What do water waves have to do with algebraic geometry?

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Abstract

This talk will discuss a remarkable and quite symmetrical interaction bwetween the applied mathematical subject of nonlinear wave motion and the pure mathematical subject of algebraic geometry. As an example, we will talk about how algebraic geometry can be used to generate solutions of nonlinear wave equations. Then we will discuss how nonlinear wave theory solves a long-standing problem of algebraic geometry. Other topics may also be discussed if there is time.

Outline

I. Modeling Water Waves
   A. Dimensionless Physical Model for Water Waves
   B. Weakly Nonlinear Long Waves: KdV and KP Approximations
   C. Real Physical Phenomena

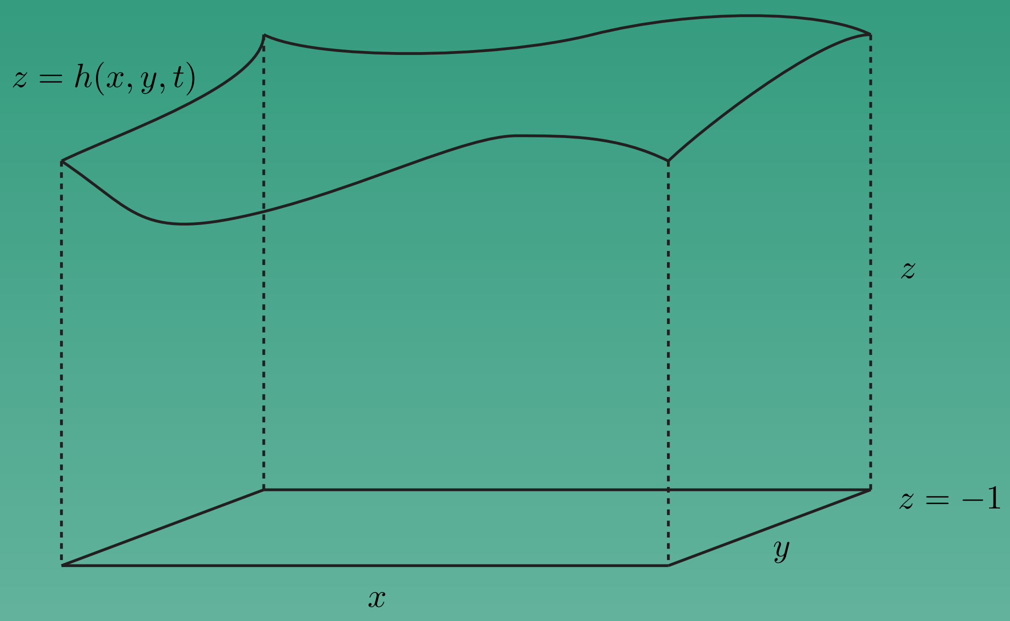
II. Solutions of KdV and KP
   A. Lax Pair for KdV
   B. Solitons in KdV
   C. Multiphase Waves for KP: Krichever Construction

III. The Schottky Problem and its Solution
   A. Riemann Matrices and the Schottky Problem
   B. The Novikov Conjecture
   C. Shiota's Theorem

IV. Segal-Wilson Theory

V. Conclusions

Modeling Water Waves: Physical Model



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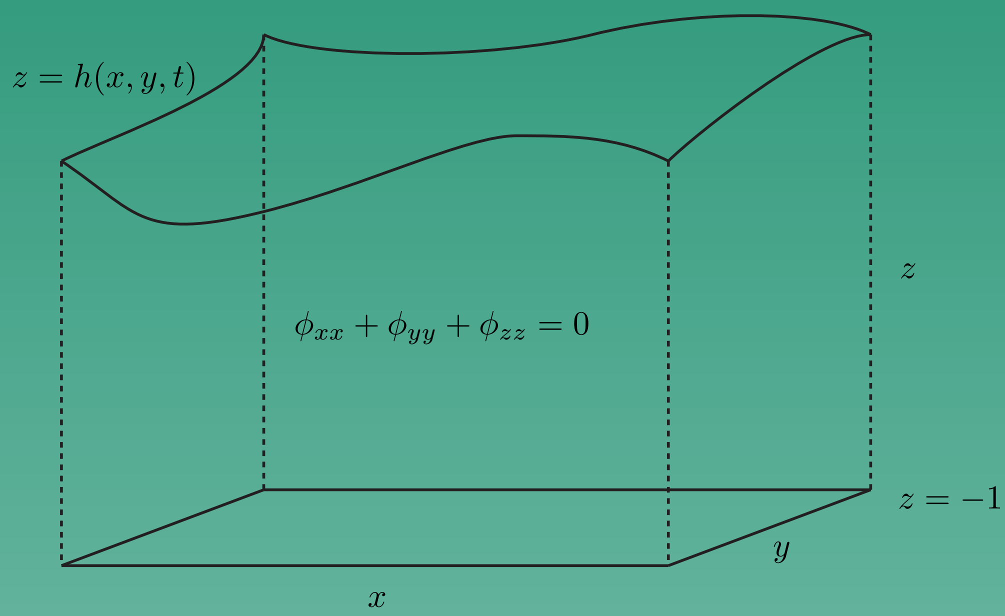

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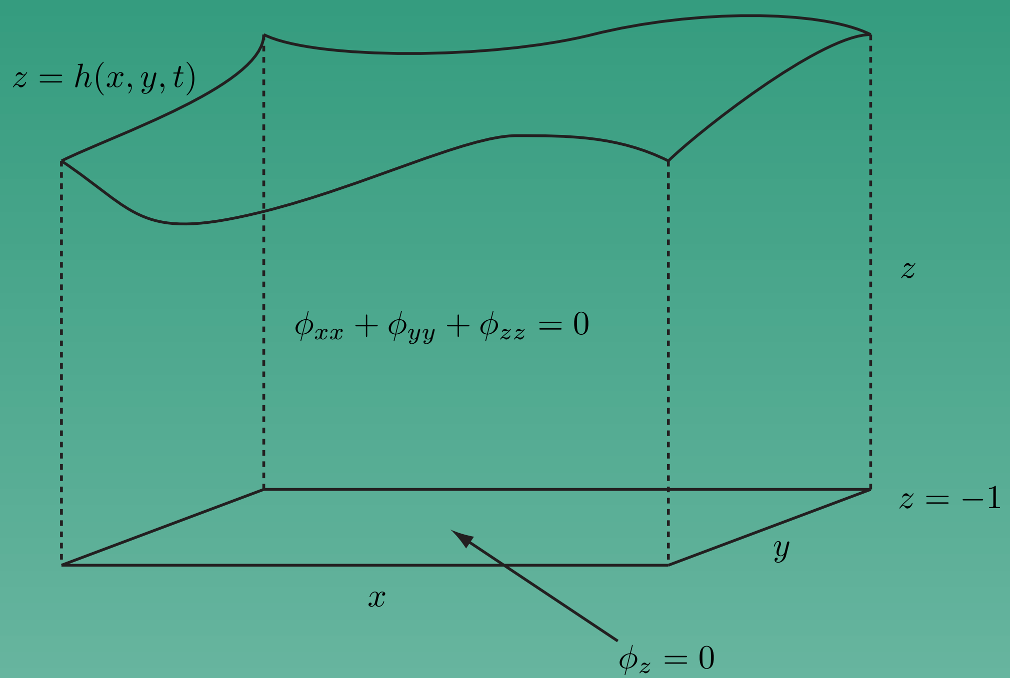

