What is . . . a soliton?

Peter D. Miller
Department of Mathematics, University of Michigan

March 5, 2009

Abstract

Norman Zabusky coined the word "soliton" in 1965 to describe a curious feature he and Martin Kruskal observed in their numerical simulations of the initial-value problem for a simple nonlinear partial differential equation. This talk will describe several of the aspects of solitons that have become important in pure and applied mathematics since their accidental discovery 40 years ago in a (by today's standards) primitive numerical experiment. In particular, a soliton is at once (i) a particular solution of one of many special "integrable" nonlinear partial differential equations, (ii) an eigenvalue of a linear operator, and (iii) a robust coherent structure with particle-like properties.

Department of MATHEMATICS University of Michigan

Outline

What is . . . a wave?

The wave equation:

$$u_{tt} - c^2 u_{xx} = 0$$

was known as a mathematical model for wave propagation in vibrating strings as early as the mid-18th century.

What is . . . a wave?

The wave equation:

$$u_{tt} - c^2 u_{xx} = 0$$

was known as a mathematical model for wave propagation in vibrating strings as early as the mid-18th century.

Its success as a model for vibrations in a medium hinges on the fact that it is the simplest second-order equation exhibiting "wave-like" solutions:

$$u(x,t) = a\cos(k(x \pm ct - x_0))\,.$$

What is . . . a wave?

Each one of these wave-like solutions may be viewed as the real part of a complex-valued solution given by

$$u(x,t) = Ae^{i(kx-\omega t)}\,.$$

In this case, we have $\omega^2 = c^2k^2$.

What is . . . a wave?

Each one of these wave-like solutions may be viewed as the real part of a complex-valued solution given by

$$u(x,t) = Ae^{i(kx-\omega t)}\,.$$

In this case, we have $\omega^2 = c^2k^2$. The parameters in this solution formula are

- The (complex) amplitude A. Its magnitude determines the peak-to-trough depth of the wave and its phase determines the location of the peaks when $x = t = 0$.

What is . . . a wave?

Each one of these wave-like solutions may be viewed as the real part of a complex-valued solution given by

$$u(x,t) = Ae^{i(kx-\omega t)}\,.$$

In this case, we have $\omega^2 = c^2k^2$. The parameters in this solution formula are

- The (complex) amplitude A. Its magnitude determines the peak-to-trough depth of the wave and its phase determines the location of the peaks when $x = t = 0$.
- The wavenumber k. This determines the peak-to-peak wavelength $\lambda := 2\pi/|k|$.

What is . . . a wave?

Each one of these wave-like solutions may be viewed as the real part of a complex-valued solution given by

$$u(x,t) = Ae^{i(kx-\omega t)}\,.$$

In this case, we have $\omega^2 = c^2k^2$. The parameters in this solution formula are

- The (complex) amplitude A. Its magnitude determines the peak-to-trough depth of the wave and its phase determines the location of the peaks when $x = t = 0$.
- The wavenumber k. This determines the peak-to-peak wavelength $\lambda := 2\pi/|k|$.
- The frequency ω. This determines the period $T := 2\pi/|\omega|$ of the wave motion.

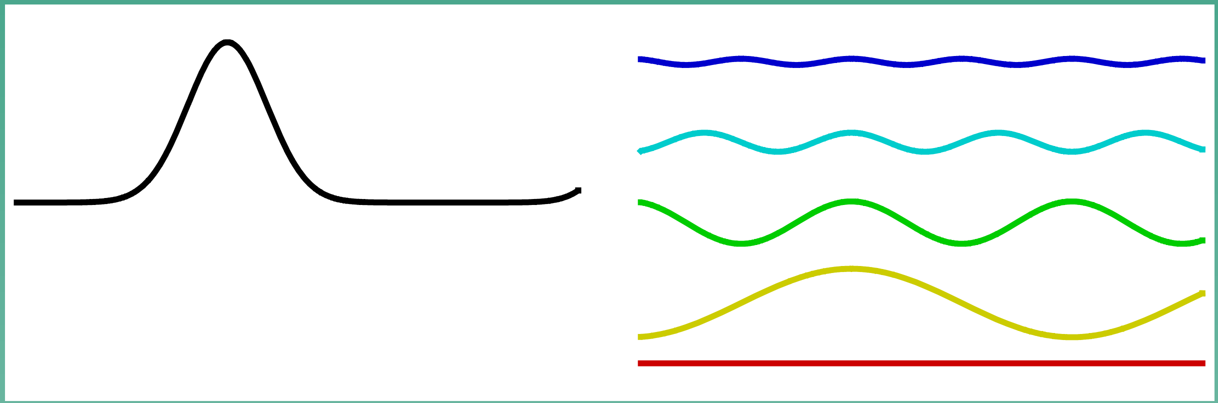
The wave propagates rigidly at the *phase velocity* $v = \omega/k$, which in this case is $\pm c$. It is a *traveling wave*.

What is . . . a wave?

The importance of solutions of the form $u = Ae^{i(kx-\omega t)}$ goes beyond the fact that they represent periodic traveling waves. Since the wave equation is a linear homogeneous equation, it obeys the *superposition principle*:

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{and} \quad v_{tt} - c^2 v_{xx} = 0 \quad \text{implies} \quad (u+v)_{tt} - c^2 (u+v)_{xx} = 0\,.$$

In this way, the simple traveling wave solutions may be combined by superposition to form more complicated solutions, often called "wavepackets".



What is . . . dispersion?

For the more accurate modeling of vibrations in various media, the wave equation becomes insufficient. The feature of its periodic traveling wave solutions that is most glaringly incorrect in many applications is that they all travel with the same speed $|c|$.

What is . . . dispersion?

For the more accurate modeling of vibrations in various media, the wave equation becomes insufficient. The feature of its periodic traveling wave solutions that is most glaringly incorrect in many applications is that they all travel with the same speed $|c|$.

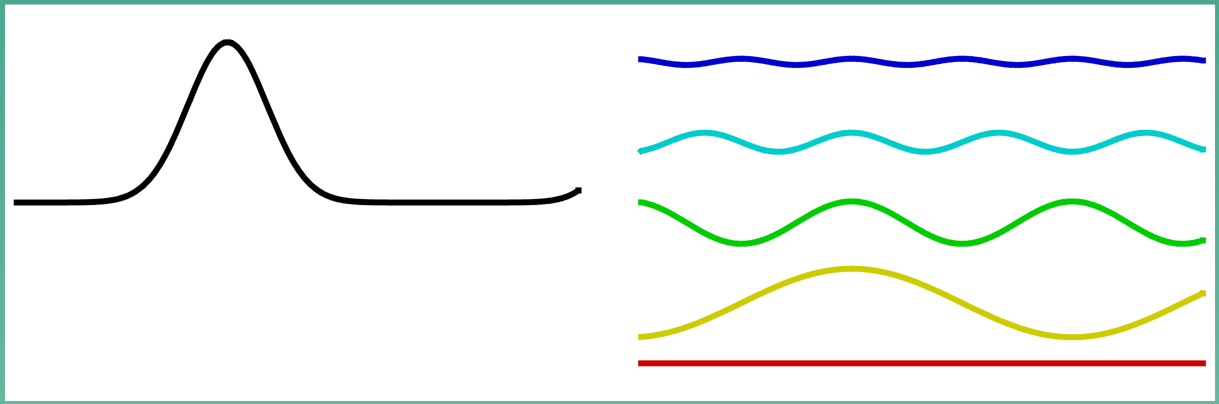
For example, in the theory of surface waves driven by gravity on deep water of depth h, it was known since about 1840 (due to Airy) that the frequency of such waves was not just a constant multiple of the wavenumber k. The correct formula is in fact:

$$\omega^2 = gk\tanh(kh)\,, \quad g = 9.8\mathrm{m/s}^2\,.$$

This means that the phase velocity $v = \omega/k$ varies with the wavenumber k for ocean waves (shorter waves travel more slowly on the ocean).

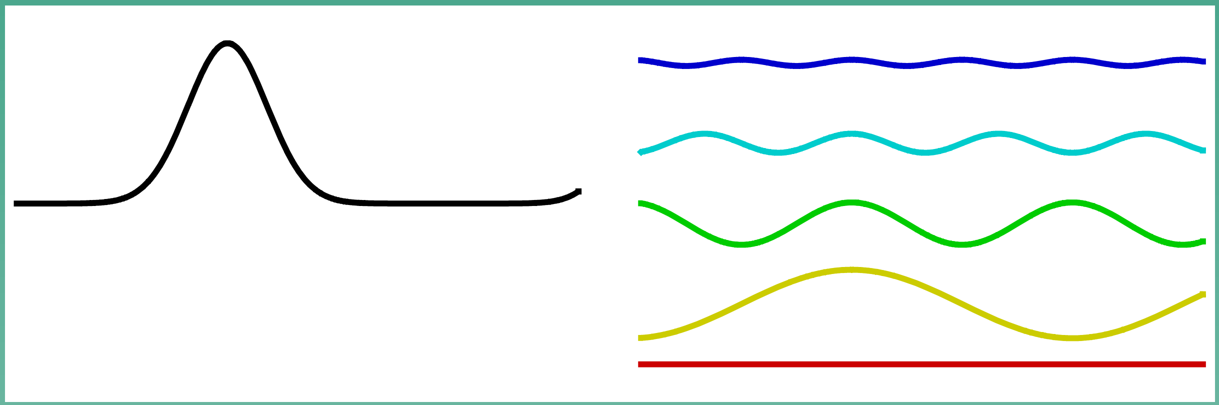
What is . . . dispersion?

