Riemann-Hilbert Problems with Lots of Discrete Spectrum: Asymptotics and Applications

Peter D. Miller
Department of Mathematics, University of Michigan
millerpd@umich.edu

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Abstract

I will discuss several situations in which an asymptotic limit of interest leads one to consider the construction of a matrix-valued meromorphic function with principal part data specified at asymptotically many poles. Applications include semiclassical asymptotics of integrable nonlinear wave problems (KdV, NLS, sine-Gordon) as well as statistical combinatorics (discrete orthogonal polynomial ensembles, i.e. discrete analogues of random matrix theory).

Outline

Introduction

Recall the analysis of Lax and Levermore (1983) of the zero-dispersion limit of the Cauchy problem for the Korteweg-de Vries equation. For each $\varepsilon > 0$ there exists a unique global solution of

$$u_t + uu_x + \varepsilon^2 u_{xxx} = 0\,, \quad x \in \mathbb{R}\,, \quad t > 0\,,$$

subject to ε-independent initial data $u(x, 0; \varepsilon) = u_0(x)$. The "zero-dispersion limit" analyzed by L&L refers to the asymptotic analysis of the family of solutions $u(x, t; \varepsilon)$ as $\varepsilon \to 0$.

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L&L used inverse-scattering to solve the Cauchy problem, assuming that $u_0(x) > 0$, rapidly decreasing as $|x| \to \infty$, and having a single critical point (local max). The first step: analyze the stationary Schrödinger equation

$$-6\varepsilon^2 \psi_{xx} + V(x)\psi = E\psi$$

where E is the spectral parameter and $V(x) := -u_0(x)$ is a potential well.

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1. The reflection coefficient for $E > 0$ fixed is "as small in ε as V is smooth".
2. The number $N(\varepsilon)$ of discrete eigenvalues (all simple) is large, proportional to ε^{-1}. The eigenvalues are approximately located according to the *Bohr-Sommerfeld quantization rule*: $E_n = E_n^0 + O(\varepsilon^2)$ where

$$\Phi(E_n^0) = \pi\varepsilon\left(n + \frac{1}{2}\right), \quad n = 0, \ldots, N(\varepsilon) - 1, \quad \Phi(E) := \frac{1}{\sqrt{6}}\int_{x_-(E)}^{x_+(E)} \sqrt{E - V(s)}\, ds .$$

   Here $x_-(E) < x_+(E)$ are the *turning points* (branches of V^{-1}). The asymptotic number of eigenvalues is

$$N(\varepsilon) = \left\lfloor \frac{1}{2} + \frac{1}{\pi\varepsilon\sqrt{6}}\int_{-\infty}^{+\infty} \sqrt{-V(x)}\, dx \right\rfloor .$$

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L&L therefore modified the spectral data associated with $V(x) = -u_0(x)$ by taking the reflection coefficient to be zero and using the approximate eigenvalues $\{E_n^0\}$. Thus, the approximate solution of the KdV Cauchy problem is a pure ensemble of $N(\varepsilon)$ solitons.

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The multisoliton solution of KdV is specified by a collection of $N(\varepsilon)$ discrete eigenvalues and a "norming constant" for each. For such purely discrete data, the inverse-scattering procedure collapses to a problem of linear algebra in dimension $N(\varepsilon)$. Cramer's rule leads to the Kay-Moses determinantal formula

$$u(x,t;\varepsilon) = 12\varepsilon^2 \frac{\partial^2}{\partial x^2}\log(\tau) \quad \text{where} \quad \tau := \det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right) .$$

Here $\kappa_n = \sqrt{-E_n}$ and $F_n = e^{(\kappa_n x - 4\kappa_n^3 t + \beta_n)/\varepsilon}$ and $\{\beta_n\}$ amount to the norming constants.

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A natural approach is to expand τ:

$$\tau = 1 + \sum_{\text{subsets } S \text{ of } \{0,\ldots,N(\varepsilon)-1\}} \det\left(\left.\frac{F_\alpha F_\beta}{\kappa_\alpha + \kappa_\beta}\right|_{\alpha,\beta\in S}\right) .$$

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L&L observed: *each term in the sum is positive*. They proved that as $N(\varepsilon) \to \infty$ the sum is dominated by its largest term. This leads to a discrete variational problem that may be further approximated by a variational problem for an absolutely continuous *equilibrium measure*. In this way, L&L proved that $u(x,t;\varepsilon)$ has a weak limit $\overline{u}(x,t)$.

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Venakides (1990) obtained strong asymptotics for $u(x,t;\varepsilon)$ by "going to higher order", in particular by quantizing the mass of the equilibrium measure. He found strongly nonlinear oscillations of unit amplitude about the mean $\overline{u}(x,t)$ modeled by algebro-geometric multiphase wave solutions of KdV.

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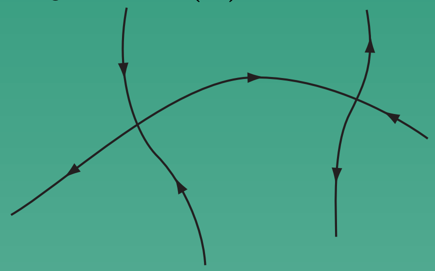

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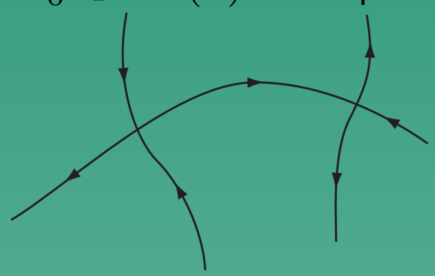

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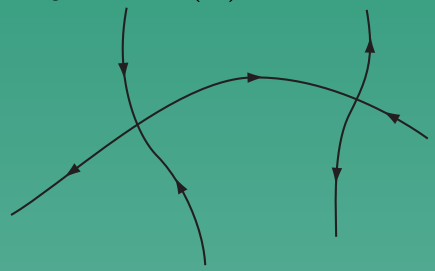

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2. $\mathbf{M}$ takes continuous boundary values $\mathbf{M}_\pm(z)$, $z \in \Sigma$, that are related by $\mathbf{M}_+(z) = \mathbf{M}_-(z)\mathbf{V}(z)$ ($+$ means left, $-$ means right).

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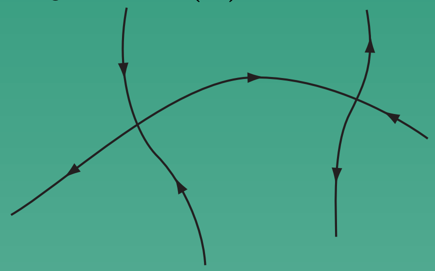

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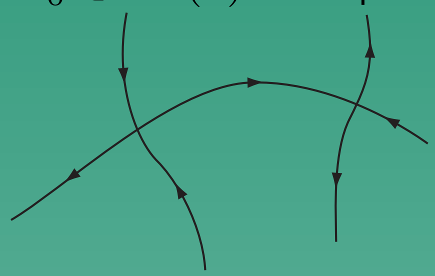

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Steepest descent in a nutshell: a systematic construction of a “global parametrix” $\hat{\mathbf{M}}(z)$ such that the linear substitution $\mathbf{M}(z) = \mathbf{E}(z)\hat{\mathbf{M}}(z)$ results in a “small-norm” Riemann-Hilbert problem for the *error* $\mathbf{E}(z)$. By definition, a small-norm problem is one for which simple estimates can establish that $\mathbf{E}(z) \approx \mathbb{I}$ in a suitable sense.

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$$\operatorname*{Res}_{z=z_p} \mathbf{M}(z) = \lim_{z \to z_p} \mathbf{M}(z) \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} .$$

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If the singularity at z_p remains separated from the rest of the spectrum in the asymptotic limit of interest, the pole may be removed at the expense of augmenting Σ and $\mathbf{V}$ as follows: make the substitution

$$\mathbf{N}(z) := \mathbf{M}(z) \begin{bmatrix} 1 & -c(z - z_p)^{-1} \\ 0 & 1 \end{bmatrix} \text{ for } |z - z_p| < \delta, \text{ and } \mathbf{N}(z) := \mathbf{M}(z) \text{ for } |z - z_p| > \delta.$$

$\mathbf{N}(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma \cup \Sigma_p$, where Σ_p is a small, positively oriented circle about z_p of radius δ.

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$$\text{Then, } \mathbf{N}_+(z) = \mathbf{N}_-(z)\mathbf{V}(z) \text{ for } z \in \Sigma \text{ and } \mathbf{N}_+(z) = \mathbf{N}_-(z) \begin{bmatrix} 1 & -c(z - z_p)^{-1} \\ 0 & 1 \end{bmatrix} \text{ for } z \in \Sigma_p.$$

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Moreover, there are a number of problems involving "lots of discrete spectrum" to which the L&L method does not apply at all, but for which the Riemann-Hilbert problem offers an alternative approach. . .

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The Semiclassical Focusing Nonlinear Schrödinger Equation

A problem with many apparent similarities:

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subject to initial data $\psi(x,0;\varepsilon) = A(x)e^{iS(x)/\varepsilon}$ where $A(\cdot)$ and $S(\cdot)$ are real and independent of ε.