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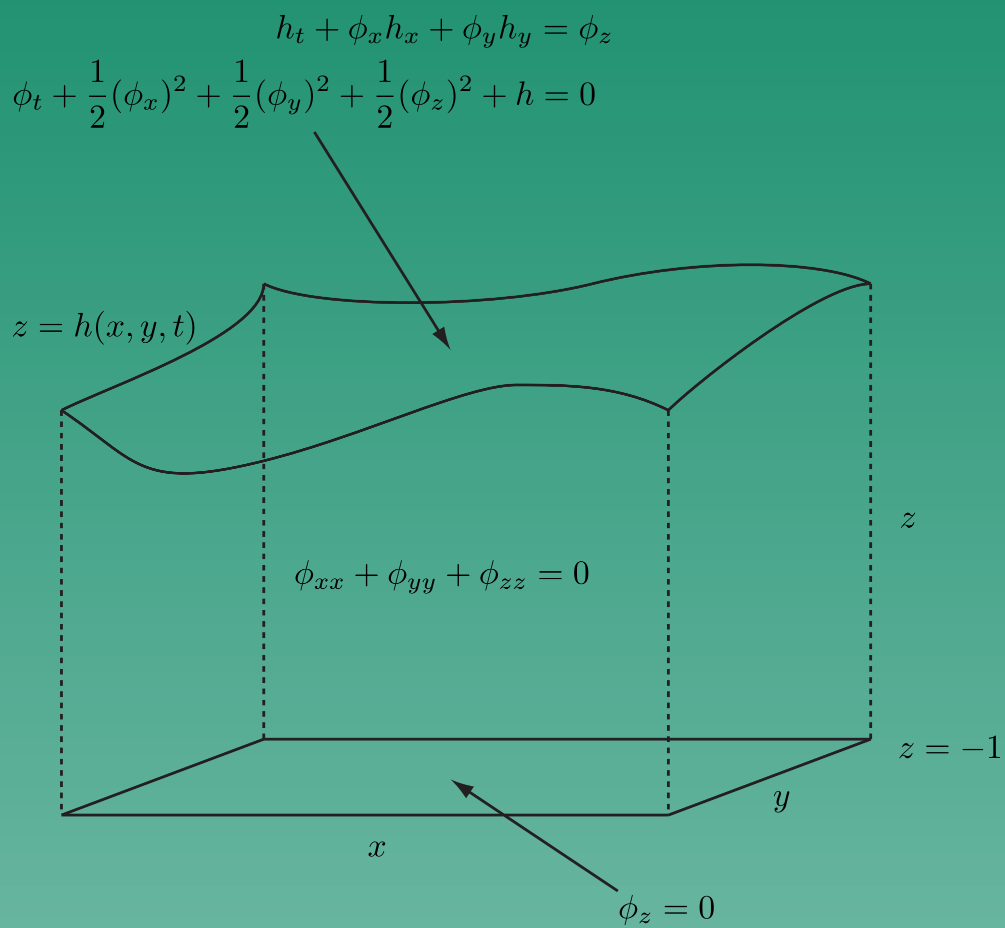

Modeling Water Waves: Physical Model

Cauchy-Kovaleskaya series solution of Laplace's equation subject to $\phi_z = 0$ for $z = -1$:

$$\phi(x, y, z, t) = \phi_0(x, y, t) + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \left[\Delta_{x,y}^k \phi_0(x, y, t) \right] (z+1)^{2k} .$$

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Here $\Delta_{x,y}$ denotes the two-dimensional (horizontal) Laplacian:

$$\Delta_{x,y} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

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Note that this series has a nonzero radius of convergence about $z = -1$ at (x, y) if $\phi_0(x, y, t)$ is an analytic function of x and y.

Modeling Water Waves: Weakly Nonlinear Long Waves

Key assumption for long waves: for some small dimensionless parameter $\epsilon > 0$ we introduce new independent variables X, Y, and T with respect to which one unit measures the size of a typical wave by

$$X := \epsilon^{1/2}x\,, \qquad Y := \epsilon^{1/2}y\,, \qquad T := \epsilon^{1/2}t\,.$$

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Considering $\phi_0 = w(X,Y,T)$, the series for ϕ gets a new interpretation as an asymptotic series in the limit $\epsilon \downarrow 0$:

$$\phi = w - \frac{\epsilon}{2}(z+1)^2\Delta w + \frac{\epsilon^2}{24}(z+1)^4\Delta^2 w + O(\epsilon^3)\,, \qquad \Delta := \Delta_{X,Y}\,.$$

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For consistency, scale the velocity potential with ϵ as well: $w = \epsilon^{1/2}N(X, Y, T)$.

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The two functions $G(X, Y, T)$ and $N(X, Y, T)$ are to be determined by imposing the kinematic and force-balance boundary conditions at $z = h = \epsilon G(X, Y, T)$.

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The kinematic boundary condition $h_t + \phi_x h_x + \phi_y h_y = \phi_z$ takes the form

$$G_T + \Delta N + \epsilon \left[\nabla G \cdot \nabla N + G \Delta N - \frac{1}{6} \Delta^2 N \right] = O(\epsilon^2) \, , \qquad \nabla := \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y} \right) \, .$$

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The force-balance boundary condition $\phi_t + \frac{1}{2}(\phi_x)^2 + \frac{1}{2}(\phi_y)^2 + \frac{1}{2}(\phi_z)^2 + h = 0$ reads

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Eliminating G (assuming derivatives of $O(\epsilon^2)$ are $O(\epsilon^2)$):

$$N_{TT} - \Delta N + \epsilon \left[\frac{1}{6}\Delta^2 N - \frac{1}{2}\Delta N_{TT} + 2\nabla N \cdot \nabla N_T + N_T \Delta N \right] = O(\epsilon^2)\,.$$

Modeling Water Waves: The KdV Equation

Korteweg and de Vries (1895) considered the case of waves in a thin channel, which means they sought solutions independent of Y. In this case, the equation for N becomes

