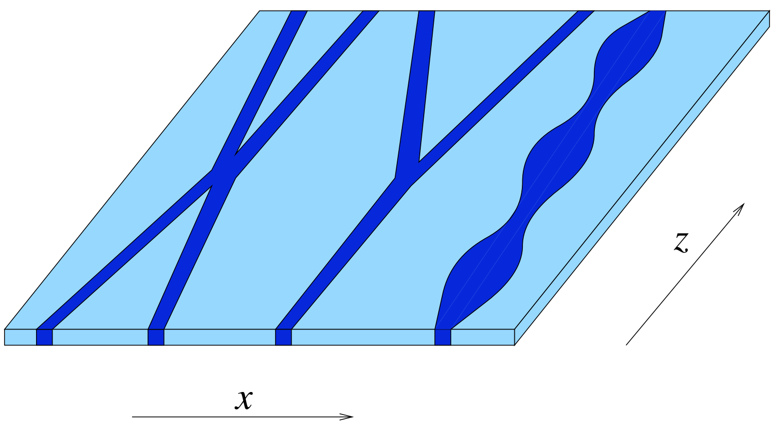
Trapping of Waves by Solitons

Peter D. Miller

Department of Mathematics
University of Michigan

Paraxial Planar Waveguide Optics



$$i\beta\frac{\partial f}{\partial z}+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}+\beta^2\Delta(x,z)f=0$$

$f(x,z)$ is the stationary envelope of the electric field.

$\Delta(x,z)$ is the refractive index distribution.

β is a frequency parameter.

M-Soliton Waveguides and Exact Solutions: $\beta = 1$

Begin with the expressions:

$$a(x,z,\lambda) = \left(\lambda^M + \sum_{p=0}^{M-1} \lambda^p a^{(p)}(x,z)\right) e^{-2i(\lambda x + \lambda^2 z)} \quad \text{and} \quad \vec{b}(x,z,\lambda) = \sum_{p=0}^{M-1} \lambda^p \vec{b}^{(p)}(x,z).$$

Choose $\lambda_1, \ldots, \lambda_M$ in $\mathbb{C}_+$, and corresponding N-vectors $\vec{g}^{(1)}, \ldots, \vec{g}^{(M)}$.

$\forall k$, impose: $a(x,z,\lambda_k) = \vec{g}^{(k)\dagger}\vec{b}(x,z,\lambda_k)$ and $\vec{b}(x,z,\lambda_k^*) = -a(x,z,\lambda_k^*)\vec{g}^{(k)}$.

Index function: set $\Delta(x,z) = 4\sum_{n=1}^{N} |b_n^{(M-1)}(x,z)|^2$.

Corresponding Exact Solutions for $f(x,z)$:

- Dispersive modes for $\lambda \in \mathbb{R}$: $\Psi_\mathrm{d}(x,z,\lambda) = \left(\pi \prod_{k=1}^{M} |\lambda - \lambda_k|^2\right)^{-1/2} a(x,z,\lambda)$
- M independent bound states: $\{\Psi_{\mathrm{b},1}(x,z), \ldots, \Psi_{\mathrm{b},M}(x,z)\}$ obtained from $\{a(x,z,\lambda_1^*), \ldots, a(x,z,\lambda_M^*)\}$ by Gram-Schmidt in $L^2(\mathbb{R})$.

In the background is a nonlinear problem...

$$i\frac{\partial\psi_k}{\partial z} + \frac{1}{2}\frac{\partial^2\psi_k}{\partial x^2} + \left(\sum_{n=1}^{N} |\psi_n|^2\right)\psi_k = 0$$

and $\psi_k(x,z) = 2ib_k^{(M-1)}(x,z)$.