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Note: formal zero-dispersion limit of KdV is inviscid Burgers' equation: $u_t + uu_x = 0$ (hyperbolic). Fact: the well-posed Cauchy problem for this equation governs the early stages of the zero-dispersion limit for KdV.

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$$\rho := |\psi|^2 \quad \text{and} \quad u := \varepsilon[\Im\{\log(\psi)\}]_x$$

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subject to $\rho(x,0;\varepsilon) = A(x)^2$ and $u(x,0;\varepsilon) = S'(x)$. The formal limit (neglecting $\varepsilon^2 F[\rho]$) is a Cauchy problem for an elliptic system. This is an ill-posed (formal) limit problem!

The relevant spectral problem for inverse-scattering is the nonselfadjoint Zakharov-Shabat system

$$\varepsilon \mathbf{u}_x = \begin{bmatrix} -i\lambda & A(x)e^{iS(x)/\varepsilon} \\ -A(x)e^{-iS(x)/\varepsilon} & i\lambda \end{bmatrix} \mathbf{u}\,,$$ where $\lambda \in \mathbb{C}$ is the spectral parameter.

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1. For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the reflection coefficient is as small in ε as $A(\cdot)$ is smooth.
2. Bohr-Sommerfeld quantization rule for eigenvalues $\lambda_n \in [0, i\max A(x)]$: $\lambda_n \approx \lambda_n^0$ where

$$\Psi(\lambda_n^0) = \pi\varepsilon\left(n + \frac{1}{2}\right)\,, \quad n = 0, \ldots, N(\varepsilon) - 1\,, \quad \Psi(\lambda) := \int_{x_-(\lambda)}^{x_+(\lambda)} \sqrt{A(s) + \lambda^2}\, ds\,.$$

Here $x_-(\lambda) < x_+(\lambda)$ are turning points. The asymptotic number of positive imaginary eigenvalues is

$$N(\varepsilon) = \left\lfloor \frac{1}{2} + \frac{1}{\pi\varepsilon}\int_{-\infty}^{+\infty} A(x)\, dx \right\rfloor\,.$$

M. Klaus and J. K. Shaw, *Phys. Rev. E*, **65**, 36607–36611, 2002.

The Semiclassical Focusing Nonlinear Schrödinger Equation

This suggests an approach completely parallel to that applied by L&L to KdV: neglect the reflection coefficient and consider the asymptotic behavior of the "semiclassical soliton ensemble" given by the reflectionless potential associated with the WKB approximations to the eigenvalues.

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$$|\psi(x,t;\varepsilon)|^2 = \varepsilon^2 \frac{\partial^2}{\partial x^2}\log(\tau)\,,\ \text{where } \tau := \det(\mathbb{I} + \mathbf{B}^*\mathbf{B}) \text{ and } B_{jk} := \frac{E_j E_k^*}{i(\lambda_j - \lambda_k)}\,.$$

Here $\{\lambda_k\}$ are the eigenvalues in the upper half-plane and $E_k := e^{i(\lambda_k x + \lambda_k^2 t + \beta_k)/\varepsilon}$ where $\{\beta_k\}$ are like the norming constants.

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The Lax-Levermore method fails because the principal minors expansion of τ consists of both positive and negative terms!

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The inverse-scattering problem was considered from the point of view of matrix Riemann-Hilbert problems by Kamvissis, McLaughlin, and M (2003). General Klaus-Shaw initial data is considered with further assumptions:

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1. The parameter $\varepsilon > 0$ is restricted to a discrete sequence decreasing to zero:

$$\varepsilon = \varepsilon_N := \frac{1}{\pi N}\int_{-\infty}^{+\infty} A(x)\, dx\,, \quad N = 1, 2, 3, \ldots\ .$$

   This makes the reflection coefficient uniformly small. Note $N(\varepsilon) = N$.

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Neglecting the reflection coefficient and taking the WKB eigenvalues and proportionality constants as exact spectral data, let

$$c_k(x,t) := \frac{1}{\gamma_k} \operatorname*{Res}_{\lambda=\lambda_k} W(\lambda)\,, \quad W(\lambda) := e^{2i(\lambda x+\lambda^2 t)/\varepsilon} \prod_{n=0}^{N-1} \frac{\lambda - \lambda_n^*}{\lambda - \lambda_n}\,.$$

The Riemann-Hilbert problem of inverse scattering is to find a 2×2 matrix $\mathbf{m}(\lambda)$, $\lambda \in \mathbb{C}$, with the following properties:

1. $\mathbf{m}(\lambda)$ is a rational function of λ with simple poles confined to $\{\lambda_n, \lambda_n^*\}$ such that for $k = 0, \ldots, N-1$:

$$\operatorname*{Res}_{\lambda=\lambda_k} \mathbf{m}(\lambda) = \lim_{\lambda\to\lambda_k} \mathbf{m}(\lambda) \begin{bmatrix} 0 & 0 \\ c_k(x,t) & 0 \end{bmatrix}, \qquad \operatorname*{Res}_{\lambda=\lambda_k^*} \mathbf{m}(\lambda) = \lim_{\lambda\to\lambda_k^*} \mathbf{m}(\lambda) \begin{bmatrix} 0 & -c_k(x,t)^* \\ 0 & 0 \end{bmatrix}.$$

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The semiclassical soliton ensemble itself is given by $\psi(x,t;\varepsilon) = 2i \displaystyle\lim_{\lambda\to\infty} \lambda m_{12}(\lambda)$.

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This is not a traditional RHP: no jumps, just poles. Indeed it seems to be simple, solvable by partial fractions:

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where $\mathbf{a}_k$ and $\mathbf{b}_k$ are matrix coefficients determined by the residue conditions, leading to a linear algebra problem for the coefficients.

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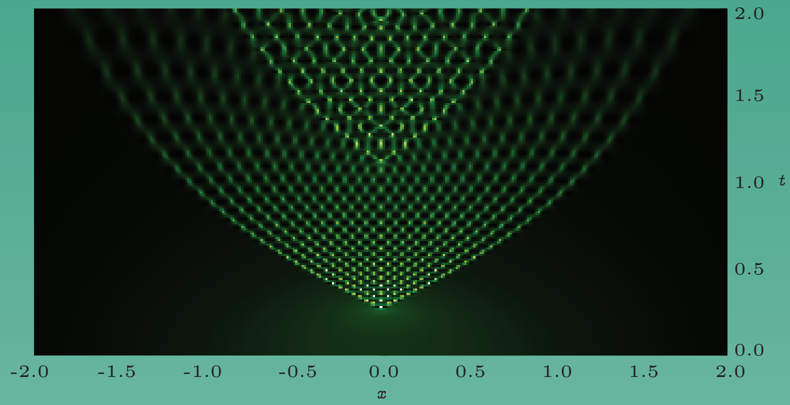
However, for N reasonably large, it can be solved numerically on a grid of independent (x, t)-values. The resulting plots reveal marvelous structures (here $A(x) = 2\,\mathrm{sech}(x)$ and $\varepsilon = 2/N$ with $N = 40$):



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$$\gamma_k = -ie^{-i\Psi(\lambda_k)/\varepsilon}, \quad k = 0, 1, 2, \ldots, N-1 .$$

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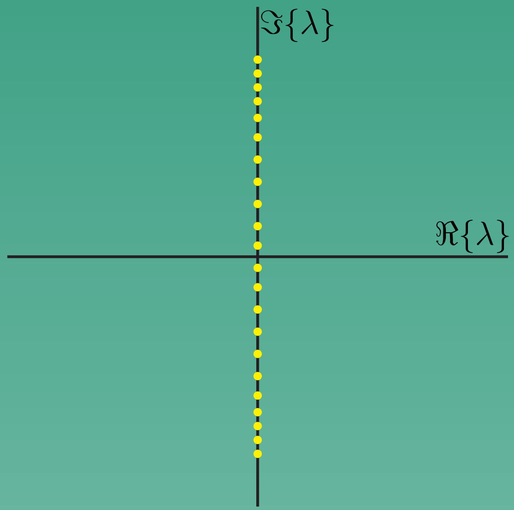

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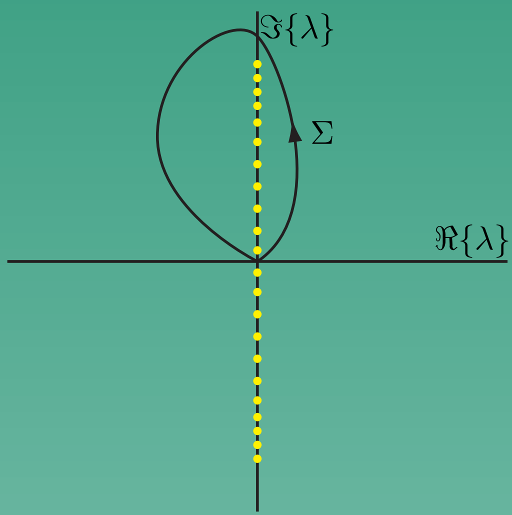