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This looks like a perturbation of the wave equation $N_{TT} = N_{XX}$, which has solutions of the form of arbitrary functions of $X \pm T$. To examine the right-going waves (say), go into a moving frame of reference with the change of coordinates $\xi := X - T$ and $\tau := \epsilon T$. The equation for N then becomes

$$2 N_{\xi\tau} + \frac{1}{3} N_{\xi\xi\xi\xi} + 3 N_\xi N_{\xi\xi} = O(\epsilon) \, .$$

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Neglecting $O(\epsilon)$ and setting $F = N_\xi$ leads to the Korteweg-de Vries (KdV) equation:

$$F_\tau + \frac{3}{2} F F_\xi + \frac{1}{6} F_{\xi\xi\xi} = 0 \, .$$

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Modeling Water Waves: The KP Equation

Recall the general equation governing weakly nonlinear long waves:

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Kadomtsev and Petviashvili (1970) were interested in instabilities of the Y-independent waves that break this symmetry. To consider waves propagating "primarily in the positive X-direction" with weak dependence on Y, use the coordinate system $\xi := X - T$, $\eta := \epsilon^{1/2} Y$, $\tau := \epsilon T$.

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Neglecting $O(\epsilon)$, taking $\partial/\partial\xi$ and setting (as in the KdV case) $F = N_\xi$, one arrives at the Kadomtsev-Petviashvili (KP) equation:

$$-\frac{1}{2}F_{\eta\eta} = \left[F_\tau + \frac{3}{2}FF_\xi + \frac{1}{6}F_{\xi\xi\xi} \right]_\xi\,.$$

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Modeling Water Waves: Physical Phenomena

John Scott-Russell, "Report on Waves" to the British Association, 1844:

> I believe I shall best introduce the phaenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

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Scott-Russell’s “solitary wave” corresponds to a special solution (the soliton) of the KdV equation in the form $u_t + uu_x + u_{xxx} = 0$:

$$u(x,t) = 3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right), \qquad \text{wavespeed is } c > 0.$$

Modeling Water Waves: Physical Phenomena

Scott-Russell's wave re-created in the Scott-Russell Aqueduct of the Union Canal (Scotland):

Modeling Water Waves: Physical Phenomena

The KdV and KP equations are universal in applied mathematics, and their importance there extends far beyond the theory of surface water waves. Another application is in the modeling of internal waves in the atmosphere. The “Morning Glory” wave (Burketown, Queensland, Australia):

Return to outline.

Solutions of KdV and KP: Lax Pair for KdV

The hallmark of integrability of a nonlinear equation is its capacity to be represented as the compatibility condition for an overdetermined system of two linear equations (a Lax pair).

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

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The Lax pair for KdV (Gardner, Greene, Kruskal, and Miura, 1967) consists of the two linear equations

$$-6\phi_{xx} - u\phi = \lambda\phi \qquad \text{and} \qquad \phi_t = -4\phi_{xxx} - u\phi_x - \frac{1}{2}u_x\phi$$

where λ is a complex parameter (eigenvalue) and $u = u(x, t)$ is a nonconstant coefficient.

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where λ is a complex parameter (eigenvalue) and $u = u(x,t)$ is a nonconstant coefficient.

The condition for the compatibility of this Lax pair amounts to an equation governing $u = u(x,t)$, namely, the KdV equation in the form

$$u_t + uu_x + u_{xxx} = 0\,.$$

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Solutions of KdV and KP: Solitons in KdV

This means: whenever $u(x,t)$ satisfies $u_t + uu_x + u_{xxx} = 0$ there is a basis of functions ϕ that simultaneously satisfy both linear equations of the Lax pair.

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Example 1: the simplest solution of KdV is $u(x,t) \equiv 0$. In this case, by direct calculation, the simultaneous solution is

$$\phi(x,t,k) = e^{kx-4k^3t}, \qquad \text{where} \qquad k := \sqrt{-\frac{\lambda}{6}}.$$

A second solution, linearly independent for $k \neq 0$, is obtained by replacing k with $-k$.

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Example 2: another solution of KdV is the soliton $u(x,t) = 3c\,\mathrm{sech}^2(\xi)$ with $\xi := \sqrt{c}(x - x_0 - ct)/2$. Since $u \to 0$ for large x and t, it is reasonable to seek ϕ in the form $\phi = \psi e^{kx-4k^3t}$, and it is easy to solve for ψ:

$$\phi(x,t,k) = \left(1 - \frac{\sqrt{c}}{2k}\tanh\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right)\right) e^{kx-4k^3t}\,.$$

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This formula has an interesting property: the exponential factor is a part of the $\tanh$ whenever $k = \pm\sqrt{c}/2$. This implies a relation between the two functions $\phi(x,t,\pm\sqrt{c}/2)$:

$$\frac{\phi(x,t,-\sqrt{c}/2)}{\phi(x,t,\sqrt{c}/2)} = e^{-\sqrt{c}x_0} =: \gamma_1 = \text{constant (independent of } x \text{ and } t)\,.$$

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We can try to generalize these features in a straightforward manner. . .

Solutions of KdV and KP: Solitons in KdV

Perhaps for $N = 0, 1, 2, \ldots,$ there are solutions of KdV corresponding to ϕ of the form

$$\phi(x,t,k) = \left(1 + \phi_1^-(x,t)k^{-1} + \cdots + \phi_N^-(x,t)k^{-N}\right) e^{kx-4k^3t} \tag{1}$$

that, for some data $0 < \kappa_1 < \cdots < \kappa_N$ and $\{\gamma_n > 0\}_{n=1}^N$ satisfy relations of the form

$$\phi(x,t,-\kappa_n) = (-1)^{N-n}\gamma_n\phi(x,t,\kappa_n)\,, \qquad n = 1, \ldots, N\,. \tag{2}$$

That this works is the consequence of some simple facts.

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That this works is the consequence of some simple facts.

Proposition 1. *Fix $(x,t) \in \mathbb{R}^2$. The set Λ of functions $f(k)$ of a complex variable k having the form*

$$f(k) = \left(f_0 + f_1^- k^{-1} + \cdots + f_N^- k^{-N}\right) e^{kx-4k^3t}$$

is a vector space of dimension $N+1$ over $\mathbb{C}$. Given a set of data as above, the subspace Λ_0 of functions obeying (2) satisfies $\dim(\Lambda_0) = 1$. In particular, if $f \in \Lambda_0$ is normalized by $f_0 = 1$, then it is uniquely determined, and if $f \in \Lambda_0$ satisfies $f_0 = 0$, then $f(k) \equiv 0$.