It is easy to derive from physical laws simple equations that like the wave equation are low-order linear, but that exhibit traveling wave solutions with variable phase velocity.



What is . . . dispersion?

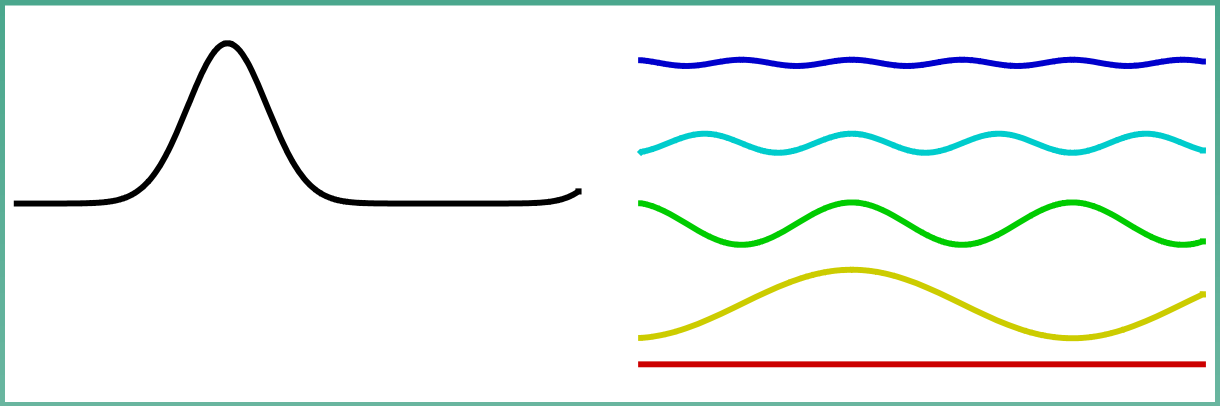
It is easy to derive from physical laws simple equations that like the wave equation are low-order linear, but that exhibit traveling wave solutions with variable phase velocity. For example, $u_t - u_{xxx} = 0$ has solutions $u(x,t) = Ae^{i(kx-\omega t)}$ with $\omega = k^3$, so the phase velocity is $v = v(k) = k^2$.



What is . . . dispersion?

It is easy to derive from physical laws simple equations that like the wave equation are low-order linear, but that exhibit traveling wave solutions with variable phase velocity. For example, $u_t - u_{xxx} = 0$ has solutions $u(x,t) = Ae^{i(kx-\omega t)}$ with $\omega = k^3$, so the phase velocity is $v = v(k) = k^2$.

The dependence of phase velocity on wavenumber is called *dispersion*. This is because while the superposition principle applies to linear dispersive wave equations, relative motion of the wave components leads to distortion of wavepackets:



Who was . . . John Scott-Russell?

The phenomenon of wave dispersion was well-known by the early 19th century. One of the key principles of linear dispersive wave theory is that the only *traveling waves*, i.e. solutions of the form $u(x,t) = F(x - vt)$ for some velocity v, are spatially extended (in particular, periodic in x). The whole subject was driven by the analysis of periodic traveling waves (AKA wavetrains) and the dispersion of wavepackets.

Who was . . . John Scott-Russell?

The phenomenon of wave dispersion was well-known by the early 19th century. One of the key principles of linear dispersive wave theory is that the only *traveling waves*, i.e. solutions of the form $u(x,t) = F(x - vt)$ for some velocity v, are spatially extended (in particular, periodic in x). The whole subject was driven by the analysis of periodic traveling waves (AKA wavetrains) and the dispersion of wavepackets.

John Scott-Russell (1808–1882) was a Scottish engineer specializing in water waves and their influence on boats. In 1834 he made an accidental discovery that would change the theory of waves forever. He observed a surface water wave in the Union Canal between Edinburgh and Glasgow that appearred to be a *spatially localized* traveling wave. Given his expertise with the existing wave theory, he was very, very surprised; so much that his excitement is still clear in this account written ten years later:

Who was . . . John Scott-Russell?

> I believe I shall best introduce the phaenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

John Scott-Russell, "Report on Waves" to the British Association, 1844

Who was . . . John Scott-Russell?

Here is a recent re-creation of Scott-Russell's "solitary wave" in the Scott-Russell Aqueduct of the Union Canal, named in the honor of this portentious observation:

Who were . . . Korteweg and de Vries?

One of the most obvious features of Scott-Russell's "solitary wave" is its enormous size. The wave height is a significant fraction of the channel depth.

D. J. Korteweg and G. de Vries, *Philos. Mag. Ser. 5*, **39**, 422–443, 1895. Also due independently to Boussinesq, about 20 years earlier.

Who were . . . Korteweg and de Vries?

One of the most obvious features of Scott-Russell's "solitary wave" is its enormous size. The wave height is a significant fraction of the channel depth.

This suggests that a mathematical model capable of reproducing "solitary waves" should be nonlinear: a higher wave would feel the effect of the bottom more, so any equation modeling such large waves should *not* be invariant under scaling $u \longrightarrow \alpha u$.

D. J. Korteweg and G. de Vries, *Philos. Mag. Ser. 5*, **39**, 422–443, 1895. Also due independently to Boussinesq, about 20 years earlier.

Who were . . . Korteweg and de Vries?

One of the most obvious features of Scott-Russell's "solitary wave" is its enormous size. The wave height is a significant fraction of the channel depth.

This suggests that a mathematical model capable of reproducing "solitary waves" should be nonlinear: a higher wave would feel the effect of the bottom more, so any equation modeling such large waves should *not* be invariant under scaling $u \longrightarrow \alpha u$.

In 1895, D. J. Korteweg and G. de Vries published a paper in which they derived from the physical laws governing water wave motion in a channel the following equation governing the height $u(x, t)$ of a disturbance:

$$u_t + uu_x + u_{xxx} = 0 \, .$$

D. J. Korteweg and G. de Vries, *Philos. Mag. Ser. 5*, **39**, 422–443, 1895.
Also due independently to Boussinesq, about 20 years earlier.

Who were . . . Korteweg and de Vries?

One of the most obvious features of Scott-Russell's "solitary wave" is its enormous size. The wave height is a significant fraction of the channel depth.

This suggests that a mathematical model capable of reproducing "solitary waves" should be nonlinear: a higher wave would feel the effect of the bottom more, so any equation modeling such large waves should *not* be invariant under scaling $u \longrightarrow \alpha u$.

In 1895, D. J. Korteweg and G. de Vries published a paper in which they derived from the physical laws governing water wave motion in a channel the following equation governing the height $u(x, t)$ of a disturbance:

$$u_t + uu_x + u_{xxx} = 0\,.$$

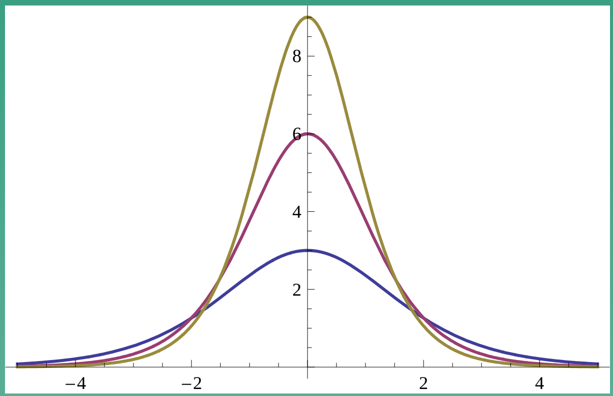
This nonlinear equation is now called the *Korteweg-de Vries equation* or, KdV for short.

D. J. Korteweg and G. de Vries, *Philos. Mag. Ser. 5*, **39**, 422–443, 1895.
Also due independently to Boussinesq, about 20 years earlier.

Who were . . . Korteweg and de Vries?

In their paper, Korteweg and de Vries noted that for any speed $v > 0$ their equation admits localized traveling wave solutions of the exact form

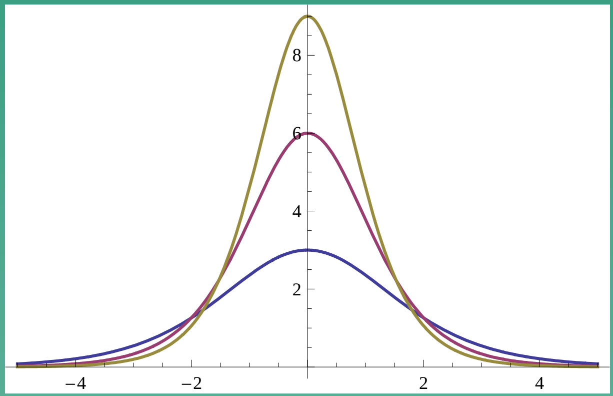
$$u(x,t) = 3v\operatorname{sech}^2\left(\frac{\sqrt{v}}{2}(x - x_0 - vt)\right) .$$



Who were . . . Korteweg and de Vries?