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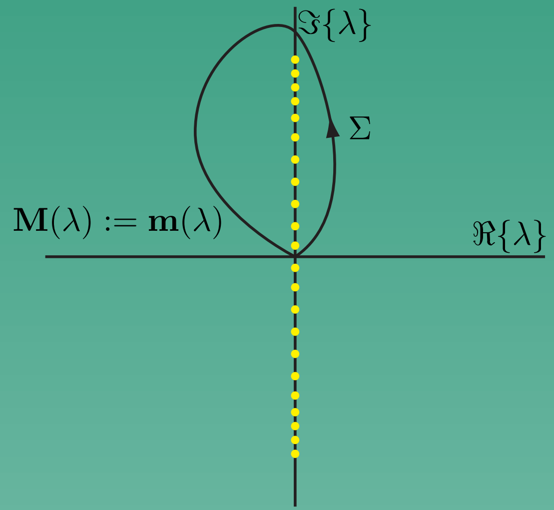

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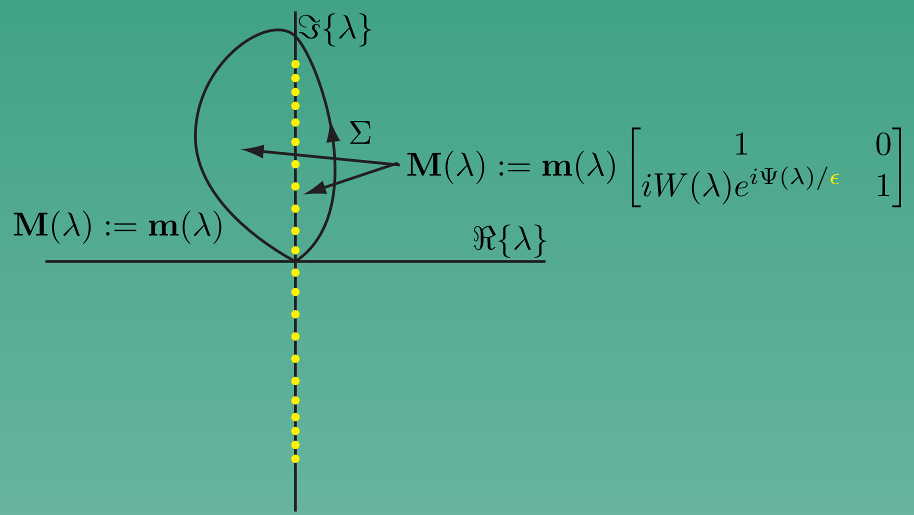

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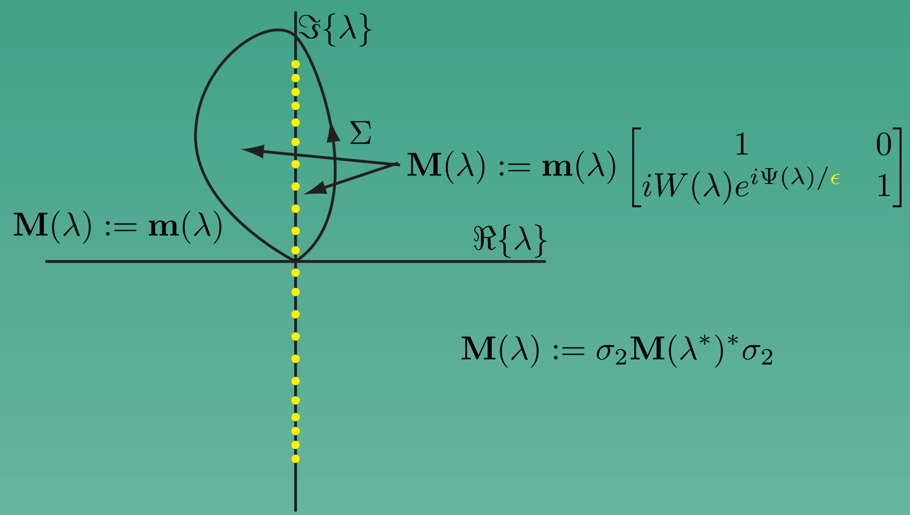

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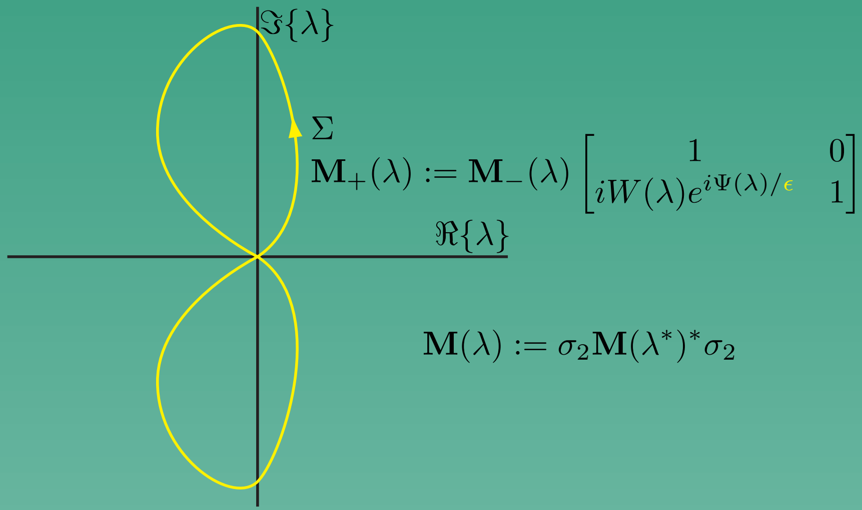

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The Deift-Zhou technique may now be applied, the key step of which is to stabilize the problem with a "g-function". Supposing that $g(\lambda)$ is a function analytic in $\mathbb{C} \setminus \Sigma \cup \Sigma^*$ with $g \to 0$ as $\lambda \to \infty$ and $g(\lambda) + g(\lambda^*)^* = 0$ define a new unknown: $\mathbf{N}(\lambda) := \mathbf{M}(\lambda)e^{-g(\lambda)\sigma_3/\varepsilon}$.

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$$W(\lambda) := \exp\left(\frac{1}{\varepsilon}\left[2i\lambda x + 2i\lambda^2 t + \sum_{n=0}^{N-1} \varepsilon \log(\lambda - \lambda_n^*) - \sum_{n=0}^{N-1} \varepsilon \log(\lambda - \lambda_n)\right]\right)$$

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One then aims to choose the contour Σ and the function $g(\lambda)$ so that as $\varepsilon \to 0$ the jump of $\mathbf{N}$ takes two alternative forms. This *determines* both g and several arcs of $\Sigma \cup \Sigma^*$. The local dynamics of $\psi(x, t; \varepsilon)$ are given in terms of Θ for the double cover of $\mathbb{C}$ with cuts on these arcs.

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For example, when $t = 0$, the theory predicts that one arc of Σ should *be contained within* the imaginary interval $[0, i \max A(x)]$. In M (2002) it is shown how to nonetheless make use of the g-function at $t = 0$. The key is to use two different analytic interpolants of the proportionality constants $\{\gamma_k\}$ at the eigenvalues $\{\lambda_k\}$:

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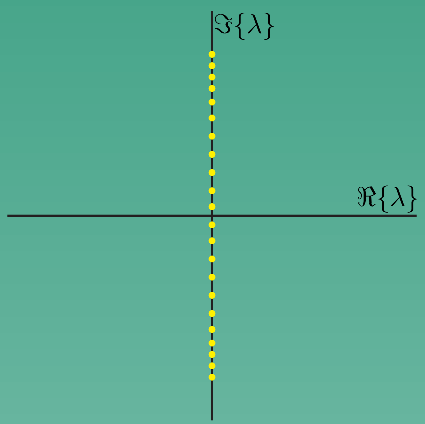
For example, when $t = 0$, the theory predicts that one arc of Σ should *be contained within* the imaginary interval $[0, i \max A(x)]$. In M (2002) it is shown how to nonetheless make use of the g-function at $t = 0$. The key is to use two different analytic interpolants of the proportionality constants $\{\gamma_k\}$ at the eigenvalues $\{\lambda_k\}$:



P. D. Miller, *Internat. Math. Res. Not.*, 383–454, 2002.

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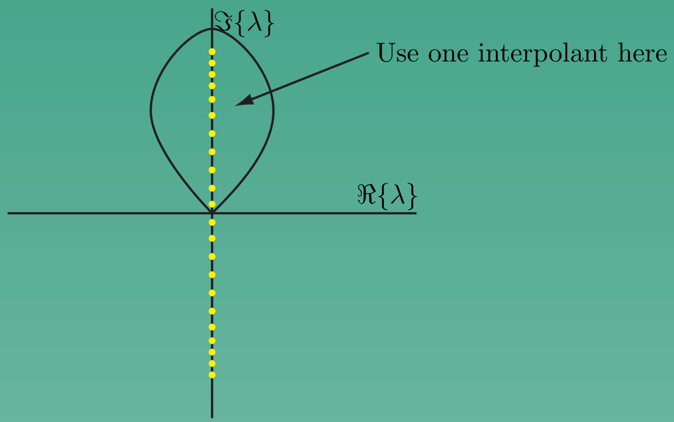

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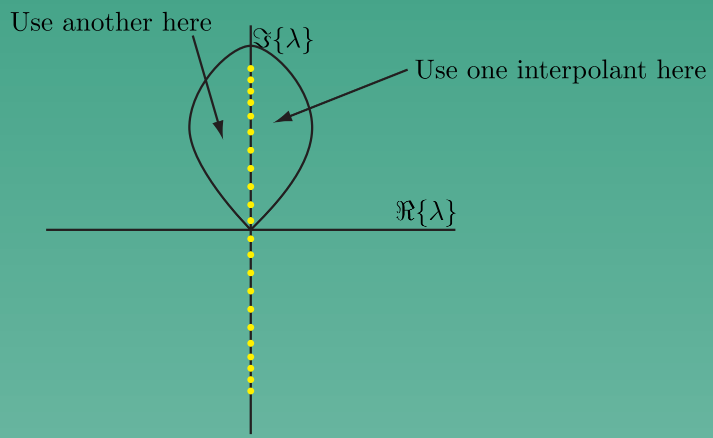

