}^2+(b_3-b_1)^2]}{[\Delta_{31}^2+(b_3+b_1)^2][\Delta_{32}^2+(b_3+b_2)^2]}\ , \end{aligned} \qquad (47)$$

and the diagonal elements are given in terms of the others as

$$\begin{aligned} P_{11}&=1-P_{21}-P_{31}\ ,\\ P_{22}&=1-P_{12}-P_{32}\ ,\\ P_{33}&=1-P_{13}-P_{23}\ , \end{aligned} \qquad (48)$$

which expresses conservation of power. We have again used the notation $\Delta_{ij}=a_i-a_j$, and it is again clear that there is no dependence of the overall angle of the waveguide structure in the medium with respect to the Z axis. Note that the power transmission matrix for the three-soliton waveguide is less symmetrical than it was for a soliton X junction in the sense that $P_{11}\neq P_{22}$. The reason for the broken symmetry is that there now is a distinguished waveguide corresponding to the eigenvalue λ_2 that is between the other two. Also, observe that the power transfer matrix for the soliton X junction can be recovered from the upper left corner of this matrix upon setting $b_3=0$. There are five independent parameters in the above formulas (three amplitudes and two angles between the solitons), which feed into the three independent elements of the power transfer matrix. This means that we can design a 3×3 switch with diverse properties by controlling the independent parameters. Examples will be considered elsewhere.

V. CONCLUSIONS

The main results of this work are the analytical formulas for the transfer matrices (35) of linear wave propagation through the impact area of multisoliton collisions. Using them, we can calculate the complex amplitudes (and powers) of linear waves in each of the output channels from the amplitudes of the input linear waves. Clearly, this is the response function of a $2N$-port device made from a multisoliton collision. The elements of these matrices depend only on the amplitudes of colliding solitons and the angles of collision between them—there is never any dependence on the initial relative optical phases of the solitons or on the overall spatial geometry of the soliton interaction. Hence we can control the switching properties of this $2N$-port device by specifying only the angles of collision and amplitudes of the solitons. The lack of dependence on the relative phases of the solitons also means that the linear properties of an N-soliton collision can be completely described in terms of $(N^2-N)/2$ pairwise collisions.

We can also comment on the generality of multiport linear devices made from the interaction of N solitons. A symmetric $N\times N$ power transfer matrix that conserves power can only have $(N^2-N)/2$ independent real elements. We have seen that exactly $2N$ real quantities (the real and imaginary parts of the soliton eigenvalues) feed into the power transfer matrix for a multiport linear device made from solitons. Thus, for $N<5$, we expect multiport devices made from solitons to have quite general properties, while for $N>5$ the multiport devices made from solitons will be very special cases of the general linear multiport device.

The scattering description encoded in the transfer matrices is only strictly valid for signals that are injected into the device at $Z=-\infty$ with form (37) for arbitrary α_j. More general signals, or signals that are imperfectly matched to the device at $Z=-\infty$ will contain a radiation component whose evolution in Z is described by the functions $\phi^-(X,Z;\lambda)$ that will carry away a small amount of the input power.

Regarding our mathematical procedure, one issue that we have not addressed here is the question of the completeness of the solutions to the linear Schrödinger equation (3) represented by formulas (15) and (16). In the future, we would like to prove that the space of functions with convergent expansions (for fixed but arbitrary Z) of the form

$$f(X)=\sum_{k=1}^{N}\alpha_k\phi^+(X,Z;\lambda_k)+\int_{-\infty}^{\infty}g(\lambda)\phi^-(X,Z;\lambda)d\lambda \tag{49}$$

is exactly $L^2(\mathbb{R})$ for appropriate functions $g(\lambda)$. We would also like to establish orthogonality relations among solutions (15) and (16) of (3) that allow the easy projection of $f(X)$ onto its components. With these tools, in fact the general solution to the initial value problem for (3) will be written in the form of (49).

We also plan to treat the problem of solving the linear Schrödinger equation (3) in the case when the potential $V(X,Z)$ is equal to the negative square modulus of a N dark soliton solution of the defocusing cubic nonlinear Schrödinger equation. The basic calculations are much the same, but the techniques for writing down the exact N-soliton solution and its associated basis of simulations solutions to the linear problems (5) and (6) are slightly more involved.

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