In their paper, Korteweg and de Vries noted that for any speed $v > 0$ their equation admits localized traveling wave solutions of the exact form

$$u(x,t) = 3v\text{sech}^2\left(\frac{\sqrt{v}}{2}(x - x_0 - vt)\right) .$$



They also made the observation that this formula has a shape similar to the "solitary wave" shape described by Scott-Russell.

What is . . . a solitary wave?

On one level, a solitary wave is simply a localized traveling-wave solution of a nonlinear wave equation.

What is . . . a solitary wave?

On one level, a solitary wave is simply a localized traveling-wave solution of a nonlinear wave equation. For example, for the equation $u_t + u^3 u_x + u_{xxx} = 0$ we seek a solution of the form $u = f(x - vt)$ where v is a wave speed parameter. Thus,

$$-vf' + f^3 f' + f''' = 0\,, \qquad f' := \frac{df}{d\xi}\,, \quad \xi = x - vt\,.$$

What is . . . a solitary wave?

On one level, a solitary wave is simply a localized traveling-wave solution of a nonlinear wave equation. For example, for the equation $u_t + u^3 u_x + u_{xxx} = 0$ we seek a solution of the form $u = f(x - vt)$ where v is a wave speed parameter. Thus,

$$-vf' + f^3 f' + f''' = 0\,, \qquad f' := \frac{df}{d\xi}\,, \quad \xi = x - vt\,.$$

Integrate once using f and f'' tending to zero as $\xi \to \infty$ (for localization):

$$f'' + \tfrac{1}{4} f^4 - vf = 0\,.$$

What is . . . a solitary wave?

On one level, a solitary wave is simply a localized traveling-wave solution of a nonlinear wave equation. For example, for the equation $u_t + u^3 u_x + u_{xxx} = 0$ we seek a solution of the form $u = f(x - vt)$ where v is a wave speed parameter. Thus,

$$-vf' + f^3 f' + f''' = 0\,, \qquad f' := \frac{df}{d\xi}\,, \quad \xi = x - vt\,.$$

Integrate once using f and f'' tending to zero as $\xi \to \infty$ (for localization):

$$f'' + \tfrac{1}{4}f^4 - vf = 0\,.$$

Multiply by f' and integrate once again:

$$\left(\frac{df}{d\xi}\right)^2 = vf^2 - \tfrac{1}{10}f^5\,.$$

What is . . . a solitary wave?

On one level, a solitary wave is simply a localized traveling-wave solution of a nonlinear wave equation. For example, for the equation $u_t + u^3 u_x + u_{xxx} = 0$ we seek a solution of the form $u = f(x - vt)$ where v is a wave speed parameter. Thus,

$$-vf' + f^3 f' + f''' = 0\,, \qquad f' := \frac{df}{d\xi}\,, \quad \xi = x - vt\,.$$

Integrate once using f and f'' tending to zero as $\xi \to \infty$ (for localization):

$$f'' + \tfrac{1}{4} f^4 - vf = 0\,.$$

Multiply by f' and integrate once again:

$$\left(\frac{df}{d\xi}\right)^2 = vf^2 - \tfrac{1}{10} f^5\,.$$

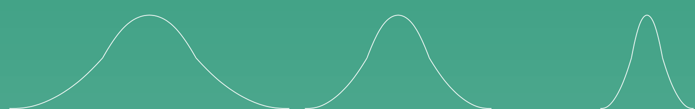
This equation is separable. For any $v \geq 0$, $f(\xi) = (10v)^{1/3} \mathrm{sech}^{2/3}(\frac{3}{2}\sqrt{v}\xi + K)$.

What is . . . a solitary wave?

A solitary wave is a localized traveling wave achieving a dynamical balance between dispersion and nonlinear effects:

What is . . . a solitary wave?

A solitary wave is a localized traveling wave achieving a dynamical balance between dispersion and nonlinear effects:

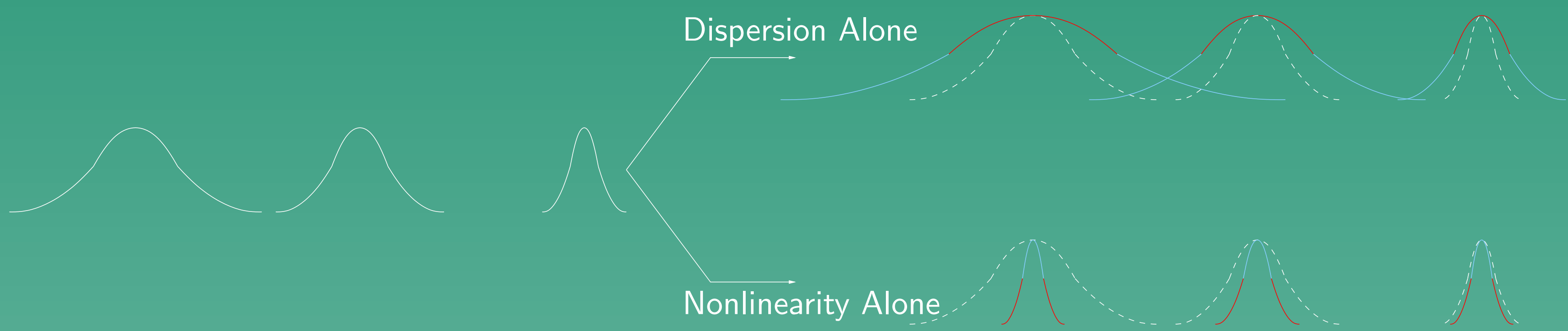


What is . . . a solitary wave?

A solitary wave is a localized traveling wave achieving a dynamical balance between dispersion and nonlinear effects:

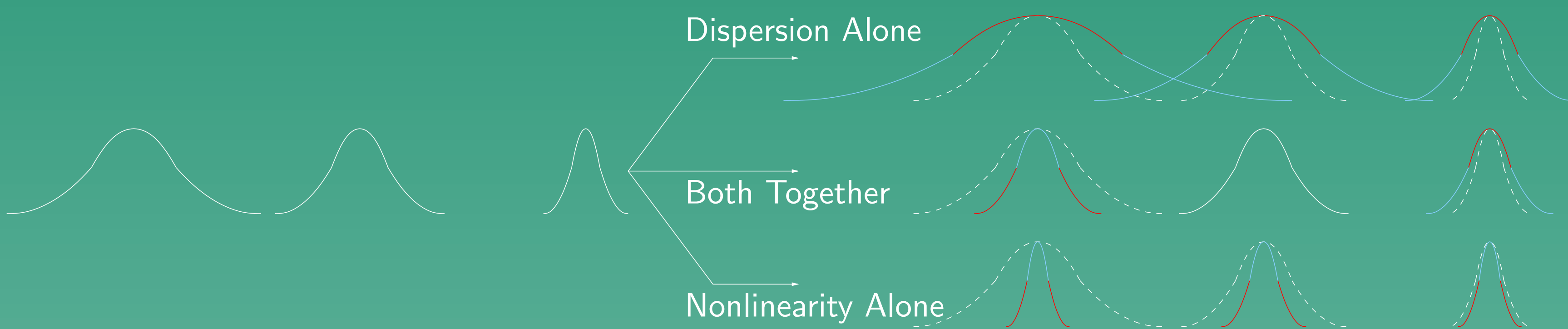
What is . . . a solitary wave?

A solitary wave is a localized traveling wave achieving a dynamical balance between dispersion and nonlinear effects:



What is . . . a solitary wave?

A solitary wave is a localized traveling wave achieving a dynamical balance between dispersion and nonlinear effects:



What is . . . a solitary wave?

As another way to think about the competition of nonlinearity and dispersion leading to the formation of solitary waves, consider a group of kids walking on a sidewalk:

What is . . . a solitary wave?

The kids are all listening to their iPods and not paying any attention to each other as they walk. Naturally, the taller kids take bigger steps and eventually get ahead of the shorter kids. The group disperses:

What is . . . a solitary wave?

But put those same kids on a giant trampoline, or a rubber sidewalk:

What is . . . a solitary wave?

But put those same kids on a giant trampoline, or a rubber sidewalk:

The dispersion is inhibited because now the taller kids have to walk uphill, while the shorter kids get to run downhill! The kids walk as a single collective object.

What is . . . a solitary wave?

But put those same kids on a giant trampoline, or a rubber sidewalk:

The dispersion is inhibited because now the taller kids have to walk uphill, while the shorter kids get to run downhill! The kids walk as a single collective object. This is an essentially nonlinear effect: the size of the deformation is greater if there are more kids.

What is . . . the Fermi-Pasta-Ulam problem?

In the 1950's, E. Fermi, J. Pasta, and S. Ulam observed that the energy *equipartition* principle appears to be at odds with the simplest models for solid materials.

What is . . . the Fermi-Pasta-Ulam problem?

In the 1950's, E. Fermi, J. Pasta, and S. Ulam observed that the energy *equipartition* principle appears to be at odds with the simplest models for solid materials.

The principle of equipartition of energy states that a mechanical system in thermodynamic equilibrium has its energy distributed equally among all of its "normal modes" of vibration.