Completeness Relation

Theorem 1 (M. and Akhmediev, Physica D, 1998) *The functions $\Psi_{\mathrm{d}}(x,z,\lambda)$ for $\lambda \in \mathbb{R}$ and $\{\Psi_{\mathrm{b},1}(x,z),\ldots,\Psi_{\mathrm{b},M}(x,z)\}$ form an orthonormal basis of $L^2(\mathbb{R})$ for any fixed z. Thus, for $\phi(x) \in L^2(\mathbb{R})$, we have*

$$\phi(x) = \sum_{k=1}^{M} \widehat{\phi}_k \Psi_{\mathrm{b},k}(x,z) + \int_{-\infty}^{\infty} \widehat{\phi}(\lambda)\Psi_{\mathrm{d}}(x,z,\lambda)\,d\lambda$$

where

$$\widehat{\phi}(\lambda) = \int_{-\infty}^{\infty} \Psi_{\mathrm{d}}(x,z,\lambda)^*\phi(x)\,dx \quad \textit{and} \quad \widehat{\phi}_k = \int_{-\infty}^{\infty} \Psi_{\mathrm{b},k}(x,z)^*\phi(x)\,dx\,.$$

Solution of Initial Value Problem: $\beta = 1$

$$i\frac{\partial f}{\partial z} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \Delta(x,z)f = 0$$

$\Delta(x,z)$ specified by discrete data $\{\lambda_1, \ldots, \lambda_M\}$ and $\{\vec{g}^{(1)}, \ldots, \vec{g}^{(M)}\}$.

1. Project the initial data $f(x,0)$ orthogonally onto the basis elements $\Psi_{\text{d}}(x,z,\lambda)$ and $\{\Psi_{\text{b},k}(x,z)\}$ at $z=0$.

2. Fix the expansion coefficients and let the basis elements evolve explicitly in "time" z.

3. Recover $f(x,z)$ for $z > 0$ by the completeness relation.

Bound State Scattering for $\beta = 1$

Let $\lambda_k = \sigma_k + i\rho_k$. If $\sigma_1, \ldots, \sigma_M$ are distinct, then

$$\Delta(x,z) \sim \sum_{k=1}^{M} 4\rho_k^2 \text{sech}^2(2\rho_k(x + 2\sigma_k z) - \delta_k^{\pm}) \quad \text{as} \quad z \to \pm\infty \,.$$

Superpositions of $\Psi_{\text{b},k}(x,z)$ have a similar asymptotic form:

$$f(x,z) \sim \sum_{k=1}^{M} 4u_k^{\pm}\rho_k \text{sech}(2\rho_k(x + 2\sigma_k z) - \delta_k^{\pm}) \quad \text{as} \quad z \to \pm\infty \,,$$

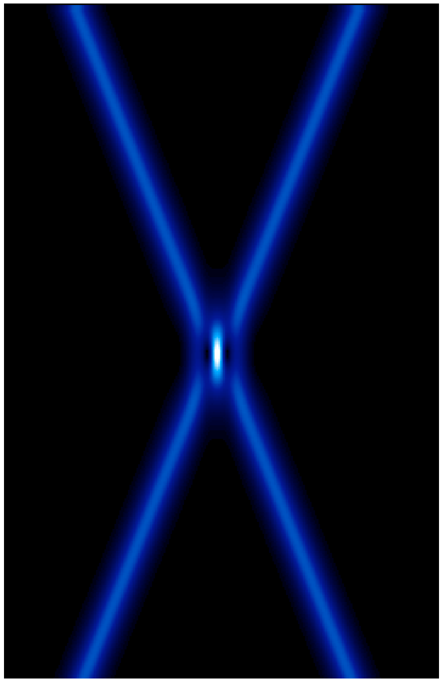
for some constants $u_k^{\pm}$. Linear relationship $u_j^+ = \sum_{k=1}^{M} T_{jk} u_k^-$ is explicitly computable. Matrix elements depend only on $\{\lambda_n\}$. For example ($M = 2$):

$$T = \frac{1}{\lambda_1^* - \lambda_2} \begin{bmatrix} \lambda_1^* - \lambda_2^* & \lambda_2^* - \lambda_2 \\ \lambda_1^* - \lambda_1 & \lambda_1 - \lambda_2 \end{bmatrix} .$$