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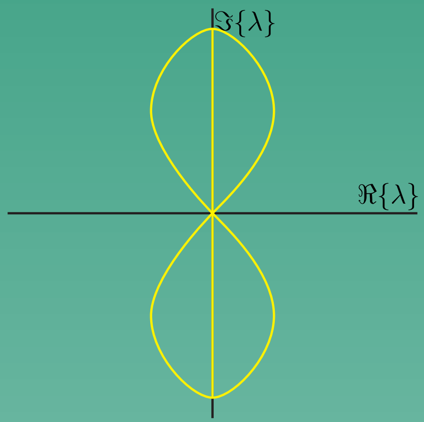

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The two interpolants used correspond to two choices of $j \in \mathbb{Z}$ in the more general formula

$$\gamma_k = -i(-1)^j e^{-i(2j+1)\Psi(\lambda_k)/\varepsilon}, \quad k = 0, 1, 2, \ldots, N-1 .$$

We use $j = 0$ on the left and $j = -1$ on the right.

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G. D. Lyng and P. D. Miller, *Comm. Pure Appl. Math.*, **60**, 951–1026, 2007.

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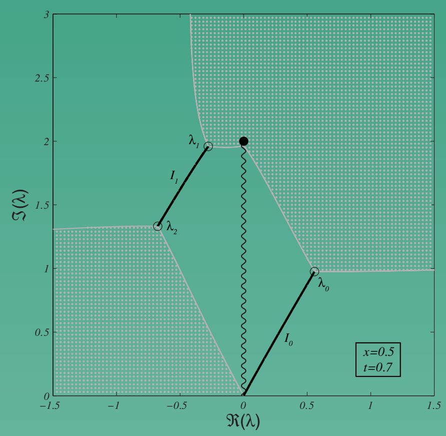

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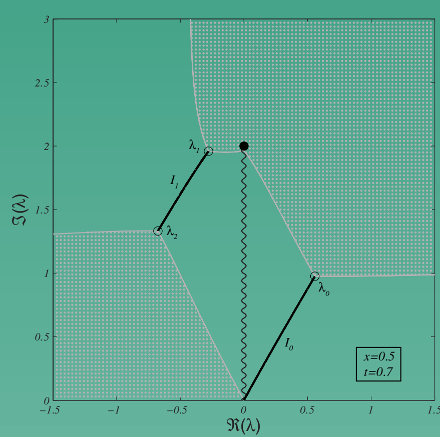

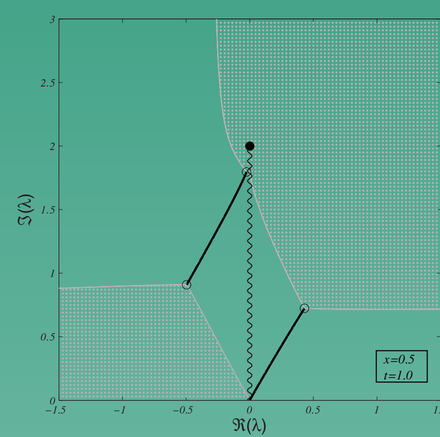

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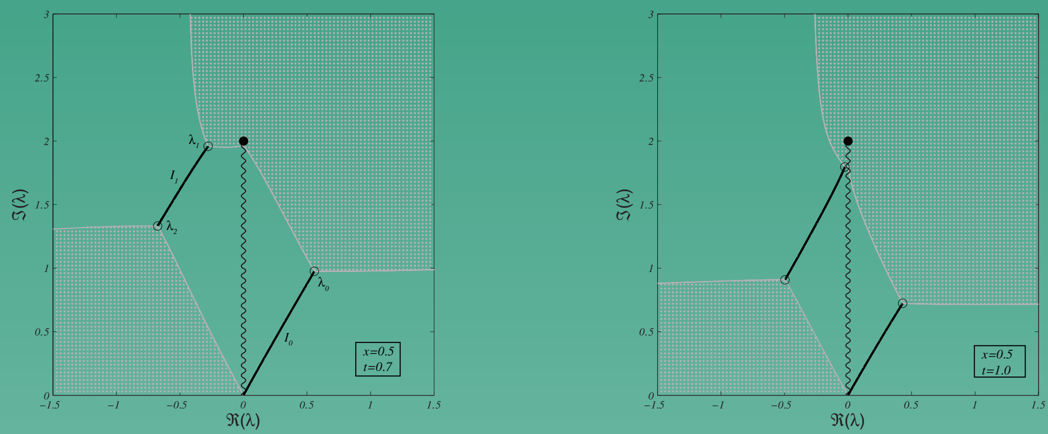

The situation is repaired using *three* interpolants: $j = -1$, $j = 0$, and $j = 1$. Return to outline.

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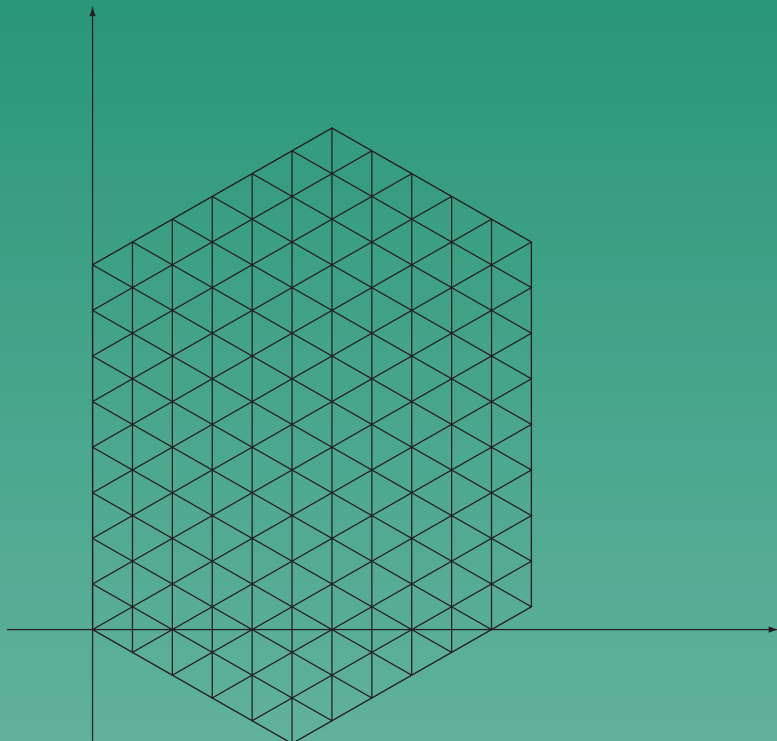
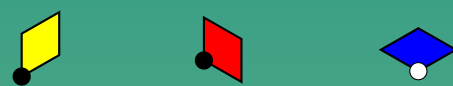
Discrete Orthogonal Polynomials

Consider random tilings of a hexagon with rhombus-shaped tiles:

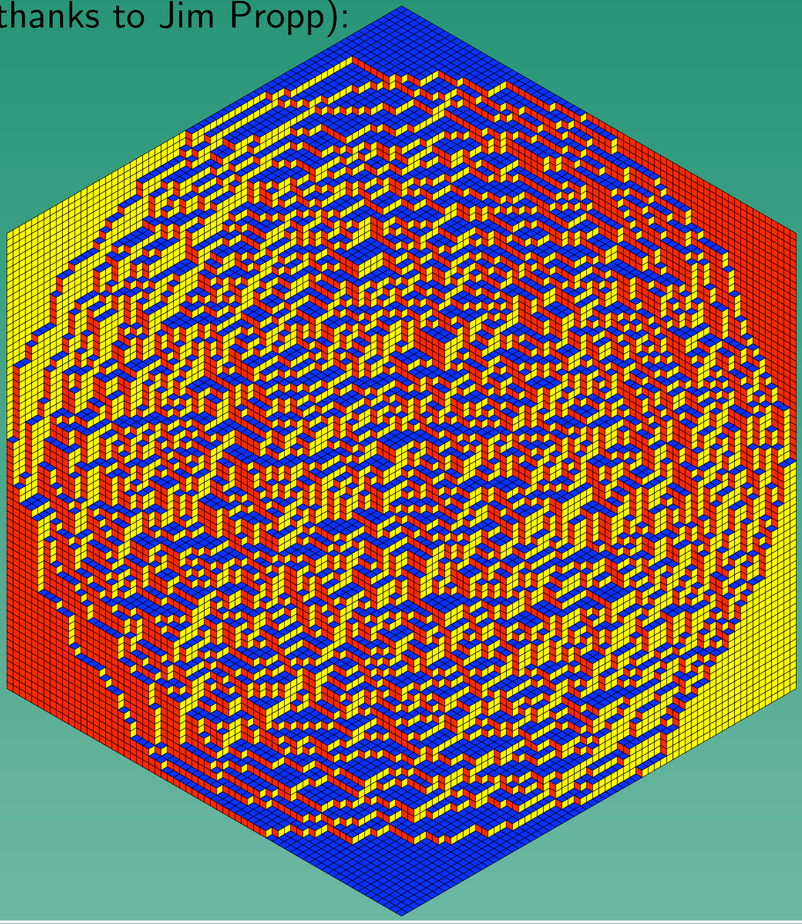




We suppose that all tilings are equally likely.