What is . . . the Fermi-Pasta-Ulam problem?

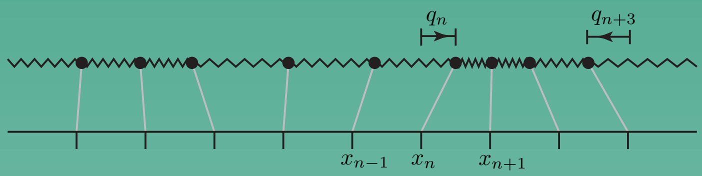
In the 1950's, E. Fermi, J. Pasta, and S. Ulam observed that the energy *equipartition* principle appears to be at odds with the simplest models for solid materials.

The principle of equipartition of energy states that a mechanical system in thermodynamic equilibrium has its energy distributed equally among all of its "normal modes" of vibration.

The simplest atomic model for a solid is the mass-and-spring system is

$$m\frac{d^2q_n}{dt^2} = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1})$$

where q_n is the displacement from equilibrium position x_n of the nth atom, m is the atomic mass, and V is the potential energy of the springs.



What is . . . the Fermi-Pasta-Ulam problem?

If one assumes that Hooke's law holds: $V(x) = \frac{a}{2}x^2$

then the equations of motion are linear, and the normal modes are just the Fourier modes:

$$m\frac{d^2q_n}{dt^2} = a\left(q_{n+1} - 2q_n + q_{n-1}\right) \quad \text{is solved by} \quad q_n(t) = e^{i(kn-\omega t)}$$

where $\omega^2 = 2a(1-\cos(k))/m$. The energy in mode k is proportional to the square of its Fourier coefficient. As these are all constants, unless the system is in equipartition at $t = 0$ it never approaches equipartition!

What is . . . the Fermi-Pasta-Ulam problem?

FPU reasoned that nonlinearity is essential for transport of energy among the normal (Fourier) modes and carried out very simple numerical experiments on the Maniac computer at Los Alamos with the anharmonic potential

$$V(x) = \frac{a}{2}x^2 + \frac{b}{3}x^3 .$$

The numerical experiments were actually performed by a rarely credited woman assistant, Mary Tsingou.

What is . . . the Fermi-Pasta-Ulam problem?

FPU reasoned that nonlinearity is essential for transport of energy among the normal (Fourier) modes and carried out very simple numerical experiments on the Maniac computer at Los Alamos with the anharmonic potential

$$V(x) = \frac{a}{2}x^2 + \frac{b}{3}x^3 .$$

FPU put all of the energy into only one mode at $t = 0$ (one Fourier coefficient nonzero) and observed:

- There was an initial phase of motion in which energy leaked into other Fourier modes.

The numerical experiments were actually performed by a rarely credited woman assistant, Mary Tsingou.

What is . . . the Fermi-Pasta-Ulam problem?

FPU reasoned that nonlinearity is essential for transport of energy among the normal (Fourier) modes and carried out very simple numerical experiments on the Maniac computer at Los Alamos with the anharmonic potential

$$V(x) = \frac{a}{2}x^2 + \frac{b}{3}x^3 .$$

FPU put all of the energy into only one mode at $t = 0$ (one Fourier coefficient nonzero) and observed:

- There was an initial phase of motion in which energy leaked into other Fourier modes.
- But after an unexpectedly short time most of the energy returned to one mode.

The numerical experiments were actually performed by a rarely credited woman assistant, Mary Tsingou.

What is . . . the Fermi-Pasta-Ulam problem?

FPU reasoned that nonlinearity is essential for transport of energy among the normal (Fourier) modes and carried out very simple numerical experiments on the Maniac computer at Los Alamos with the anharmonic potential

$$V(x) = \frac{a}{2}x^2 + \frac{b}{3}x^3 .$$

FPU put all of the energy into only one mode at $t = 0$ (one Fourier coefficient nonzero) and observed:

- There was an initial phase of motion in which energy leaked into other Fourier modes.
- But after an unexpectedly short time most of the energy returned to one mode.

Even with the nonlinearity, equipartition seemed short-lived.

The numerical experiments were actually performed by a rarely credited woman assistant, Mary Tsingou.

What is . . . the Zabusky-Kruskal experiment?

In 1965, Norman Zabusky and Martin Kruskal observed that by taking a suitable continuum limit of the Fermi-Pasta-Ulam chain with cubic potential, the displacement $q_n(t)$ is approximated by a function $\epsilon q(x,t)$ where $x = \epsilon t$ and ϵ is the equilibrium particle spacing, and $u(x,t) = q_x(x,t)$ satisfies (approximately) the KdV equation in the form

$$u_t + uu_x + \epsilon^2 u_{xxx} = 0\,.$$

What is . . . the Zabusky-Kruskal experiment?

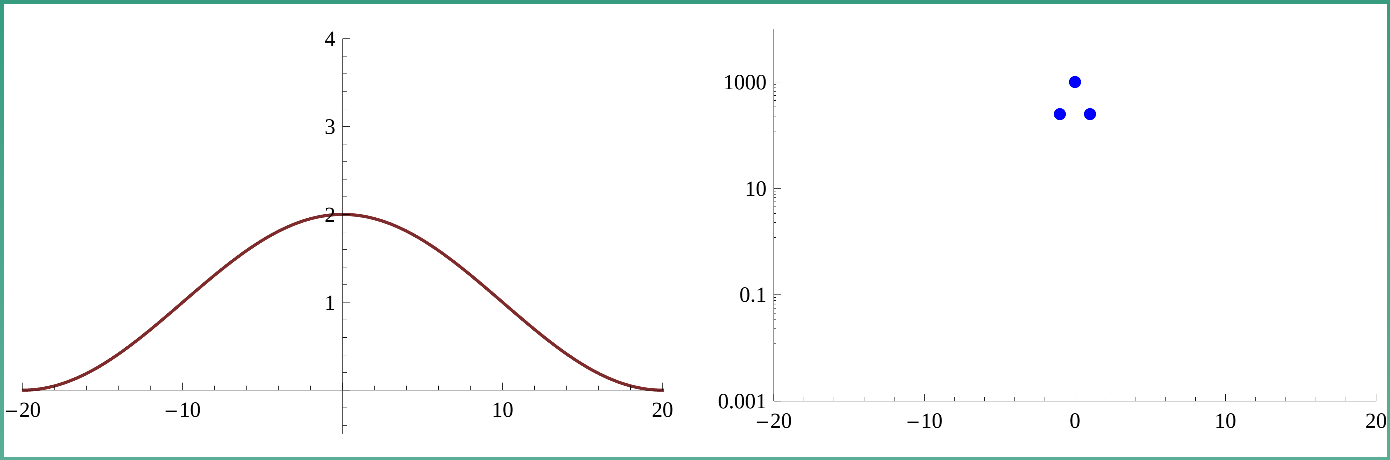
In 1965, Norman Zabusky and Martin Kruskal observed that by taking a suitable continuum limit of the Fermi-Pasta-Ulam chain with cubic potential, the displacement $q_n(t)$ is approximated by a function $\epsilon q(x,t)$ where $x = \epsilon t$ and ϵ is the equilibrium particle spacing, and $u(x,t) = q_x(x,t)$ satisfies (approximately) the KdV equation in the form

$$u_t + uu_x + \epsilon^2 u_{xxx} = 0\,.$$

Incidentally, this fact illustrates the *universality* of the KdV equation. The same equation arises from many different physical settings by taking an appropriate asymptotic limit.

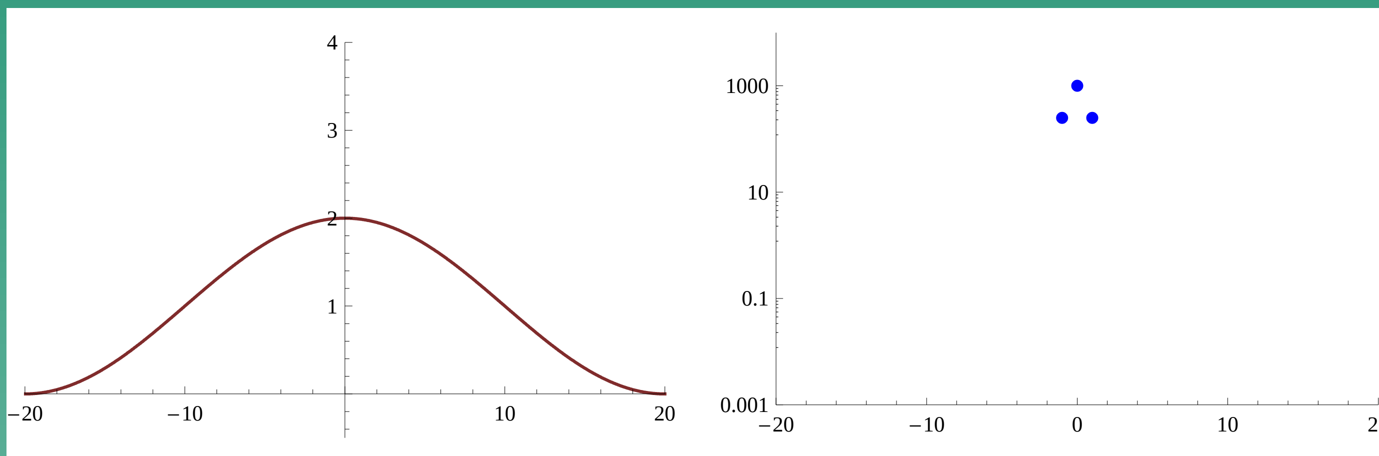
What is . . . the Zabusky-Kruskal experiment?