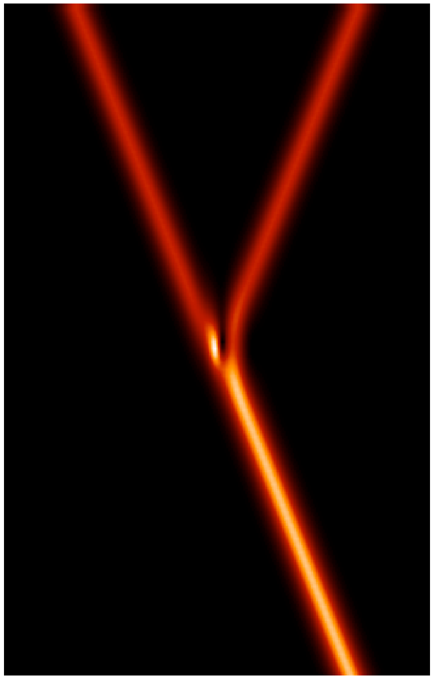
M. and Akhmediev (*Phys. Rev. E*, 1996)

Bound State Scattering for $\beta = 1$

A “Solitonic” 50%-50% Power Splitter



X-junction



linear mode

A Minus Sign: Zero-Crosstalk X-Junctions

Can also consider couplings where $\Delta(x,z) = -\sum_{n=1}^{N} |\psi_n|^2$ and

$$i\frac{\partial \psi_k}{\partial z} + \frac{1}{2}\frac{\partial^2 \psi_k}{\partial x^2} - \left(\sum_{n=1}^{N} |\psi_n|^2\right)\psi_k = 0\,.$$

The nonlinear properties of solutions of this *defocusing* equation are different, and lead to qualitatively different behavior of the waveguide $\Delta(x,z)$.

Absolutely zero "crosstalk" between intersecting waveguide channels. Useful in dense optical circuitry.

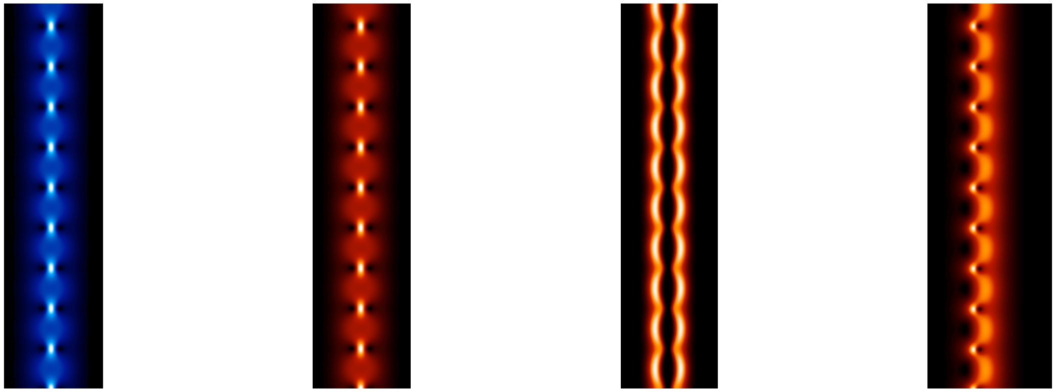
M. (*Phys. Rev. E*, 1996)

OSC = Optical Sciences Centre, Australian National University

Periodic Waveguiding Structures: $\beta = 1$

- Recall $\lambda_k = \sigma_k + i\rho_k$. If some of the σ_k are identical, the waveguide can have a periodic or quasiperiodic character.

- The bound states $\Psi_{\mathrm{b},k}(x,z)$ are exact independent Floquet solutions of a linear Schrödinger equation with z-periodic coefficients. The potential $\Delta(x,z)$ is like an isolated island in a sea of parametric resonances!