Solutions of KdV and KP: Solitons in KdV

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Proposition 2. *For each set of data, the function* $\phi(x, t, k)$ *satisfies*

$$-6\phi_{xx} - u\phi = \lambda\phi$$

where $\lambda = -6k^2$*, and where the potential function* $u(x, t)$ *is given in terms of* ϕ *by*

$$u(x, t) = -12\frac{\partial \phi_1^-}{\partial x}\,.$$

Solutions of KdV and KP: Solitons in KdV

This means that a function of the form (1) is uniquely determined by the conditions (2). Now we claim that this function satisfies some linear differential equations.

Proposition 2. *For each set of data, the function $\phi(x,t,k)$ satisfies*

$$-6\phi_{xx} - u\phi = \lambda\phi$$

where $\lambda = -6k^2$, and where the potential function $u(x,t)$ is given in terms of ϕ by

$$u(x,t) = -12\frac{\partial\phi_1^-}{\partial x}\,.$$

Proof. The function $w(x,t,k) := 6\phi_{xx} + u\phi + \lambda\phi$ lies in the subspace Λ_0 associated with its data; indeed, by direct calculation, w has the form

$$w = \left(w_0 + \cdots + w_N^- k^{-N}\right) e^{kx-4k^3t}$$

and it satisfies the relations (2) because they are linear and independent of x. Moreover, the same calculation shows that $w_0 = 12\phi_{1,x} + u$, so by the choice of u we have $w_0 = 0$ and hence $w \equiv 0$. □

Solutions of KdV and KP: Solitons in KdV

Proposition 3. *For each set of data, the function $\phi(x,t,k)$ satisfies*

$$\phi_t = -4\phi_{xxx} - u\phi_x - \frac{1}{2}u_x\phi$$

where again the potential function $u(x,t)$ is given in terms of ϕ by

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Proof. The proof is similar. The function $z(x,t,k) := \phi_t + 4\phi_{xxx} + u\phi_x + \frac{1}{2}u_x\phi$ is shown to have the form

$$z = \left(z_0 + \cdots + z_N^- k^{-N}\right) e^{kx-4k^3t}$$

and satisfy the conditions (2) and therefore lies in the subspace Λ_0. One then checks that $z_0 = 0$, which implies that $z \equiv 0$. □

Solutions of KdV and KP: Solitons in KdV

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These results imply that the function $u(x,t)$ built from the data $0 < \kappa_1 < \cdots < \kappa_N$ and $\{\gamma_n > 0\}_{n=1}^N$ makes the Lax pair compatible. That is, $u(x,t)$ solves KdV.

Solutions of KdV and KP: Solitons in KdV

Finding the coefficient ϕ_1^- given the data amounts to solving a system of N linear equations in N unknowns. Playing with Cramer's rule yields the Kay-Moses formula:

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$$u(x,t) = -12\frac{\partial \phi_1^-}{\partial x} = 12\frac{\partial^2}{\partial x^2}\log(\tau)\,, \quad \text{where} \quad \tau := \det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right)\,.$$

(The $\{x_n\}$ are related to the $\{\gamma_n\}$.)

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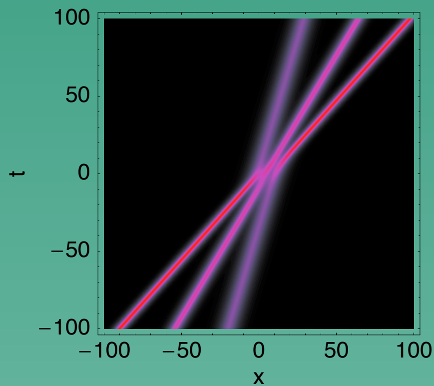
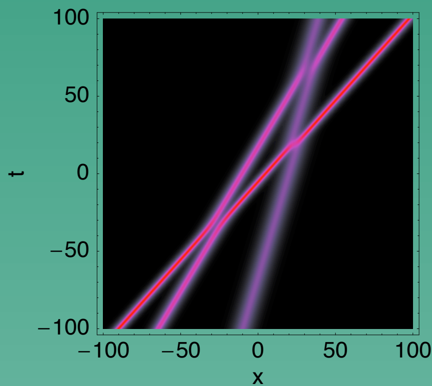
(The $\{x_n\}$ are related to the $\{\gamma_n\}$.) *E.g.* $N = 3$ with $\kappa_1 = 0.274$, $\kappa_2 = 0.387$, $\kappa_3 = 0.474$:



$x_1 = x_2 = x_3 = 0$



$x_1 = 10$, $x_2 = -10$, $x_3 = 0$

Solutions of KdV and KP: Krichever Construction

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$\phi = e^{kx - 4k^3 t}$ where $k^2 = -\lambda/6$.

Solutions of KdV and KP: Krichever Construction

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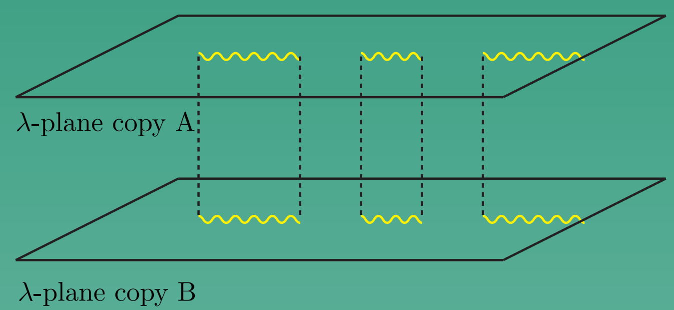

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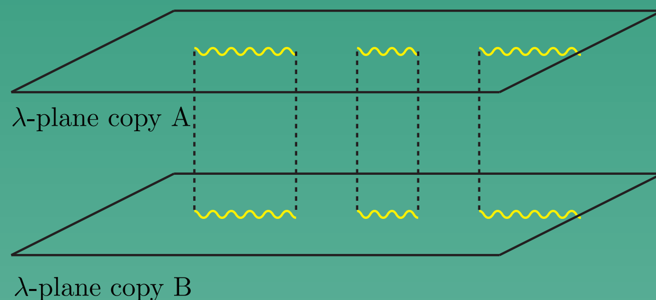

Note that $k = \lambda = \infty$ is a single point $P_\infty \in \Gamma$, and the function $z = k^{-1}$ is a holomorphic local coordinate in a neighborhood of this point.

Solutions of KdV and KP: Krichever Construction

It is easiest to explain the theory of solutions parametrized by Riemann surfaces by further generalizing from the KdV context to the KP context. Krichever proposed the following construction of solutions of KP.

I. M. Krichever, *Russ. Math. Surveys*, **32**, 185–213, 1977.

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2. A point $P_\infty \in \Gamma$ and a local coordinate $z = k^{-1}$ in a neighborhood U_∞ of P_∞ with $z(P_\infty) = 0$.