Zabusky and Kruskal wanted to see if they could observe the FPU recurrence in a numerical simulation of the initial-value problem for the KdV equation. Here is their simulation with $\epsilon = 2/\sqrt{6} \approx 0.816$ and initial data $u(x,0) = 1 + \cos(\pi x/20)$.



What is . . . the Zabusky-Kruskal experiment?

Zabusky and Kruskal wanted to see if they could observe the FPU recurrence in a numerical simulation of the initial-value problem for the KdV equation. Here is their simulation with $\epsilon = 2/\sqrt{6} \approx 0.816$ and initial data $u(x,0) = 1 + \cos(\pi x/20)$.



Note that the FPU recurrence effect is also obvious in the KdV simulation.

What is . . . the Zabusky-Kruskal experiment?

The following features were rather less expected:

- The waves that are generated from the breaking all have the correct shape and speed/height relationship as the solitary wave solutions of KdV:

$$u(x,t) = 3v\operatorname{sech}^2\left(\frac{\sqrt{v}}{2\epsilon}(x - vt)\right) .$$

What is . . . the Zabusky-Kruskal experiment?

The following features were rather less expected:

- The waves that are generated from the breaking all have the correct shape and speed/height relationship as the solitary wave solutions of KdV:

$$u(x,t) = 3v\mathrm{sech}^2\left(\frac{\sqrt{v}}{2\epsilon}(x - vt)\right) .$$

- The solitary waves appear to survive interactions with one another.

What is . . . the Zabusky-Kruskal experiment?

The following features were rather less expected:

- The waves that are generated from the breaking all have the correct shape and speed/height relationship as the solitary wave solutions of KdV:

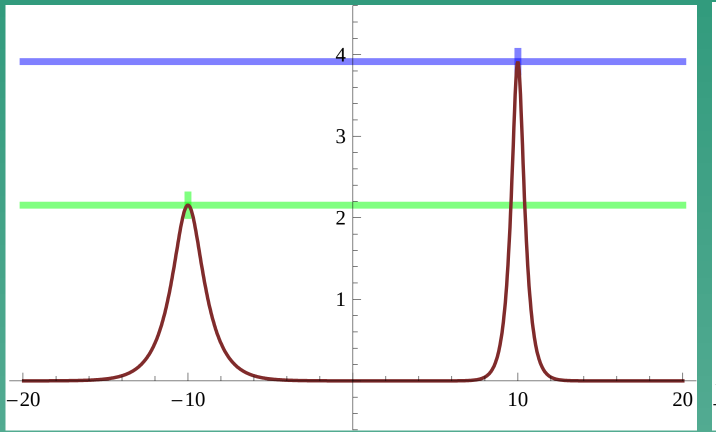
$$u(x,t) = 3v\operatorname{sech}^2\left(\frac{\sqrt{v}}{2\epsilon}(x-vt)\right) .$$

- The solitary waves appear to survive interactions with one another.

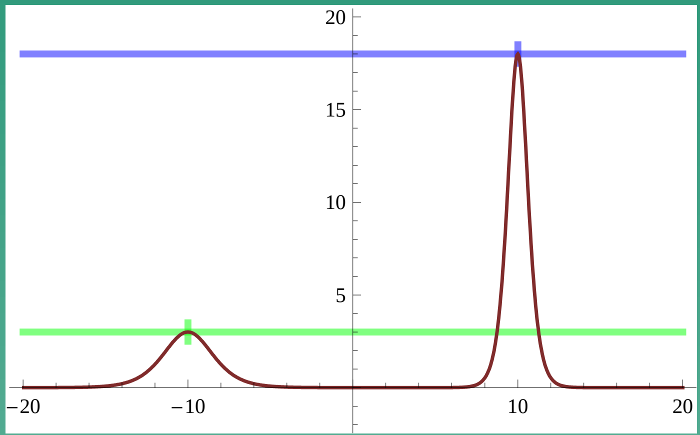
This latter feature is not to be expected from a nonlinear equation (no superposition principle). Since the KdV solitary waves seem to have identity and behave as indestructible particles, Zabusky and Kruskal named them *solitons*.

What is . . . the Zabusky-Kruskal experiment?

To give a better idea of what distinguishes solitons from solitary waves, compare these two simulations.



$$u_t + u^3 u_x + u_{xxx} = 0$$



$$u_t + u u_x + u_{xxx} = 0$$

Unlike solitons, solitary waves generally do not survive collisions with one another (completely intact, at least). The KdV equation is apparently special among nonlinear dispersive wave equations supporting solitary waves.

Hamiltonian Structure of KdV

The KdV equation can be formulated as a Hamiltonian system on the phase space of fields $u(x)$:

$$\frac{\partial u}{\partial t} = \mathscr{J} \frac{\delta H}{\delta u}$$

where the Hamiltonian operator $\mathscr{J}$ and Hamiltonian functional H are

$$\mathscr{J} := \frac{\partial}{\partial x}\,, \qquad H[u] := -\int_{-\infty}^{+\infty} \left[\frac{1}{6}u(x)^3 + \frac{1}{2}u(x)u_{xx}(x)\right]\,dx\,.$$

Hamiltonian Structure of KdV

The KdV equation can be formulated as a Hamiltonian system on the phase space of fields $u(x)$:

$$\frac{\partial u}{\partial t} = \mathscr{J} \frac{\delta H}{\delta u}$$

where the Hamiltonian operator $\mathscr{J}$ and Hamiltonian functional H are

$$\mathscr{J} := \frac{\partial}{\partial x} , \qquad H[u] := -\int_{-\infty}^{+\infty} \left[\frac{1}{6} u(x)^3 + \frac{1}{2} u(x) u_{xx}(x) \right] dx \, .$$

This is an infinite-dimensional analogue of the classical situation: $\dfrac{du}{dt} = \mathscr{J} \nabla_u H$, where

$$u = (q_1, \ldots, q_N, p_1, \ldots, p_N)^T \in \mathbb{R}^{2N} , \quad \mathscr{J} := \begin{bmatrix} \mathbf{0} & \mathbb{I}_N \\ -\mathbb{I}_N & \mathbf{0} \end{bmatrix} , \quad H = H(u) \, .$$

Hamiltonian Structure of KdV

According to the Liouville-Arnold Theorem, a Hamiltonian system on $\mathbb{R}^{2N}$ is *completely integrable* if there exist N "sufficiently independent" functions of u whose values remain constant as u evolves. Complete integrability implies a canonical map to "action-angle variables" in terms of which the initial-value problem can be solved explicitly.

Hamiltonian Structure of KdV

According to the Liouville-Arnold Theorem, a Hamiltonian system on $\mathbb{R}^{2N}$ is *completely integrable* if there exist N "sufficiently independent" functions of u whose values remain constant as u evolves. Complete integrability implies a canonical map to "action-angle variables" in terms of which the initial-value problem can be solved explicitly.

The Zabusky-Kruskal experiment suggests that there could be a large number of independent constants of motion for the KdV equation: the amplitudes of the solitons.

Hamiltonian Structure of KdV

According to the Liouville-Arnold Theorem, a Hamiltonian system on $\mathbb{R}^{2N}$ is *completely integrable* if there exist N "sufficiently independent" functions of u whose values remain constant as u evolves. Complete integrability implies a canonical map to "action-angle variables" in terms of which the initial-value problem can be solved explicitly.

The Zabusky-Kruskal experiment suggests that there could be a large number of independent constants of motion for the KdV equation: the amplitudes of the solitons.

A mini-industry sprung up in the late 1960's to seek conserved quantities. For example,

$$F_1[u] := \int_{-\infty}^{+\infty} u(x)\, dx\,, \quad F_2[u] := \int_{-\infty}^{+\infty} u(x)^2\, dx\,, \quad F_3[u] := H[u]$$

are conserved by KdV. Several more were soon found.

Hamiltonian Structure of KdV

According to the Liouville-Arnold Theorem, a Hamiltonian system on $\mathbb{R}^{2N}$ is *completely integrable* if there exist N "sufficiently independent" functions of u whose values remain constant as u evolves. Complete integrability implies a canonical map to "action-angle variables" in terms of which the initial-value problem can be solved explicitly.

The Zabusky-Kruskal experiment suggests that there could be a large number of independent constants of motion for the KdV equation: the amplitudes of the solitons.

A mini-industry sprung up in the late 1960's to seek conserved quantities. For example,

$$F_1[u] := \int_{-\infty}^{+\infty} u(x)\,dx\,, \quad F_2[u] := \int_{-\infty}^{+\infty} u(x)^2\,dx\,, \quad F_3[u] := H[u]$$

are conserved by KdV. Several more were soon found. Eventually, a pattern emerged, and it was shown that KdV conserves an infinite number of functionals.