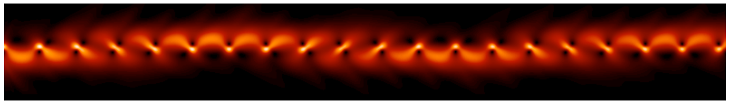


waveguide even mode odd mode superposition

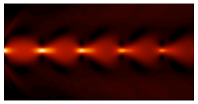
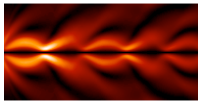
Periodic Waveguiding Structures: Perturbation Theory for $\beta \approx 1$

Modal decomposition provides an excellent starting point for perturbation theory. Frequency detuning: $\beta = 1 + \epsilon$ with $\epsilon \ll 1$.

- Modal beating is a first order effect in ϵ:



- Radiative decay is a second order effect in ϵ:

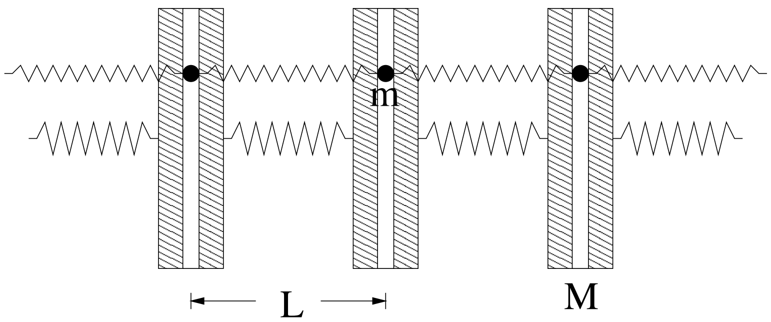
even mode:  odd mode: 

Besley, Akhmediev, and M. (*Opt. Lett.*, 1997), (*Stud. Appl. Math.*, 1998)

M., Soffer, and Weinstein (*Nonlinearity*, 2000)

Besley, M., and Akhmediev (*Phys. Rev. E*, 2000), (*Opt. Quantum Electron.*, 2001)

A Mechanical Model



$$H = H_{\text{kinetic}} + H_{\text{potential}}$$

$$H_{\text{kinetic}} = \sum_n \left[\frac{1}{2} M \dot{u}_n^2 + \frac{1}{2} m \dot{v}_n^2 \right]$$

$$H_{\text{potential}} = \sum_n W(L + u_{n+1} - u_n)$$

$$+ \sum_n V \left(\sqrt{(L + u_{n+1} - u_n)^2 + (v_{n+1} - v_n)^2} \right)$$

A Continuum Limit

- Scaling assumptions: $m = \mu M$ and V scales as $V = \mu U$ for $\mu \ll 1$.
- Small-amplitude long-wave ansatz: for $h \ll 1$, assume

$$u_n(t) = hu(X = hn, T = ht) \quad \text{and} \quad v_n(t) = hv(X = hn, T = ht)\,.$$

- Assume group velocity matching condition $LW''(L) = U'(L)$

$$\text{common velocity:} \quad c := \sqrt{W''(L)/M}\,.$$

- Change to traveling frame variables: $x = \sqrt{\tfrac{24}{c}}(X - cT)$ and $t = \sqrt{\tfrac{24}{c}}h^2T$
- Formal limit $h \downarrow 0$ with $\mu \ll h^2$:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}A^2 + \frac{\partial^2 A}{\partial x^2}\right] = 0 \quad \text{and} \quad \frac{\partial B}{\partial t} + \frac{\partial}{\partial x}\left[\kappa AB + \frac{\partial^2 B}{\partial x^2}\right] = 0\,,$$

$$A = \frac{W'''(L)}{Mc}\sqrt{\frac{6}{c}}\frac{\partial u}{\partial x} \quad \text{and} \quad B = \sqrt{\frac{24}{c}\frac{\partial v}{\partial x}} \quad \text{and} \quad \kappa := \frac{LU''(L) - U'(L)}{L^2W'''(L)}\,.$$

Integrable Cases

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}A^2 + \frac{\partial^2 A}{\partial x^2}\right] = 0 \quad \text{and} \quad \frac{\partial B}{\partial t} + \frac{\partial}{\partial x}\left[\kappa AB + \frac{\partial^2 B}{\partial x^2}\right] = 0$$

$A(x,t)$ satisfies the Korteweg-de Vries (KdV) equation.