Discrete Orthogonal Polynomials

Here is a typical tiling (thanks to Jim Propp):



Discrete Orthogonal Polynomials

Using an interpretation of random tilings as a problem of nonintersecting random walks, Johansson (2000) proved the following about the positions of the blue tiles in any vertical slice of the figure:

Theorem 1. [Johansson] *Let* $\xi_1 < \cdots < \xi_L$ *denote the integer-valued positions (counted from the bottom) of the blue tiles in the* m^{th} *vertical slice (from the left) of the hexagon with side lengths* $\mathfrak{a} \geq \mathfrak{b}$ *and* $\mathfrak{c}$*. This slice contains*

$$N := \mathfrak{c} + \frac{\mathfrak{a} - |m - \mathfrak{a}|}{2} + \frac{\mathfrak{b} - |m - \mathfrak{b}|}{2}$$

possible positions of tiles, and $L = N - \mathfrak{c}$*. Then, the probability* $P_m(\xi_1, \ldots, \xi_L)$ *of finding this configuration of blue tiles is*

$$P_m(\xi_1, \ldots, \xi_L) = \frac{1}{Z} \prod_{1 \leq j < k \leq L} (\xi_j - \xi_k)^2 \prod_{j=1}^{L} w(\xi_j) \, ,$$

where

$$w(\xi) := \frac{(\xi + |m - \mathfrak{a}|)!(N - \xi - 1 + |m - \mathfrak{b}|)!}{\xi!(N - \xi - 1)!} \, .$$

K. Johansson, *Comm. Math. Phys.*, **209**, 437–476, 2000.

Discrete Orthogonal Polynomials

Note: $w(\xi)$ is the weight function for the family of "discrete orthogonal" Hahn polynomials $\{p_{N,k}(z)\}_{k=0}^{N-1}$ defined by the orthogonality conditions

$$\sum_{j=0}^{N-1} p_{N,k}(x_{N,j})p_{N,l}(x_{N,j})^* w(x_{N,j}) = \delta_{kl} \,, \quad x_{N,j} := j \,.$$

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This leads to the general question of how to correctly formulate and compute asymptotics for families of discrete orthogonal polynomials, a question we will now discuss.

Discrete Orthogonal Polynomials

Consider the discrete measure

$$\mu(x) = \sum_{n=0}^{N-1} w_{N,n}\delta(x - x_{N,n})$$

where the nodes of orthogonality $\{x_{N,n}\}$ are prescribed for each N by giving a fixed real-analytic function $\rho^0(x) > 0$ on an interval $[a, b]$ with

$$\int_a^b \rho^0(x)\, dx = 1\,,$$

and then imposing the conditions (resembling a Bohr-Sommerfeld quantization rule)

$$\int_a^{x_{N,n}} \rho^0(x)\, dx = \frac{2n+1}{2N}\,,\ \ n = 0, \ldots, N-1\,.$$

The weights $\{w_{N,n}\}$ are assumed to be of the form

$$w_{N,n} = e^{-NV(x_{N,n})} \prod_{\substack{m=0 \\ m\neq n}}^{N-1} |x_{N,n} - x_{N,m}|^{-1}$$

for some real analytic function $V(x)$ (only weakly dependent on N, if at all).

Discrete Orthogonal Polynomials

Let $\mathbb{Z}_N := \{0, \ldots, N-1\}$, $\Delta \subset \mathbb{Z}_N$, and $\nabla := \mathbb{Z}_N \setminus \Delta$. Set $W(z) := e^{-NV(z)} \dfrac{\displaystyle\prod_{n\in\Delta}(z - x_{N,n})}{\displaystyle\prod_{n\in\nabla}(z - x_{N,n})}$.

A. Borodin, *Internat. Math. Res. Not.*, **9**, 467–494, 2000.

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The DOPs satisfy a (fully discrete) RHP. Seek a 2×2 matrix $\mathbf{Q}(z)$, $z \in \mathbb{C}$ such that:

1. $\mathbf{Q}(z)$ is a rational function of z with simple poles confined to the nodes $\{x_{N,n}\}_{n=0}^{N-1}$ such that

$$\operatorname*{Res}_{z=x_{N,n}} \mathbf{Q}(z) = \lim_{z\to x_{N,n}} \mathbf{Q}(z) \begin{bmatrix} 0 & (-1)^{N-1-n} \operatorname*{Res}_{\zeta=x_{N,n}} W(\zeta) \\ 0 & 0 \end{bmatrix}, \quad n \in \nabla$$

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The monic scaling of $p_{N,k}(z)$ is given by $\pi_{N,k}(z) = z^k + \cdots = Q_{11}(z) \displaystyle\prod_{n\in\Delta}(z - x_{N,n})$.

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Discrete Orthogonal Polynomials

To analyze this RHP, we know based on experience that we should remove the poles. The key is to find an analytic function that interpolates the signs $\{(-1)^{N-1-n}\}$ at the nodes $\{x_{N,n}\}$. Setting

$\theta^0(z) := 2\pi \int_z^b \rho^0(s)\, ds$, we have the identities $ie^{-iN\theta^0(x_{N,n})/2} = -ie^{iN\theta^0(x_{N,n})/2} = (-1)^{N-1-n}$.

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$$\mathbf{R}(z) := \mathbf{Q}(z) \begin{bmatrix} 1 & \mp ie^{\mp iN\theta^0(z)/2}W(z) \\ 0 & 1 \end{bmatrix}, z \in \Omega_\pm^\nabla$$

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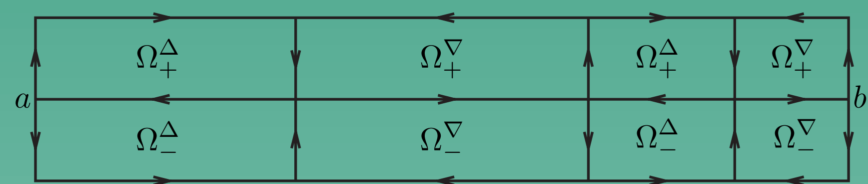

and for all other $z \in \mathbb{C}$ set $\mathbf{R}(z) := \mathbf{Q}(z)$. We obtain a traditional RHP! Return to outline.

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The Semiclassical Sine-Gordon Equation

Going back to nonlinear waves, another interesting Cauchy problem is:

$$\varepsilon^2 u_{tt} - \varepsilon^2 u_{xx} + \sin(u) = 0\,, \quad x \in \mathbb{R}\,, \quad t > 0$$

subject to initial data of the form $u(x, 0; \varepsilon) = f(x)$ and $\varepsilon u_t(x; 0; \varepsilon) = g(x)$, independent of ε.

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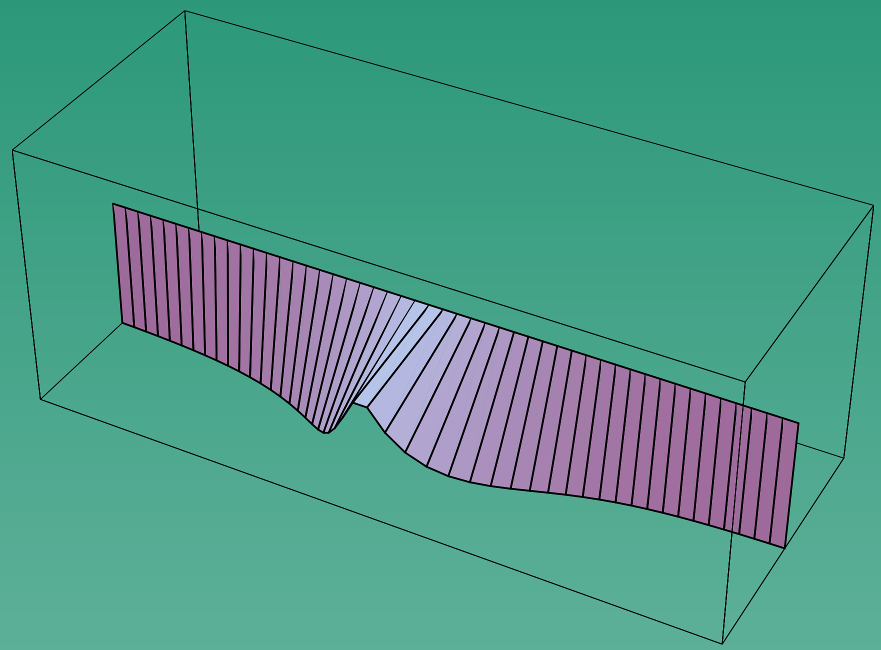
At a more elementary level, it is the equation of motion of a simple mechanical system: a chain of coupled pendula all attached to the same rod:



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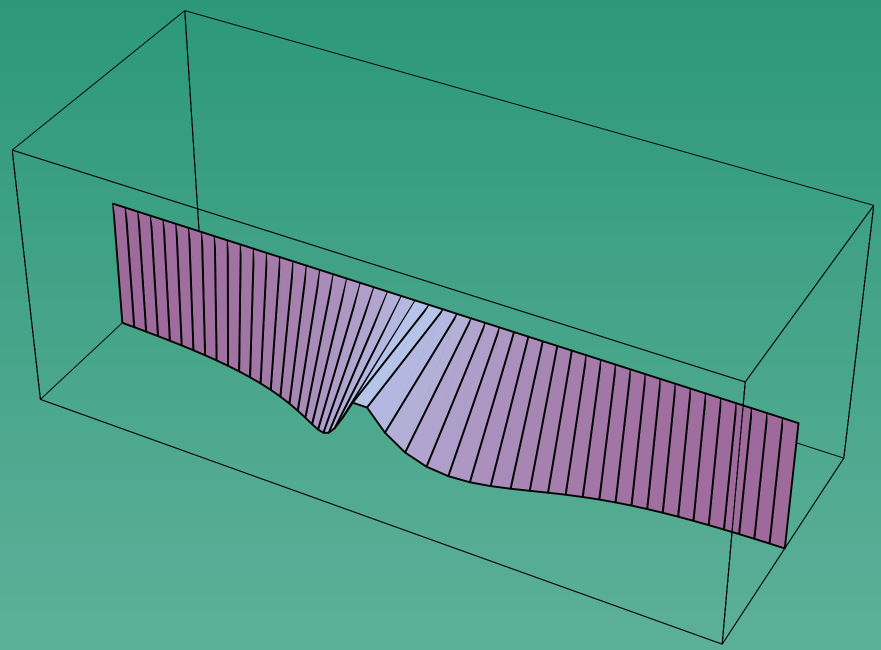

This configuration has "topological charge" equal to zero. Small ε means

The Semiclassical Sine-Gordon Equation

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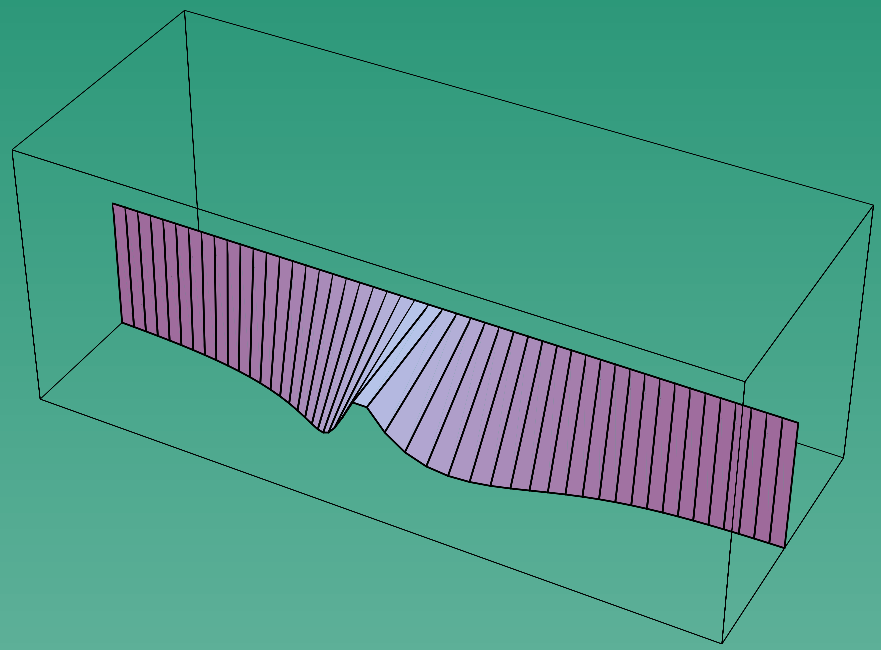

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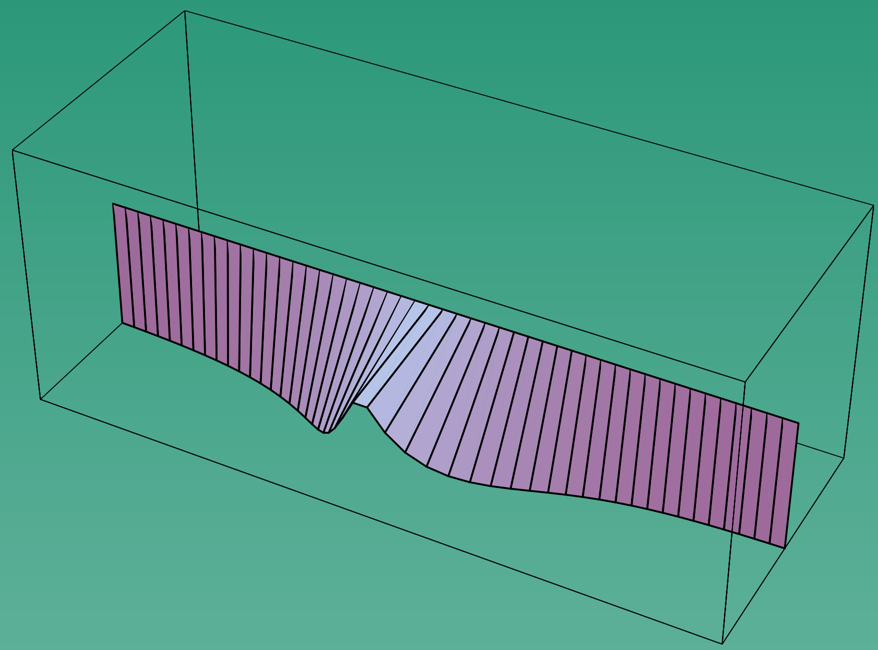
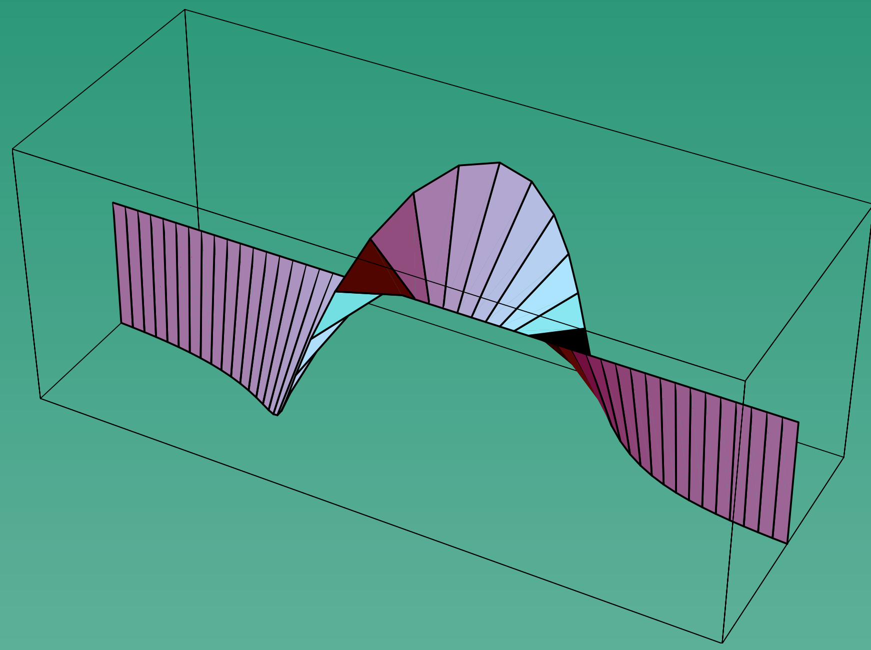


This configuration has "topological charge" equal to zero. Small ε means (i) small mass of each pendulum bob and (ii) weak coupling between neighborhing pendula.

The Semiclassical Sine-Gordon Equation

A configuration with unit topological charge is a "kink":



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As a first step in analyzing this problem, Buckingham and M (2007) considered special initial data:

$$\sin\left(\frac{f}{2}\right) := \operatorname{sech}(x)\,, \quad \cos\left(\frac{f}{2}\right) := \tanh(x)\,, \quad g := 2\mu\,\operatorname{sech}(x)$$

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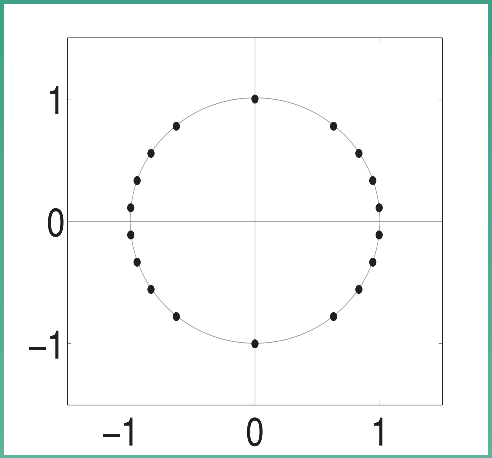

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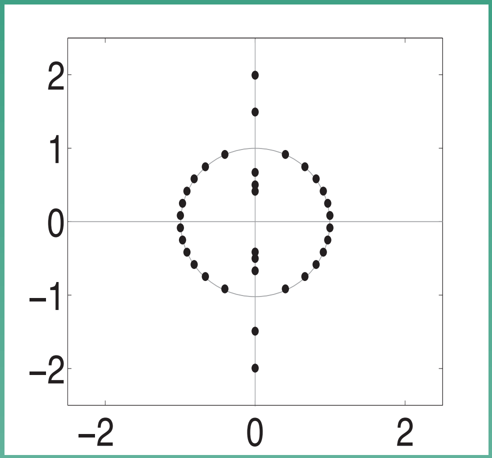