Hamiltonian Structure of KdV

According to the Liouville-Arnold Theorem, a Hamiltonian system on $\mathbb{R}^{2N}$ is *completely integrable* if there exist N "sufficiently independent" functions of u whose values remain constant as u evolves. Complete integrability implies a canonical map to "action-angle variables" in terms of which the initial-value problem can be solved explicitly.

The Zabusky-Kruskal experiment suggests that there could be a large number of independent constants of motion for the KdV equation: the amplitudes of the solitons.

A mini-industry sprung up in the late 1960's to seek conserved quantities. For example,

$$F_1[u] := \int_{-\infty}^{+\infty} u(x)\, dx \,, \quad F_2[u] := \int_{-\infty}^{+\infty} u(x)^2\, dx \,, \quad F_3[u] := H[u]$$

are conserved by KdV. Several more were soon found. Eventually, a pattern emerged, and it was shown that KdV conserves an infinite number of functionals.

So could KdV be an infinite-dimensional analogue of a Liouville-Arnold integrable system? If so, how could KdV be solved? What could the action-angle variables be?

Burgers' Equation and the Cole-Hopf Transformation

A nonlinear equation that superficially resembles KdV is Burgers' equation:

$$u_t + uu_x - 3u_{xx} = 0\,.$$

Here, instead of dispersion, we have diffusion to balance nonlinear effects.

Burgers' Equation and the Cole-Hopf Transformation

A nonlinear equation that superficially resembles KdV is Burgers' equation:

$$u_t + uu_x - 3u_{xx} = 0\,.$$

Here, instead of dispersion, we have diffusion to balance nonlinear effects.

The *Cole-Hopf transformation* is the substitution $u = -6\log(\psi)_x$ which converts Burgers' equation (nonlinear) into the heat equation (linear):

$$\psi_t - 3\psi_{xx} = 0\,.$$

Burgers' Equation and the Cole-Hopf Transformation

A nonlinear equation that superficially resembles KdV is Burgers' equation:

$$u_t + uu_x - 3u_{xx} = 0\,.$$

Here, instead of dispersion, we have diffusion to balance nonlinear effects.

The *Cole-Hopf transformation* is the substitution $u = -6\log(\psi)_x$ which converts Burgers' equation (nonlinear) into the heat equation (linear):

$$\psi_t - 3\psi_{xx} = 0\,.$$

This effectively solves Burgers' equation, since the initial-value problem for the heat equation can be solved by a Fourier transform.

The Breakthrough: Spectral Theory for Schrödinger Operators

Since KdV involves more derivatives, perhaps instead of

$$u = -6\frac{\psi_x}{\psi} \quad \text{(Cole-Hopf)} \qquad \text{one should try} \qquad u = -6\frac{\psi_{xx}}{\psi}\,.$$

Or, since a symmetry of KdV is $u \mapsto u + E$ corresponding to a Galilean boost by velocity E, a more appropriate substitution might be

$$u + E = -6\frac{\psi_{xx}}{\psi}\,.$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

Since KdV involves more derivatives, perhaps instead of

$$u = -6\frac{\psi_x}{\psi} \quad \text{(Cole-Hopf)} \qquad \text{one should try} \qquad u = -6\frac{\psi_{xx}}{\psi}\,.$$

Or, since a symmetry of KdV is $u \mapsto u + E$ corresponding to a Galilean boost by velocity E, a more appropriate substitution might be

$$u + E = -6\frac{\psi_{xx}}{\psi}\,.$$

In fact, this substitution does not lead to a linear equation for ψ.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

Since KdV involves more derivatives, perhaps instead of

$$u = -6\frac{\psi_x}{\psi} \quad \text{(Cole-Hopf)} \qquad \text{one should try} \qquad u = -6\frac{\psi_{xx}}{\psi}\,.$$

Or, since a symmetry of KdV is $u \mapsto u + E$ corresponding to a Galilean boost by velocity E, a more appropriate substitution might be

$$u + E = -6\frac{\psi_{xx}}{\psi}\,.$$

In fact, this substitution does not lead to a linear equation for ψ.

In 1967, Gardner, Greene, Kruskal, and Miura thought to rewrite the "substitution" as

$$-6\psi_{xx} - u\psi = E\psi\,,$$

which is the Schrödinger equation for the wavefunction of a quantum particle with energy eigenvalue E moving in a potential $V(x) = -u(x)$.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

GGKM considered calculating well-known *scattering data* for this problem consisting of:

- discrete eigenvalues $E = E_j = -\kappa_j^2 < 0$, $j = 1, \ldots, n$.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

GGKM considered calculating well-known *scattering data* for this problem consisting of:

- discrete eigenvalues $E = E_j = -\kappa_j^2 < 0$, $j = 1, \ldots, n$.
- norming constants c_j associated to corresponding "bound state" eigenfunctions $\psi_j(x)$ normalized so that $\|\psi_j\| = 1$: $\psi_j(x) = c_j e^{-\kappa_j x/\sqrt{6}}(1 + o(1))$ as $x \to +\infty$.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

GGKM considered calculating well-known *scattering data* for this problem consisting of:

- discrete eigenvalues $E = E_j = -\kappa_j^2 < 0$, $j = 1, \ldots, n$.
- norming constants c_j associated to corresponding "bound state" eigenfunctions $\psi_j(x)$ normalized so that $\|\psi_j\| = 1$: $\psi_j(x) = c_j e^{-\kappa_j x/\sqrt{6}}(1 + o(1))$ as $x \to +\infty$.
- reflection coefficient $r(k)$ for $E = k^2 > 0$ corresponding to "scattering" solutions satisfying

$$\psi(x) = e^{ikx/\sqrt{6}} + r(k)e^{-ikx/\sqrt{6}} + o(1)\,, \quad x \to -\infty$$

and

$$\psi(x) = t(k)e^{ikx/\sqrt{6}} + o(1)\,, \quad x \to +\infty$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

GGKM considered calculating well-known *scattering data* for this problem consisting of:

- discrete eigenvalues $E = E_j = -\kappa_j^2 < 0$, $j = 1, \ldots, n$.
- norming constants c_j associated to corresponding "bound state" eigenfunctions $\psi_j(x)$ normalized so that $\|\psi_j\| = 1$: $\psi_j(x) = c_j e^{-\kappa_j x/\sqrt{6}}(1 + o(1))$ as $x \to +\infty$.
- reflection coefficient $r(k)$ for $E = k^2 > 0$ corresponding to "scattering" solutions satisfying

$$\psi(x) = e^{ikx/\sqrt{6}} + r(k)e^{-ikx/\sqrt{6}} + o(1)\,, \quad x \to -\infty$$

  and

$$\psi(x) = t(k)e^{ikx/\sqrt{6}} + o(1)\,, \quad x \to +\infty$$

If the potential $V(x) = -u(x)$ is not fixed but depends on a parameter t as u evolves according to the KdV equation, one expects that the *scattering data* $\mathcal{S} = (\{E_j\}, \{c_j\}, r(\cdot))$ will also vary with t.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

The miracle found by GGKM is this: by direct calculation, when $u = u(x,t)$ solves KdV, the scattering data $\mathcal{S} = \mathcal{S}(t)$ evolves in a very simple manner:

- The discrete eigenvalues are constants of motion: $E_j(t) = E_j(0)$.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

The miracle found by GGKM is this: by direct calculation, when $u = u(x,t)$ solves KdV, the scattering data $\mathcal{S} = \mathcal{S}(t)$ evolves in a very simple manner:

- The discrete eigenvalues are constants of motion: $E_j(t) = E_j(0)$.
- The norming constants obey $c_j(t) = c_j(0)e^{A\kappa_j^3 t}$ for some overall constant $A \in \mathbb{R}$.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

The miracle found by GGKM is this: by direct calculation, when $u = u(x, t)$ solves KdV, the scattering data $\mathcal{S} = \mathcal{S}(t)$ evolves in a very simple manner:

- The discrete eigenvalues are constants of motion: $E_j(t) = E_j(0)$.
- The norming constants obey $c_j(t) = c_j(0)e^{A\kappa_j^3 t}$ for some overall constant $A \in \mathbb{R}$.
- The reflection coefficient satisfies $r(k;t) = r(k;0)e^{iBk^3 t}$ for some overall constant $B \in \mathbb{R}$.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

The miracle found by GGKM is this: by direct calculation, when $u = u(x,t)$ solves KdV, the scattering data $\mathcal{S} = \mathcal{S}(t)$ evolves in a very simple manner:

- The discrete eigenvalues are constants of motion: $E_j(t) = E_j(0)$.
- The norming constants obey $c_j(t) = c_j(0)e^{A\kappa_j^3 t}$ for some overall constant $A \in \mathbb{R}$.
- The reflection coefficient satisfies $r(k;t) = r(k;0)e^{iBk^3 t}$ for some overall constant $B \in \mathbb{R}$.