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2. A point $P_\infty \in \Gamma$ and a local coordinate $z = k^{-1}$ in a neighborhood U_∞ of P_∞ with $z(P_\infty) = 0$.
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A *Baker-Akhiezer function* associated with this data is a function $\phi : \Gamma \longrightarrow \mathbb{C}$ that is meromorphic on $\Gamma \setminus U_\infty$ where it satisfies $(\phi) + \mathcal{D} \geq 0$ and that has a representation in U_∞ in the form of a convergent series:

$$\phi = \left(1 + \phi_1^-(x, y, t)k^{-1} + \phi_2^-(x, y, t)k^{-2} + \cdots\right) e^{kx+2ik^2y-4k^3t} .$$

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Solutions of KdV and KP: Krichever Construction

The analogue of Proposition 1 in this case is

Proposition 4. *For generic* $(x, y, t) \in \mathbb{C}^3$*, the space* Λ_0 *of functions* $f : \Gamma \to \mathbb{C}$ *meromorphic on* $\Gamma \setminus U_\infty$ *satisfying* $(f) + \mathcal{D} \geq 0$ *and having a representation in* U_∞ *of the form*

$$f = \left(f_0 + f_1^- k^{-1} + f_2^- k^{-2} + \cdots\right) e^{kx+2ik^2y-4k^3t}, \quad k^{-1} = z(P)$$

has dimension $\dim(\Lambda_0) = 1$*. Hence if* $f \in \Lambda_0$ *and* $f_0 = 1$ *then* f *is uniquely determined, and if* $f \in \Lambda_0$ *and* $f_0 = 0$ *then* $f \equiv 0$*.*

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The proof that $\dim(\Lambda_0) \leq 1$ follows from the Riemann-Roch Theorem. The proof that $\dim(\Lambda_0) > 0$ (that is, there is a nontrivial element of Λ_0) is by direct construction.

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Before giving Krichever's formula for the (unique) Baker-Akhiezer function ϕ, we examine some further consequences of $\dim(\Lambda_0) = 1$.

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Solutions of KdV and KP: Krichever Construction

The Baker-Akhiezer function satisfies some linear differential equations.

Proposition 5. *Define*

$$u(x,y,t) := -12\frac{\partial \phi_1^-}{\partial x} \qquad \text{and} \qquad v(x,y,t) := 12\frac{\partial \phi_2^-}{\partial x} + 12\frac{\partial^2 \phi_1^-}{\partial x^2} - 12\phi_1^-\frac{\partial \phi_1^-}{\partial x}$$

where $\phi_1^-(x,y,t)$ and $\phi_2^-(x,y,t)$ are the first two expansion coefficients in the Baker-Akhiezer function $\phi(x,y,t,P)$ associated with a set of algebro-geometric data. Then for each $P \in \Gamma$, the Baker-Akhiezer function satisfies

$$3i\phi_y + 6\phi_{xx} + u\phi = 0 \qquad \text{and} \qquad \phi_t + 4\phi_{xxx} + u\phi_x + v\phi = 0\,.$$

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where $\phi_1^-(x,y,t)$ and $\phi_2^-(x,y,t)$ are the first two expansion coefficients in the Baker-Akhiezer function $\phi(x,y,t,P)$ associated with a set of algebro-geometric data. Then for each $P \in \Gamma$, the Baker-Akhiezer function satisfies

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Proof. The functions $f(P) := 3i\phi_y + 6\phi_{xx} + u\phi$ and $g(P) := \phi_t + 4\phi_{xxx} + u\phi_x + v\phi$ are meromorphic on $\Gamma \setminus U_\infty$ with poles in $\mathcal{D}$ because this was true of ϕ for all $(x,y,t) \in \mathbb{C}^3$. Moreover, by direct differentiation of the expansion of ϕ for $P \in U_\infty$, one easily checks that

$$f(P) = O(k^{-1})e^{kx+2ik^2y-4k^3t} \qquad \text{and} \qquad g(P) = O(k^{-1})e^{kx+2ik^2y-4k^3t}$$

by choice of u and v in terms of the $\{\phi_j^-\}$. By Proposition 4 it then follows that f and g are both the zero element of Λ_0. □

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Solutions of KdV and KP: Krichever Construction

This result implies a kind of compatibility of the two linear problems satisfied by ϕ. Indeed, ϕ is in the kernel of the commutator:

$$[L_3, L_2]\,\phi := \left[\partial_t + 4\partial_x^3 + u\partial_x + v, 3i\partial_y + 6\partial_x^2 + u\right]\phi = 0\,, \qquad P \in \Gamma\,.$$

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And, by direct calculation, the commutator $[L_3, L_2]$ is a first-order operator:

$$[L_3, L_2] = (6u_{xx} - 12v_x - 3iu_y)\,\partial_x + (u_t + uu_x + 4u_{xxx} - 6v_{xx} - 3iv_y)\ .$$

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As P varies in Γ, the function ϕ spans a linear space of dimension greater than one, and hence we must have $[L_3, L_2] = 0$. This amounts to two nonlinear equations for u and v:

$$6u_{xx} - 12v_x - 3iu_y = 0 \qquad \text{and} \qquad u_t + uu_x + 4u_{xxx} - 6v_{xx} - 3iv_y = 0\,.$$

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Eliminating v between these two gives the KP equation in the form

$$(u_t + uu_x + u_{xxx})_x = \tfrac{3}{4}u_{yy}\,.$$

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Solutions of KdV and KP: KP Hierarchy

This whole construction generalizes to more than three independent variables:

Simply replace $e^{kx+2ik^2y-4k^3t}$ with $\exp\left(\sum_{n=1}^{M} t_n k^n\right)$, $M < \infty$ arbitrary.

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Then (as in Proposition 5) there is a linear differential operator in $x = t_1$ of order n:

$$B_n := \partial_x^n + \sum_{k=0}^{n-2} b_{n,k}[\{\phi_j^-\}]\partial_x^k\,, \quad n \geq 2$$

such that the Baker-Akhiezer function ϕ is, for all $P \in \Gamma$, in the kernel of $\partial_{t_n} - B_n$.

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The equations $\partial_{t_n}\phi = B_n\phi$ satisfied by the Baker-Akhiezer function are called the *linear equations of the KP hierarchy*. They are compatible because the kernel of the commutator

$$[L_j, L_k] := [\partial_{t_j} - B_j, \partial_{t_k} - B_k] = (\partial_{t_k} B_j) - (\partial_{t_j} B_k) + [B_j, B_k]$$

contains enough functions ϕ (parametrized by $P \in \Gamma$) to make $[L_j, L_k] = 0$. These are nonlinear equations on the coefficients $\{b_{n,k}\}$ that may also be simultaneously solved, and they are said to be the *nonlinear equations of the KP hierarchy*.