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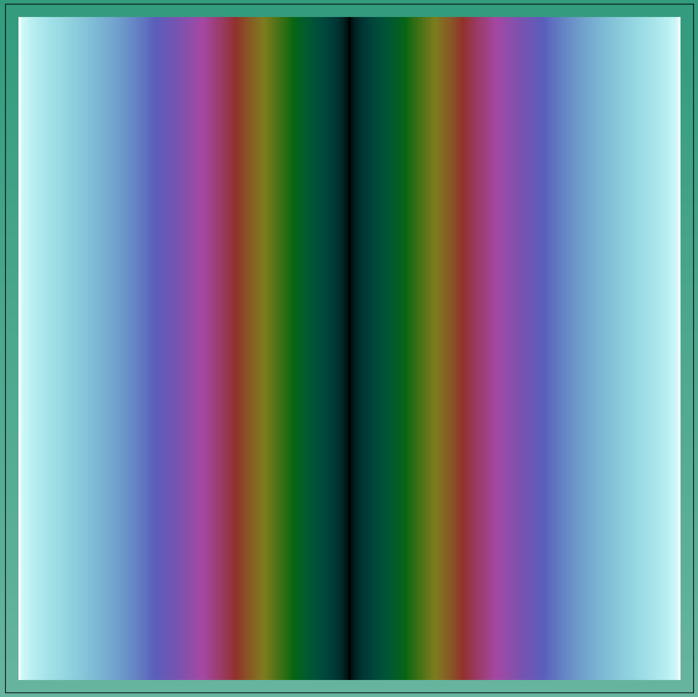
Among other things, this allows us to numerically handle the inverse-scattering. Here is a plot of $\cos(u)$ over $-2.5 < x < 2.5$ and $0 < t < 5$. Here $\mu = 0$ and $N = 0$ ($\varepsilon = 1$).



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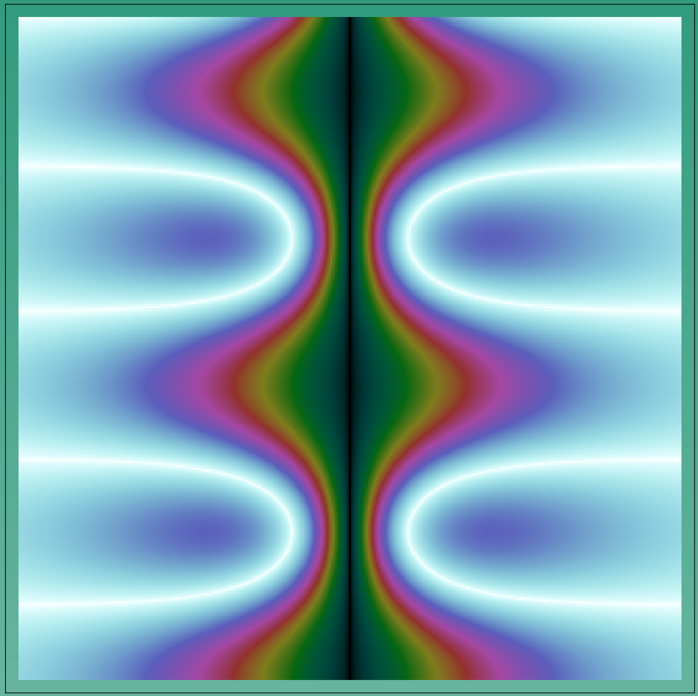

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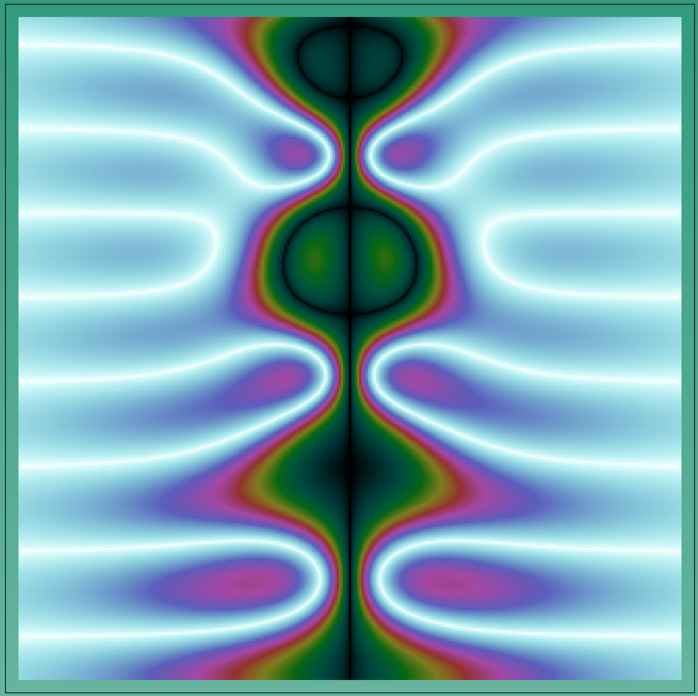

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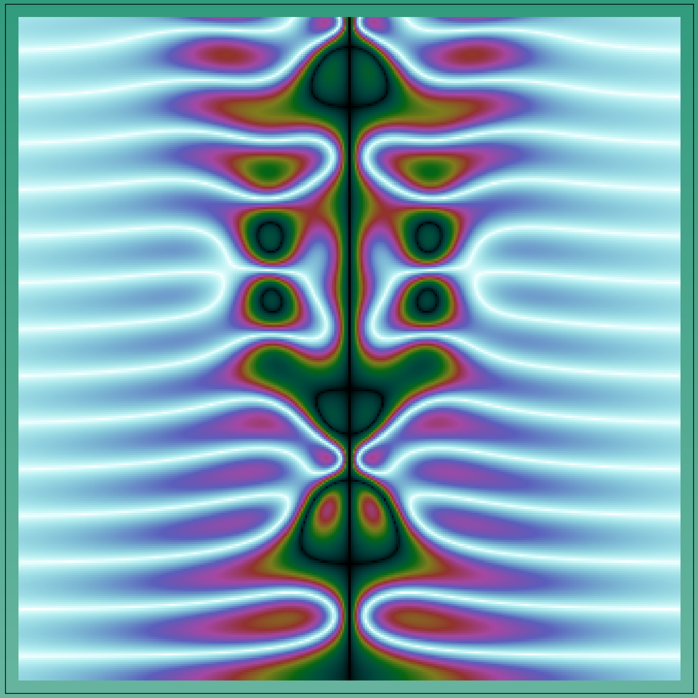

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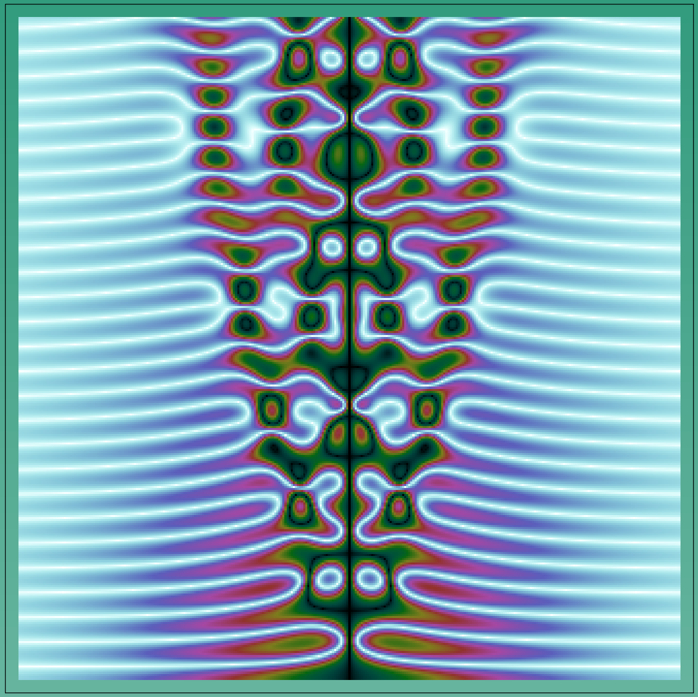

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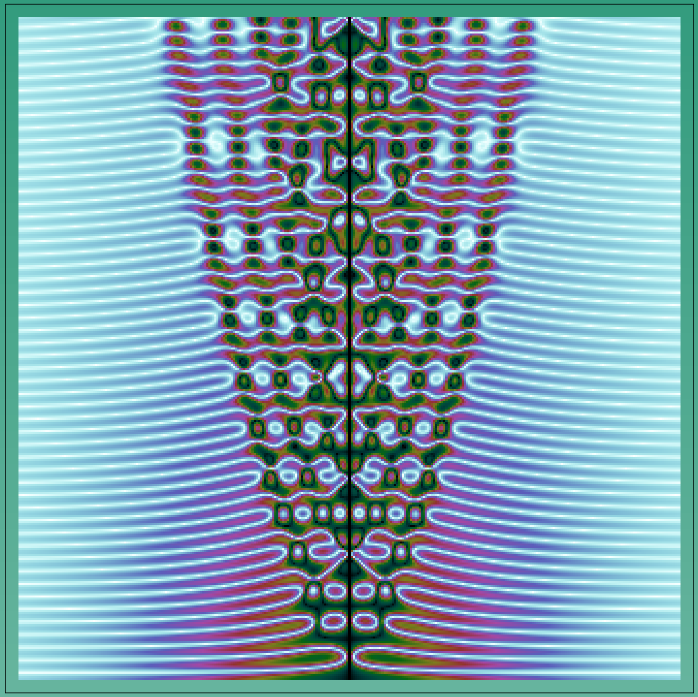