Moreover, it turns out that the discrete eigenvalues correspond to solitons in the solution. Even if they are not obvious in the potential $V(x) = -u(x,t)$ at a given time t, they become visible as $t \to \pm\infty$ as they separate from one another and from "radiative" components of $u(x,t)$ associated with the continuous spectrum.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Breakthrough: Spectral Theory for Schrödinger Operators

The miracle found by GGKM is this: by direct calculation, when $u = u(x,t)$ solves KdV, the scattering data $\mathcal{S} = \mathcal{S}(t)$ evolves in a very simple manner:

- The discrete eigenvalues are constants of motion: $E_j(t) = E_j(0)$.
- The norming constants obey $c_j(t) = c_j(0)e^{A\kappa_j^3 t}$ for some overall constant $A \in \mathbb{R}$.
- The reflection coefficient satisfies $r(k;t) = r(k;0)e^{iBk^3 t}$ for some overall constant $B \in \mathbb{R}$.

Moreover, it turns out that the discrete eigenvalues correspond to solitons in the solution. Even if they are not obvious in the potential $V(x) = -u(x,t)$ at a given time t, they become visible as $t \to \pm\infty$ as they separate from one another and from "radiative" components of $u(x,t)$ associated with the continuous spectrum. Thus KdV is an isospectral flow for the Schrödinger operator, and a soliton is . . . an eigenvalue of the Schrödinger operator.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

To solve the Cauchy initial-value problem for KdV:

$$\boxed{u(x,0)} \xrightarrow{\;u_t + uu_x + u_{xxx} = 0\;} \boxed{u(x,t)}$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

To solve the Cauchy initial-value problem for KdV:

$$\begin{tikzcd}[column sep=huge, row sep=huge] u(x,0) \arrow[r, dashed, "u_t + uu_x + u_{xxx} = 0"] \arrow[d, "\substack{\text{Direct Scattering} \\ \mathscr{S}}"'] & u(x,t) \\ \mathcal{S}(0) & \end{tikzcd}$$

Scattering map $\mathscr{S}$ calculated via spectral analysis of Schrödinger operator with potential $V(x) = -u(x,0)$.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

To solve the Cauchy initial-value problem for KdV:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{u(x,0)} \arrow[r, dashed, "u_t + uu_x + u_{xxx} = 0"] \arrow[d, "\substack{\text{Direct Scattering} \\ \mathscr{S}}"'] & \boxed{u(x,t)} \\
\boxed{\mathcal{S}(0)} \arrow[r, "e^{A\kappa_j^3 t},\quad e^{iBk^3 t}"] & \boxed{\mathcal{S}(t)}
\end{tikzcd}
$$

Explicit and simple time evolution of the scattering data $\mathcal{S}(t)$ as $u(x,t)$ evolves according to KdV.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

To solve the Cauchy initial-value problem for KdV:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{u(x,0)} \arrow[r, dashed, "u_t + uu_x + u_{xxx} = 0"] \arrow[d, "\substack{\text{Direct Scattering} \\ \mathscr{S}}"'] & \boxed{u(x,t)} \\
\boxed{\mathcal{S}(0)} \arrow[r, "{e^{A\kappa_j^3 t},\quad e^{iBk^3 t}}"] & \boxed{\mathcal{S}(t)} \arrow[u, "\substack{\text{Inverse Scattering} \\ \mathscr{S}^{-1}}"']
\end{tikzcd}
$$

Construction of the inverse scattering map $\mathscr{S}^{-1}$ was a previously solved problem: Gelfand, Levitan, and Marchenko in the 1950's.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

A special case occurs when the initial data is *reflectionless*, i.e. $r(k) \equiv 0$, $k \in \mathbb{R}$.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

A special case occurs when the initial data is *reflectionless*, i.e. $r(k) \equiv 0$, $k \in \mathbb{R}$.

- This condition is preserved under time evolution.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

A special case occurs when the initial data is *reflectionless*, i.e. $r(k) \equiv 0$, $k \in \mathbb{R}$.

- This condition is preserved under time evolution.
- The only relevant data is discrete: $\{\kappa_j\}_{j=1}^n$ and $\{c_j\}_{j=1}^n$.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

A special case occurs when the initial data is *reflectionless*, i.e. $r(k) \equiv 0$, $k \in \mathbb{R}$.

- This condition is preserved under time evolution.
- The only relevant data is discrete: $\{\kappa_j\}_{j=1}^{n}$ and $\{c_j\}_{j=1}^{n}$.
- Moreover, the inverse scattering map $\mathscr{S}^{-1}$ reduces to a problem in n-dimensional linear algebra.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

A special case occurs when the initial data is *reflectionless*, i.e. $r(k) \equiv 0$, $k \in \mathbb{R}$.

- This condition is preserved under time evolution.
- The only relevant data is discrete: $\{\kappa_j\}_{j=1}^n$ and $\{c_j\}_{j=1}^n$.
- Moreover, the inverse scattering map $\mathscr{S}^{-1}$ reduces to a problem in n-dimensional linear algebra.

This linear algebra problem can be solved by Cramer's rule.

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

Computing $\mathscr{S}^{-1}$ in the reflectionless case yields an explicit formula for $u(x,t)$ in terms of determinants known as the *Kay-Moses formula* (the $\{x_j\}$ are related to the $\{c_j\}$):

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

Computing $\mathscr{S}^{-1}$ in the reflectionless case yields an explicit formula for $u(x,t)$ in terms of determinants known as the *Kay-Moses formula* (the $\{x_j\}$ are related to the $\{c_j\}$):

$$u(x,t) = 12\frac{\partial^2}{\partial x^2}\log(\tau)\,, \quad \tau := \det\left(\delta_{jk} + \frac{F_j F_k}{\kappa_j + \kappa_k}\right)\,, \quad F_j := e^{\kappa_j(x-x_j)-4\kappa_j^3 t}\,.$$

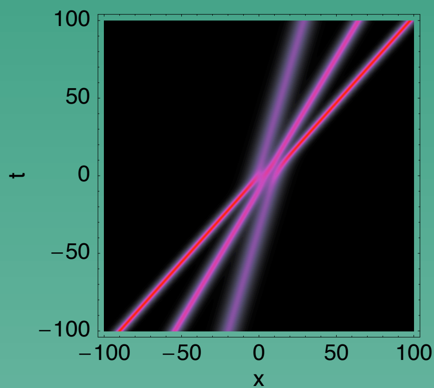
C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

The Inverse Scattering Transform

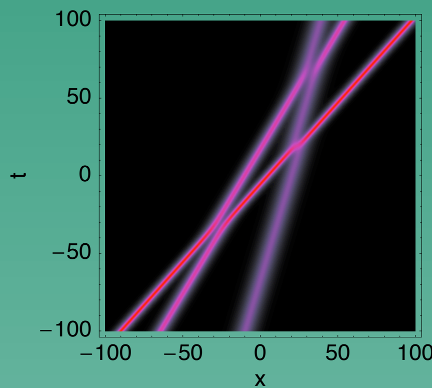
Computing $\mathscr{S}^{-1}$ in the reflectionless case yields an explicit formula for $u(x,t)$ in terms of determinants known as the *Kay-Moses formula* (the $\{x_j\}$ are related to the $\{c_j\}$):

$$u(x,t) = 12\frac{\partial^2}{\partial x^2}\log(\tau)\,, \quad \tau := \det\left(\delta_{jk} + \frac{F_jF_k}{\kappa_j+\kappa_k}\right)\,, \quad F_j := e^{\kappa_j(x-x_j)-4\kappa_j^3 t}\,.$$

E.g. $n=3$ with $\kappa_1 = 0.274$, $\kappa_2 = 0.387$, $\kappa_3 = 0.474$:



$x_1 = x_2 = x_3 = 0$



$x_1 = 10$, $x_2 = -10$, $x_3 = 0$

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.*, **19**, 1095–1097, 1967.

Lax Formalism

In 1968, Peter Lax observed that if one defines linear differential operators by

$$L := -6\frac{d^2}{dx^2} - u\,, \qquad B := -4\frac{d^3}{dx^3} - u\frac{d}{dx} - \frac{1}{2}u_x\,,$$

then the KdV equation is equivalent to the operator equation $\dfrac{dL}{dt} + [L, B] = 0.$

Lax Formalism

In 1968, Peter Lax observed that if one defines linear differential operators by

$$L := -6\frac{d^2}{dx^2} - u\,, \qquad B := -4\frac{d^3}{dx^3} - u\frac{d}{dx} - \frac{1}{2}u_x\,,$$

then the KdV equation is equivalent to the operator equation $\dfrac{dL}{dt} + [L, B] = 0$.

In particular, this implies that the spectrum $\sigma(L)$ is independent of t, and also

$$\frac{d}{dt}(L - \lambda\mathbb{I})^{-1} + [(L - \lambda\mathbb{I})^{-1}, B] = 0\,, \quad \lambda \notin \sigma(L)\,.$$

Lax Formalism

In 1968, Peter Lax observed that if one defines linear differential operators by

$$L := -6\frac{d^2}{dx^2} - u\,, \qquad B := -4\frac{d^3}{dx^3} - u\frac{d}{dx} - \frac{1}{2}u_x\,,$$

then the KdV equation is equivalent to the operator equation $\dfrac{dL}{dt} + [L, B] = 0$.