- $\kappa = 1$. $B(x,t)$ satisfies the *linearized KdV equation*.
  1. Simplest nontrivial solution: $B(x,t) = \dfrac{\partial A}{\partial x}(x,t)$.
  2. Particular solutions in terms of "squared eigenfunctions".
  3. Completeness of squared eigenfunctions proved by R. L. Sachs (*SIAM J. Math. Anal.*, 1983).

- $\kappa = 1/2$. Equation for $B(x,t)$ is not a linearized KdV equation for any solution $A(x,t)$.
  1. Simplest nontrivial solution: $B(x,t) = A(x,t)$.
  2. Other facts to follow...

Parametric Instability of Co-propagating Waves

(General values of coupling κ)

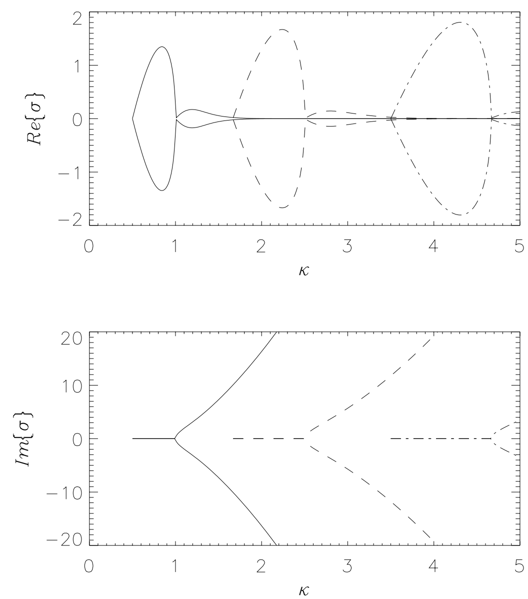
One-soliton solution for KdV:

$$A(x,t) = 12\eta^2 \text{sech}^2(\eta\chi)$$

where $\chi = x - 4\eta^2 t - \alpha$. Ansatz for $B(x,t)$:

$$B(x,t) = e^{\sigma\eta^3 t} b_\sigma(\eta\chi)$$

Leads to a third-order eigenvalue problem for $b_\sigma(\cdot)$ parametrized by κ.



Bifurcation points: $\kappa = \kappa_n = (n+1)(n+2)/12$.

Stable wave trapping appears possible only for $\kappa = 1/2$ and $\kappa = 1$.

Solution Formulas: $\kappa = 1/2$

- Lax pair:

$$12 f_{xx} = -3\lambda^2 f - 2A(x,t) f$$

$$6 f_t = A_x(x,t) f + (6\lambda^2 - 2A(x,t)) f_x$$

  $f = f(x,t,\lambda)$ (Lax eigenfunction) exists when $A(x,t)$ solves KdV.

- Pick $\lambda \in \mathbb{C}$. Define

$$B(x,t) := \frac{\partial}{\partial x}\left[f e^{\pm i(\lambda x + \lambda^3 t)/2} \right].$$

- Exact elementary solutions in terms of *linear forms* in the Lax eigenfunctions.