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Solutions of KdV and KP: Krichever Construction

To give Krichever's formula for the Baker-Akhiezer function, we need some ingredients.

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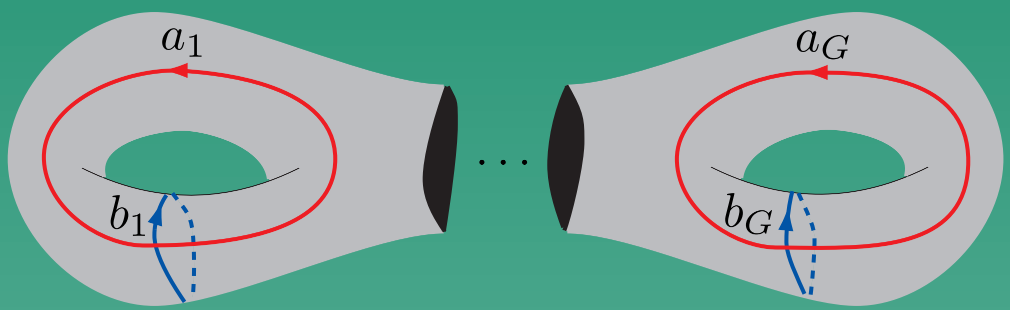

- A basis $\mathcal{H}$ of homology cycles $a_1, \ldots, a_G$ and $b_1, \ldots, b_G$ on Γ.

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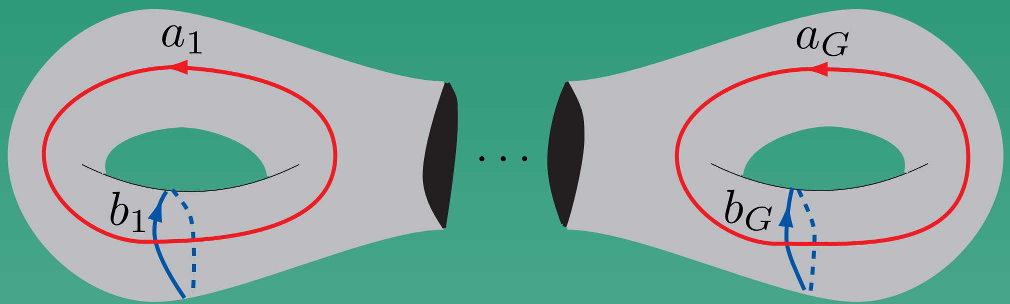

- A basis $\mathcal{H}$ of homology cycles $a_1, \ldots, a_G$ and $b_1, \ldots, b_G$ on Γ.
- The basis $\omega_1, \ldots, \omega_G$ of holomorphic differentials on Γ normalized with respect to $\mathcal{H}$, and the coincident Riemann matrix $\mathbf{B}$:

$$\oint_{a_j} \omega_k = 2\pi i \delta_{j,k}\,, \quad B_{j,k} := \oint_{b_j} \omega_k = \oint_{b_k} \omega_j\,. \quad \text{(Note } \mathbf{v}^\dagger \Re\{\mathbf{B}\}\mathbf{v} < 0 \text{ for } \mathbf{v} \neq 0.\text{)}$$

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Solutions of KdV and KP: Krichever Construction

- A base point $P_0 \in \Gamma$, the Abel mapping $A : \Gamma \longrightarrow \mathrm{Jac}(\Gamma)$ given by

$$A(P) = \left[\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_G \right]^T,$$

and its linear extension to divisors.

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  and its linear extension to divisors.
- The vector $\mathbf{k}$ of Riemann constants associated with Γ.
- The theta function (multiple Fourier series) with matrix $\mathbf{B}$

$$\Theta(\mathbf{w}; \mathbf{B}) := \sum_{\mathbf{n} \in \mathbb{Z}^G} e^{\frac{1}{2}\mathbf{n}^T \mathbf{B} \mathbf{n}} e^{\mathbf{n}^T \mathbf{w}} .$$

  Note for $\mathbf{e}_k$ a unit vector and $\mathbf{b}_k$ a column of $\mathbf{B}$ the automorphic relations:

$$\Theta(\mathbf{w} + 2\pi i \mathbf{e}_k; \mathbf{B}) = \Theta(\mathbf{w}; \mathbf{B}) \quad \text{and} \quad \Theta(\mathbf{w} + \mathbf{b}_k; \mathbf{B}) = e^{-\frac{1}{2}B_{kk}} e^{-w_k} \Theta(\mathbf{w}; \mathbf{B}) .$$

I. M. Krichever, *Russ. Math. Surveys*, **32**, 185–213, 1977.

Solutions of KdV and KP: Krichever Construction

- The differentials Ω_n holomorphic on $\Gamma \setminus \{P_\infty\}$ with principal parts

$$\Omega_n = \left[n k^{n-1} + O(k^{-2}) \right] dk \,, \qquad n = 1, 2, 3, \ldots \,,$$

(here $z(P) = k(P)^{-1}$ is the local parameter in U_∞) made unique by the normalization

$$\oint_{a_j} \Omega_n = 0 \quad \text{for} \quad j = 1, \ldots, G \,.$$

Finally define the corresponding vectors $\mathbf{u}_n \in \mathbb{C}^G$ by

$$\mathbf{u}_n := \left[\oint_{b_1} \Omega_n, \ldots, \oint_{b_G} \Omega_n \right]^T \,.$$

I. M. Krichever, *Russ. Math. Surveys*, **32**, 185–213, 1977.

Solutions of KdV and KP: Krichever Construction

Then, Krichever's formula for the Baker-Akhiezer function is

$$\phi(P) = c\frac{\Theta(A(P) - A(\mathcal{D}) - \mathbf{k} + \sum_{n=1}^{M} t_n \mathbf{u}_n; \mathbf{B})}{\Theta(A(P) - \mathcal{D} - \mathbf{k}; \mathbf{B})} \exp\left(\sum_{n=1}^{M} t_n \int_{P_0}^{P} \Omega_n\right) .$$

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Expanding this formula near P_∞ we obtain the Its-Matveev formula ($x = t_1$):

$$u(t_1, \ldots, t_M) = -12\partial_x \phi_1 = u_0 + 12\partial_x^2 \log \Theta(A(P_\infty) - A(\mathcal{D}) - \mathbf{k} + \textstyle\sum_{n=1}^{M} t_n \mathbf{u}_n; \mathbf{B}) .$$

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Note: u is a quasiperiodic "multiphase wave".