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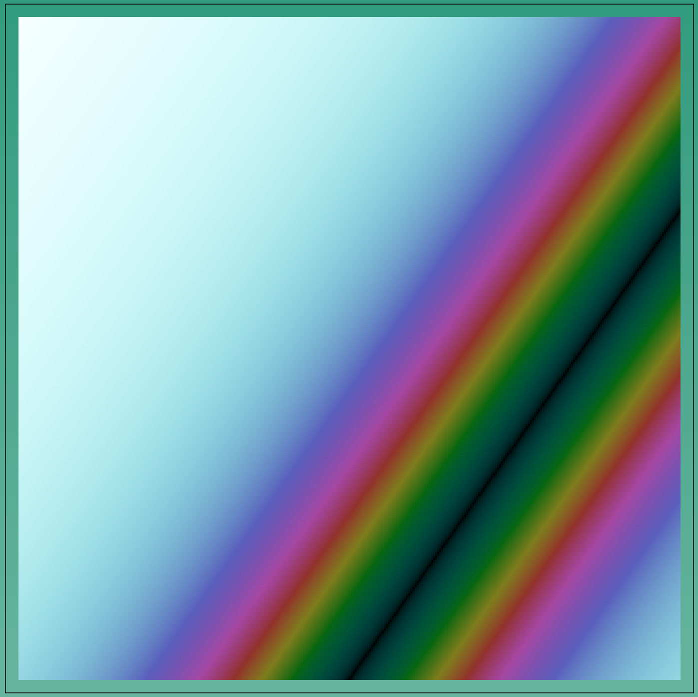
And here is a plot of $\cos(u)$ over $-2.5 < x < 2.5$ and $0 < t < 5$, now for $\mu = 1$ and $N = 0$ ($\varepsilon = \sqrt{2}$), bringing some kinks into the mix.



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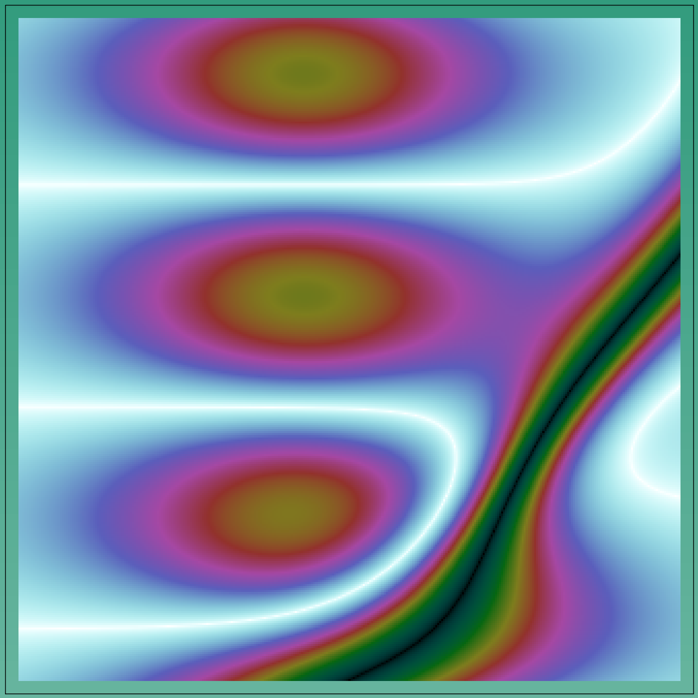

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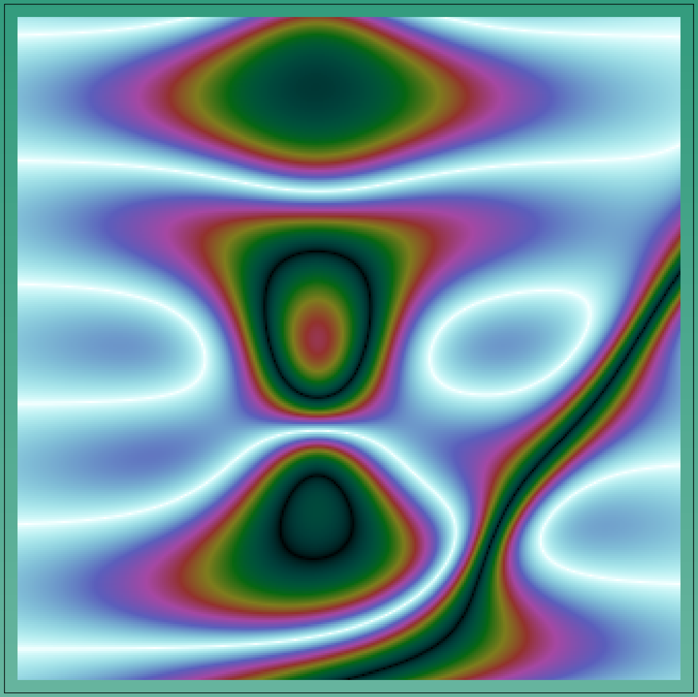

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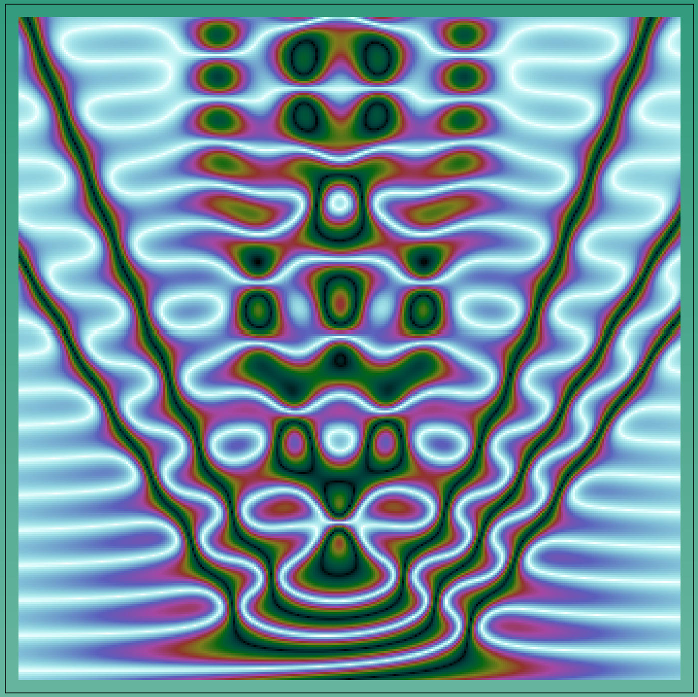

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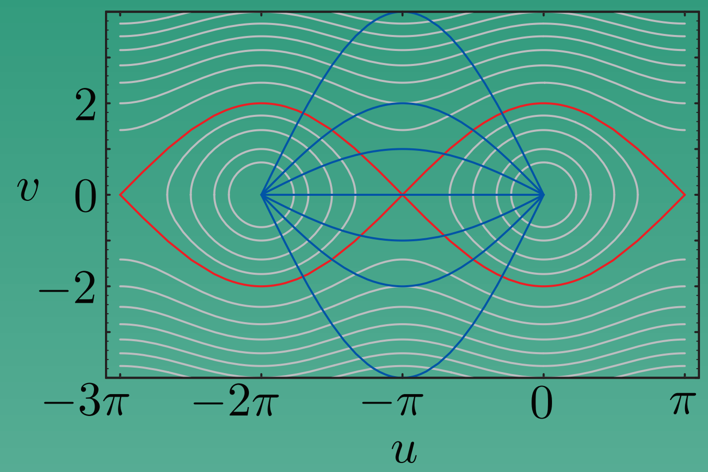

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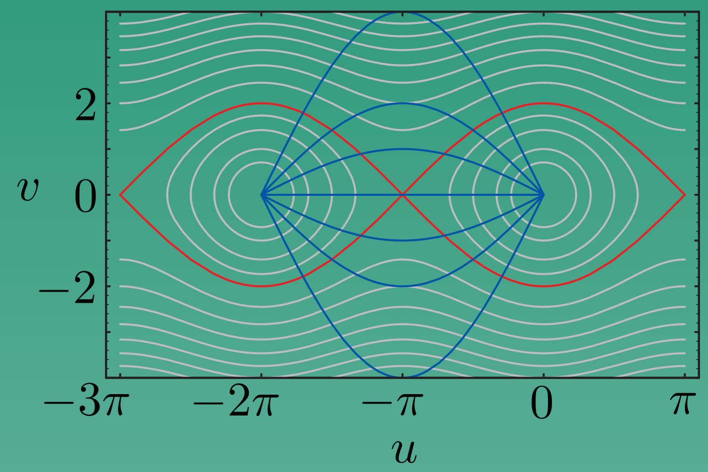

We are now in a position to analyze the discrete RHP associated with these reflectionless potentials and generalizations thereof (work in progress).

Return to outline.

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Thank You!