Algebraic Nature of N-Soliton Solutions

Kay and Moses (*J. Appl. Phys.*, 1956): consider

$$f(x,t,\lambda) = \left(1 + \sum_{n=0}^{N-1} \lambda^{n-N} f_n(x,t)\right) \exp\left(-\frac{i}{2}(\lambda x + \lambda^3 t)\right) .$$

Pick $\eta_1 > \eta_2 > \ldots > \eta_N > 0$ and $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$. Impose

$$f(x,t,2i\eta_n) = (-1)^{n+1} \exp(2\eta_n \alpha_n) f(x,t,-2i\eta_n) \, ,$$

for $n = 1, \ldots, N$. This determines $f_n(x,t)$ for all n.

$$\begin{aligned} A(x,t) \quad &:= \quad 6i \frac{\partial f_{N-1}}{\partial x}(x,t) \qquad \text{solves KdV} \\ &\sim \quad \sum_{n=1}^{N} 12\eta_n^2 \operatorname{sech}^2(\eta_n(x - \alpha_n^{\pm}) - 4\eta_n^3 t) \quad \text{as} \quad t \to \pm\infty \end{aligned}$$

$f_{\pm}(x,t,\lambda) := f(x,t,\pm\lambda)$ are linearly independent solutions of the Lax pair.

Completeness Relation: $\kappa = 1/2$

Special solutions of linear PDE corresponding to N-soliton $A(x,t)$:

$$h_\pm(x,t,\lambda) := \frac{\partial g_\pm}{\partial x}(x,t,\lambda) \quad \text{where} \quad g_\pm(x,t,\lambda) := f_\pm(x,t,\lambda)\exp\left(\frac{i}{2}(\lambda x + \lambda^3 t)\right).$$

Theorem 2 (M. and Clarke, SIAM J. Math. Anal., 2001) *Let $\phi(x) \in L^1(\mathbb{R})$ be absolutely continuous. Fix $t \in \mathbb{R}$ and $w \in \overline{\mathbb{R}}$. Define the "mode function":*

$$H(x,\lambda) := \lambda^N h_-(x,t,\lambda) \quad \textit{(entire function of } \lambda\textit{)},$$

"amplitudes": $$b^\pm(\lambda) := \pm\int_w^{\pm\infty} \frac{\lambda^N g_+(z,t,\lambda)\exp(-i(\lambda z + \lambda^3 t))}{\lambda(\lambda^2+4\eta_1^2)\cdots(\lambda^2+4\eta_N^2)}\phi(z)\,dz\,,$$

$$b(\lambda) := b^+(\lambda) + b^-(\lambda)\,, \quad b_0 := \frac{1}{2}\operatorname*{Res}_{\lambda=0}\,(b^+(\lambda) - b^-(\lambda))\,, \quad b_n^\pm := \mp \operatorname*{Res}_{\lambda=\pm 2i\eta_n} b^\mp(\lambda)\,.$$

Then: $$\phi(x) = \lim_{R\to\infty}\frac{1}{2\pi i}\text{P.V.}\int_{-R}^{R} b(\lambda)H(x,\lambda)\,d\lambda$$
$$+\, b_0 H(x,0) + \sum_{n=1}^{N}\left[b_n^- H(x,-2i\eta_n) + b_n^+ H(x,2i\eta_n)\right].$$

Remarks:

- Representation of arbitrary $\phi(x)$ in terms of a sum of discrete components ("bound states") and a singular integral over a "continuous spectrum".
- Only N independent bound states.
- Asymmetrical nature of the mapping between $\phi(x)$ and its expansion coefficients. Not just inner products.

Main ideas of proof:

1. $g_\pm(x,t,\lambda)$ satisfy an ODE in x: $-i\dfrac{\partial^2 g_\pm}{\partial x^2} - i\dfrac{A(x,t)}{6}g_\pm = \lambda\dfrac{\partial g_\pm}{\partial x}$.
2. Construct "resolvent" by variation of parameters and integrate on large semicircular contours in the λ-plane.
3. Directly prove convergence to the identity operator. Similarities to Fourier expansion apparent for large λ.
4. Exploit nifty residue identities to collapse contours to $\mathbb{R}$.