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The Schottky Problem

One might notice that the Its-Matveev formula for multiphase wave solutions of KP

$$u(x,y,t) = u_0 + 12\partial_x^2 \log \Theta(x\mathbf{u} + y\mathbf{v} + t\mathbf{w} + \mathbf{z}; \mathbf{B})$$

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Krichever's construction shows that the Its-Matveev formula solves the KP equation when these parameters are associated with a Riemann surface Γ. Perhaps it works more generally?

The Schottky Problem: Novikov's Conjecture

The Schottky problem is a classical problem of algebraic geometry. The problem is: characterize the Jacobian locus in the moduli space of principally polarized abelian varieties. In other words, which $G \times G$ matrices $\mathbf{B}$ in the Siegel upper half-space are period matrices of Riemann surfaces of genus G?

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This problem is related to the question we just posed about the Its-Matveev formula, that is, whether this formula can represent a solution of the KP equation regardless of whether the matrix $\mathbf{B}$ comes from a Riemann surface. Indeed, if the answer is negative, then the KP equation itself provides the acid test that solves the Schottky problem.

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The conjecture that the Its-Matveev formula in fact solves the Schottky problem was first formulated by S. P. Novikov.

The Schottky Problem: Shiota's Theorem

In fact, Novikov's conjecture is true! The KP equation determines whether or not a period matrix $\mathbf{B}$ comes from a Riemann surface; the Its-Matveev formula only solves KP if $\mathbf{B}$ is a point in the Jacobian locus. This was proved by Shiota in 1986.

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T. Shiota, *Invent. Math.*, **83**, 333-382, 1986.

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Theorem 1 (Shiota). *The following two conditions (A) and (B) for a principally polarized abelian variety X associated with a point $\mathbf{B}$ in the Siegel upper half-space are equivalent:*

(A) There exist vectors $\mathbf{u} \neq 0$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{C}^G$, and a constant u_0 such that for any vector $\mathbf{z} \in \mathbb{C}^G$ the Its-Matveev formula

$$u = u_0 + 12\partial_x^2 \log \Theta(x\mathbf{u} + y\mathbf{v} + t\mathbf{w} + \mathbf{z}; \mathbf{B})$$

satisfies the KP equation $(u_t + uu_x + u_{xxx})_x = \frac{3}{4}u_{yy}$, and the theta divisor of X is irreducible.
(B) X is isomorphic to the Jacobian variety of a complete smooth curve of genus G over $\mathbb{C}$.

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> *(B) X is isomorphic to the Jacobian variety of a complete smooth curve of genus G over $\mathbb{C}$.*

Of note is that only the KP equation is required, not the whole KP hierarchy. Thus, water wave theory gives something back to pure mathematics.

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T. Shiota, *Invent. Math.*, **83**, 333-382, 1986.

Segal-Wilson Theory

Observe:

G. Segal and G. Wilson, *Publ. Math. IHES*, **63**, 1–64, 1985.

Segal-Wilson Theory

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- For N-soliton solutions of KdV, the function ϕ (and hence the corresponding solution of KdV via the coefficient ϕ_1^-) was specified by the discrete data $0 < \kappa_1 < \cdots < \kappa_N$ and $\{\gamma_n > 0\}_{n=1}^N$.

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- For multiphase wave solutions of KP, the function ϕ (and hence the corresponding solution of KP via the coefficient ϕ_1^-) was specified by the algebro-geometric data Γ, $P_\infty \in \Gamma$, $k(P)$, and $\mathcal{D}$.

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In 1985, Segal and Wilson proposed a theory in which both types of solutions are put into a common framework. Both types of data correspond to points in an infinite-dimensional Grassmannian.

G. Segal and G. Wilson, *Publ. Math. IHES*, **63**, 1–64, 1985.

Segal-Wilson Theory

Let $H := L^2(S^1)$ be the Hilbert space of square integrable functions on the unit circle in the k-plane. There is a natural orthogonal decomposition $H = H_+ \oplus H_-$ where

$$H_+ := \left\{ f \in H\,, \quad f = f_0 + f_1^+ k + f_2^+ k^2 + \cdots \right\}$$

is the Hardy space of functions in H that are boundary values of functions analytic for $|k| < 1$, and

$$H_- := \left\{ f \in H\,, \quad f = f_1^- k^{-1} + f_2^- k^{-2} + \cdots \right\}$$

is the Hardy space of functions in H that are boundary values of functions analytic for $|k| > 1$ that vanish as $k \to \infty$.

G. Segal and G. Wilson, *Publ. Math. IHES*, **63**, 1–64, 1985.

Segal-Wilson Theory

The orthogonal projection onto H_+, $\pi_+ : H \longrightarrow H_+$, is defined by:

$$\pi_+ : \sum_{n=1}^{\infty} f_n^- k^{-n} + f_0 + \sum_{n=1}^{\infty} f_n^+ k^n \mapsto f_0 + \sum_{n=1}^{\infty} f_n^+ k^n \, .$$

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An infinite-dimensional Grassmannian $\mathrm{Gr}H$ is then defined as follows:

$$\mathrm{Gr}H = \left\{ \text{subspaces } W \subset H \text{ for which } \pi_+\big|_W : W \longrightarrow H_+ \text{ is a Fredholm operator} \right\} \,.$$

Important subsets are $T \subset \mathrm{Gr}_0 H \subset \mathrm{Gr}H$:

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Important subsets are $T \subset \mathrm{Gr}_0 H \subset \mathrm{Gr}H$:

- $\mathrm{Gr}_0 H$ consists of those subspaces W for which the restricted projection operator $\pi_+\big|_W$ has index zero.
- T consists of those subspaces W for which the restricted projection operator $\pi_+\big|_W$ is a bijection. Such subspaces are called *transversal*. If W is transversal and $f \in W$, then $(1 - \pi_+) f$ is a function of $\pi_+ f$.

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Segal-Wilson Theory

Note that if $\mathbf{t} := \{t_1, t_2, t_3, \ldots\}$ is a sufficiently rapidly decaying sequence of complex numbers, then the function

$$\exp\left(\sum_{n=1}^{\infty} t_n k^n\right) = 1 + g_1^+(\mathbf{t})k + g_2^+(\mathbf{t})k^2 + \cdots$$

is an example of a function in the subspace H_+.