Solving the Initial Value Problem for $\kappa = 1/2$

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}A^2 + \frac{\partial^2 A}{\partial x^2}\right] = 0 \qquad \frac{\partial B}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}AB + \frac{\partial^2 B}{\partial x^2}\right] = 0$$

Take $A(x,t)$ to be an N-soliton solution of KdV. Solving for $B(x,t)$:

1. Project initial data $B(x,0)$ onto the modes $H(x,t,\lambda)$ using the expansion formulas.

2. Fix the expansion coefficients and let $H(x,t,\lambda)$ evolve explicitly in time.

3. Recover $B(x,t)$ for $t > 0$ by the completeness relation.

Bound State Scattering for $\kappa = 1/2$

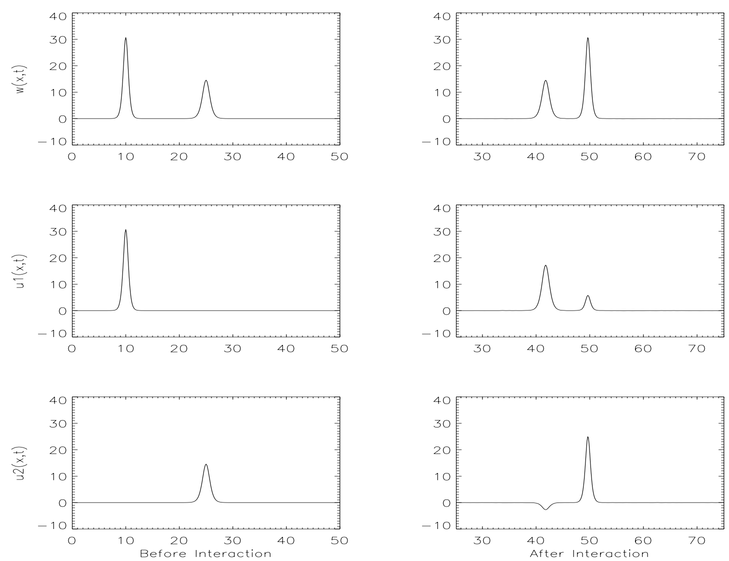
$$\begin{aligned} A(x,t) &\sim \sum_{n=1}^{N} 12\eta_n^2 \mathrm{sech}^2(\eta_n(x-\alpha_n^{\pm}) - 4\eta_n^3 t) \\ B(x,t) &\sim \sum_{n=1}^{N} 12\beta_n^{\pm}\eta_n^2 \mathrm{sech}^2(\eta_n(x-\alpha_n^{\pm}) - 4\eta_n^3 t) \end{aligned} \qquad \text{as} \quad t \to \pm\infty$$

for some constants $\beta_n^{\pm}$. Linear relationship: $\beta_j^{+} = \sum_{k=1}^{N} T_{jk}\beta_k^{-}$ is explicitly computable. Matrix elements depend only on $\{\eta_n\}$. For example ($N = 2$):

$$\mathbf{T} = \frac{1}{\eta_1^2 - \eta_2^2} \begin{bmatrix} (\eta_1-\eta_2)^2 & 2\eta_2(\eta_1-\eta_2) \\ 2\eta_1(\eta_1-\eta_2) & -(\eta_1-\eta_2)^2 \end{bmatrix}$$

Bound State Scattering for $\kappa = 1/2$

Effect of $T_{22} < 0$:



M. and Christiansen (*Physica Scripta*, 2000).

Conclusions

- Linear wave equations parametrically driven by solutions of nonlinear integrable equations arise in physical systems:
  - By fortune
  - By design
- Integrable structure can be exploited to provide general solutions to these linear equations in the form of generalized transforms.
- Waves can indeed be trapped by solitons, and their mechanics (asymptotics) explicitly calculated, including interactions among the trapping solitons.
- Integrable machinery is a useful starting point for perturbation theory.