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is an example of a function in the subspace H_+. Let $W \in \mathrm{Gr}H$. A family of functions $\phi(\mathbf{t}) \in H$ parametrized by $\mathbf{t}$ is said to be a *Baker function* associated with W if $\phi(\mathbf{t}) \in W$ and

$$\exp\left(-\sum_{n=1}^{\infty} t_n k^n\right)\phi(k;\mathbf{t}) = 1 + \phi_1^-(\mathbf{t})k^{-1} + \phi_2^-(\mathbf{t})k^{-2} + \cdots \in 1 + H_- .$$

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Segal-Wilson Theory

Theorem 2. *Let* $W \in \mathrm{Gr}H$*. If* $\exp(-\sum_{n=1}^{\infty} t_n k^n)W$ *is transversal for* $\mathbf{t}$ *in some open set, then for each such* $\mathbf{t}$ *there is a unique Baker function* $\phi = \phi_W(k;\mathbf{t})$*, and furthermore* ϕ *satisfies all the linear equations of the KP hierarchy.*

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Proof. Since $\exp(-\sum_{n=1}^{\infty} t_n k^n)W$ is transversal, the series $1 + \phi_1^-(\mathbf{t})k^{-1} + \phi_2^-(\mathbf{t})k^{-2} + \cdots$ is obtained uniquely by pulling back the function 1 via π_+, which proves existence and uniqueness of ϕ.

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Proof. Since $\exp(-\sum_{n=1}^{\infty} t_n k^n)W$ is transversal, the series $1 + \phi_1^-(\mathbf{t})k^{-1} + \phi_2^-(\mathbf{t})k^{-2} + \cdots$ is obtained uniquely by pulling back the function 1 via π_+, which proves existence and uniqueness of ϕ. Next, since ϕ lies in the fixed space W for all $\mathbf{t}$ in some open set, so does $(\partial_{t_j} - B_j)\phi$ as this only involves linear operations among ϕ for different $\mathbf{t}$. Moreover, by definition of the differential operators B_j, we have

$$\exp\left(-\sum_{n=1}^{\infty} t_n k^n\right) \cdot \left(\partial_{t_j} - B_j\right)\phi = O(k^{-1})$$

which vanishes identically by transversality again. $\square$

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Segal-Wilson Theory

Example 1: Let $0 < |\kappa| < 1$ and $0 < |\kappa'| < 1$, and let $\gamma \in \mathbb{C}$. Consider the subspace $W_{\kappa,\kappa',\gamma} \in \mathrm{Gr}H$ given by

$$W_{\kappa,\kappa',\gamma} := \left\{ f(k) = f_1^- k^{-1} + f_0 + f_1^+ k + f_2^+ k^2 + \cdots , \quad f(\kappa') = \gamma f(\kappa) \right\} .$$

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This subspace is transversal if $\gamma \neq \kappa/\kappa'$. Indeed, from the relation $f(\kappa') = \gamma f(\kappa)$,

$$f_1^- = \frac{\kappa\kappa'}{\kappa - \gamma\kappa'} \left[(\gamma - 1) f_0 + (\gamma\kappa - \kappa') f_1^+ + (\gamma\kappa^2 - \kappa'^2) f_2^+ + \cdots \right] ,$$

so if $\gamma \neq \kappa/\kappa'$ we can solve for f_1^- in terms of $\pi_+ f$.

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so if $\gamma \neq \kappa/\kappa'$ we can solve for f_1^- in terms of $\pi_+ f$. Also, by direct calculation,

$$\exp\left(-\sum_{n=1}^{\infty} t_n k^n \right) W_{\kappa,\kappa',\gamma} = W_{\kappa,\kappa',\gamma(\mathbf{t})} \,, \qquad \gamma(\mathbf{t}) := \gamma \exp\left(\sum_{n=1}^{\infty} t_n (\kappa^n - \kappa'^n) \right) .$$

G. Segal and G. Wilson, *Publ. Math. IHES*, **63**, 1–64, 1985.

Segal-Wilson Theory

To find the Baker function associated with $W_{\kappa,\kappa',\gamma}$ for $\gamma \neq \kappa/\kappa'$, note that the series $1 + \phi_1^-(\mathbf{t})k^{-1} + \phi_2^-(\mathbf{t})k^{-2} + \cdots$ lies by definition in $\exp(-\sum_{n=1}^{\infty} t_n k^n)W_{\kappa,\kappa',\gamma}$, a subspace we have shown to be equal to $W_{\kappa,\kappa',\gamma(\mathbf{t})}$, so

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The Baker function for $W_{\kappa,\kappa',\gamma}$ is thus determined uniquely, and in particular, the function $u := -12\partial_{t_1}\phi_1^-(\mathbf{t})$ satisfies the KP equation.

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This is the soliton solution of KP.

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Segal-Wilson Theory

In the Segal-Wilson theory, the KdV hierarchy arises as a special case by considering only subspaces $W \in \mathrm{Gr} H$ that satisfy $k^2 W \subset W$. This restriction on $W_{\kappa,\kappa',\gamma}$ forces $\kappa' = -\kappa$. The dependence on the even times t_{2j} then disappears from $\gamma(\mathbf{t})$:

$$\gamma(\mathbf{t}) = \gamma \exp\left(\sum_{j=0}^{\infty} 2t_{2j+1}\kappa^{2j+1}\right) .$$

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For $c > 0$ and $x_0 \in \mathbb{R}$, write $\kappa = \frac{\sqrt{c}}{2}$, $\gamma = e^{-\sqrt{c}x_0}$, $t_1 = x$, and $t_3 = -4t$. Then,

$$\phi_1^-(x,t) = -\frac{\sqrt{c}}{2}\tanh\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right) , \quad \text{assuming } t_{2j+1} = 0 \text{ for } j \geq 2$$

and therefore we recover the soliton solution of KdV:

$$u(x,t) = -12\partial_x\phi_1^-(x,t) = 3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right) .$$

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Segal-Wilson Theory

Example 2: Given a Riemann surface Γ of genus G, a local coordinate $z(P) = k^{-1}$ defined on U_∞ containing a point P_∞ with $z(P_\infty) = 0$, and a nonspecial integral divisor $\mathcal{D}$ on Γ of degree G, define an element of $\mathrm{Gr}H$ by:

$$W_{\Gamma,k,\mathcal{D}} := \left\{ f\big|_{|k|=1}, \quad \text{where } f \text{ is meromorphic on } \Gamma \setminus U_\infty \text{ with } (f) + \mathcal{D} \geq 0 \right\}.$$

That in fact the subspace $W_{\Gamma,k,\mathcal{D}}$ lies in $\mathrm{Gr}H$, and moreover is transversal, is a consequence of the Riemann-Roch theorem. The coincident Baker function is exactly the Baker-Akhiezer function of Krichever.

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Thus we see that both the class of soliton solutions and the class of algebro-geometric multiphase wave solutions of the KP hierarchy may be identified with transversal points in the Segal-Wilson Grassmannian.

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- Integrable systems arise as universal models in applied mathematics. Their solutions accurately predict real physical phenomena.
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Thank You!

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