In particular, this implies that the spectrum $\sigma(L)$ is independent of t, and also

$$\frac{d}{dt}(L - \lambda\mathbb{I})^{-1} + [(L - \lambda\mathbb{I})^{-1}, B] = 0\,, \quad \lambda \notin \sigma(L)\,.$$

Since the resolvent is trace-class, and commutators are traceless,

$$\frac{d}{dt}\mathrm{trace}(L - \lambda\mathbb{I})^{-1} = 0\,, \quad \lambda \notin \sigma(L)\,.$$

This explains the infinite number of constants of motion for KdV: the trace of the resolvent is a generating function for them!

Lax Formalism

The Lax formalism also shows how the mathematical structure underlying the KdV equation and its solitons may be generalized to other equations:

Lax Formalism

The Lax formalism also shows how the mathematical structure underlying the KdV equation and its solitons may be generalized to other equations:

- The operator B could be replaced by another one as long as $[L, B]$ works out to be a multiplication operator. There is an infinite sequence of such B operators, and this gives a hierarchy of integrable equations called the KdV hierarchy. This immediately shows that KdV is not alone!

Lax Formalism

The Lax formalism also shows how the mathematical structure underlying the KdV equation and its solitons may be generalized to other equations:

- The operator B could be replaced by another one as long as $[L, B]$ works out to be a multiplication operator. There is an infinite sequence of such B operators, and this gives a hierarchy of integrable equations called the KdV hierarchy. This immediately shows that KdV is not alone!
- The operator L could be replaced by another one too. For example, in 1971 Zakharov and Shabat took L to be the Dirac operator from quantum mechanics, and by constructing an appropriate sequence of B operators found a new hierarchy of which the nonlinear Schrödinger equation is a member.

Lax Formalism

The Lax formalism also shows how the mathematical structure underlying the KdV equation and its solitons may be generalized to other equations:

- The operator B could be replaced by another one as long as $[L, B]$ works out to be a multiplication operator. There is an infinite sequence of such B operators, and this gives a hierarchy of integrable equations called the KdV hierarchy. This immediately shows that KdV is not alone!
- The operator L could be replaced by another one too. For example, in 1971 Zakharov and Shabat took L to be the Dirac operator from quantum mechanics, and by constructing an appropriate sequence of B operators found a new hierarchy of which the nonlinear Schrödinger equation is a member.
- The operators L and B could be generalized to difference operators which should yield hierarchies of discrete integrable equations. Integral operators lead to integral equations.

The Zoo of Integrable Systems

Here is a very partial list of some integrable systems fitting into this framework.

PDE in 1+1 dimensions:

- KdV: $u_t + uu_x + u_{xxx} = 0$. This is the universal equation governing unidirectional weakly nonlinear dispersive long waves, applicable to: surface and internal water waves, plasma vibrations, lattice vibrations in solid state physics (acoustic phonons), and many other physical problems.

The Zoo of Integrable Systems

Here is a very partial list of some integrable systems fitting into this framework.

PDE in 1+1 dimensions:

- KdV: $u_t + uu_x + u_{xxx} = 0$. This is the universal equation governing unidirectional weakly nonlinear dispersive long waves, applicable to: surface and internal water waves, plasma vibrations, lattice vibrations in solid state physics (acoustic phonons), and many other physical problems.
- Boussinesq equation: $u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$. More general than KdV as it takes into account waves propagating in two opposite directions. Here L is a third-order differential operator.

The Zoo of Integrable Systems

Here is a very partial list of some integrable systems fitting into this framework.

PDE in 1+1 dimensions:

- KdV: $u_t + uu_x + u_{xxx} = 0$. This is the universal equation governing unidirectional weakly nonlinear dispersive long waves, applicable to: surface and internal water waves, plasma vibrations, lattice vibrations in solid state physics (acoustic phonons), and many other physical problems.
- Boussinesq equation: $u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$. More general than KdV as it takes into account waves propagating in two opposite directions. Here L is a third-order differential operator.
- Cubic nonlinear Schrödinger (NLS) equation: $i\psi_t + \frac{1}{2}\psi_{xx} \pm |\psi|^2\psi = 0$. This is the universal equation governing packets of weakly nonlinear waves in the presence of dispersion, applicable to: wavetrains on deep water, optical fiber transmission systems, optical waveguide theory, and many other physical problems. Here L is the Dirac operator.

The Zoo of Integrable Systems

Here is a very partial list of some integrable systems fitting into this framework.

PDE in 1+1 dimensions:

- KdV: $u_t + uu_x + u_{xxx} = 0$. This is the universal equation governing unidirectional weakly nonlinear dispersive long waves, applicable to: surface and internal water waves, plasma vibrations, lattice vibrations in solid state physics (acoustic phonons), and many other physical problems.
- Boussinesq equation: $u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$. More general than KdV as it takes into account waves propagating in two opposite directions. Here L is a third-order differential operator.
- Cubic nonlinear Schrödinger (NLS) equation: $i\psi_t + \frac{1}{2}\psi_{xx} \pm |\psi|^2\psi = 0$. This is the universal equation governing packets of weakly nonlinear waves in the presence of dispersion, applicable to: wavetrains on deep water, optical fiber transmission systems, optical waveguide theory, and many other physical problems. Here L is the Dirac operator.
- Modified KdV: $u_t + u^2 u_x + u_{xxx} = 0$. This equation arises nearly as often as does KdV, but requires some extra symmetry in the physical system. This equation is strangely *not* in the KdV hierarchy, but is actually a "higher" NLS equation.

The Zoo of Integrable Systems

Here is a very partial list of some integrable systems fitting into this framework.

PDE in 1+1 dimensions:

- KdV: $u_t + uu_x + u_{xxx} = 0$. This is the universal equation governing unidirectional weakly nonlinear dispersive long waves, applicable to: surface and internal water waves, plasma vibrations, lattice vibrations in solid state physics (acoustic phonons), and many other physical problems.
- Boussinesq equation: $u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$. More general than KdV as it takes into account waves propagating in two opposite directions. Here L is a third-order differential operator.
- Cubic nonlinear Schrödinger (NLS) equation: $i\psi_t + \frac{1}{2}\psi_{xx} \pm |\psi|^2\psi = 0$. This is the universal equation governing packets of weakly nonlinear waves in the presence of dispersion, applicable to: wavetrains on deep water, optical fiber transmission systems, optical waveguide theory, and many other physical problems. Here L is the Dirac operator.
- Modified KdV: $u_t + u^2u_x + u_{xxx} = 0$. This equation arises nearly as often as does KdV, but requires some extra symmetry in the physical system. This equation is strangely *not* in the KdV hierarchy, but is actually a "higher" NLS equation.
- sine-Gordon: $u_{tt} - u_{xx} + \sin(u) = 0$. This equation is a model for coupled pendulum motion and arises in the theory of superconducting Josephson junctions. It is also in the NLS hierarchy.

The Zoo of Integrable Systems

PDE in 2+1 dimensions:

- The Kadomtsev-Petviashvili (KP) equation: $\pm u_{yy} = [u_t + uu_x + u_{xxx}]_x$. This is a two-dimensional generalization of KdV, arising in all of the same application problems. The integrable theory of the KP equation has had a remarkable impact in the pure mathematical subject of algebraic geometry, where it was used to solve the Schottky problem, a long-standing problem in the field.

The Zoo of Integrable Systems

PDE in 2+1 dimensions:

- The Kadomtsev-Petviashvili (KP) equation: $\pm u_{yy} = [u_t + uu_x + u_{xxx}]_x$. This is a two-dimensional generalization of KdV, arising in all of the same application problems. The integrable theory of the KP equation has had a remarkable impact in the pure mathematical subject of algebraic geometry, where it was used to solve the Schottky problem, a long-standing problem in the field.
- The Davey-Stewartson equation:

$$i\psi_t + \frac{1}{2}(\psi_{xx} \pm \psi_{yy}) \pm |\psi|^2\psi = u\psi\,, \quad u_{xx} \mp u_{yy} = \pm 2(|\psi|^2)_{xx}\,.$$

This is a two-dimensional generalization of NLS applicable in certain problems of water wave propagation.

The Zoo of Integrable Systems

Integro-differential equations:

- The Benjamin-Ono equation: $u_t + uu_x + Hu_{xx} = 0\,,\quad Hf(x) := \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty} \frac{f(y)\,dy}{y-x}$ arising in the theory of internal waves, also in the atmosphere. Thought to model the famous "Morning Glory" wave regularly seen near Burketown, Queensland, Australia:

The Zoo of Integrable Systems

Integro-differential equations:

- The Benjamin-Ono equation: $u_t + uu_x + Hu_{xx} = 0\,,\quad Hf(x) := \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty}\frac{f(y)\,dy}{y-x}$
  arising in the theory of internal waves, also in the atmosphere. Thought to model the famous "Morning Glory" wave regularly seen near Burketown, Queensland, Australia:

- The intermediate long wave equation

$$u_t + \frac{1}{\delta}u_x + uu_x + T[u_{xx}] = 0\,,\quad Tf(x) := \frac{1}{2\delta}\int_{-\infty}^{+\infty}\coth\left[\frac{\pi}{2\delta}(y-x)\right]f(y)\,dy\,.$$

The Zoo of Integrable Systems

Differential-difference equations:

- The Toda lattice equations: $\dfrac{d^2 q_n}{dt^2} = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1})\,, \qquad V(\Delta) := e^{\Delta}.$

  This is a special case of the Fermi-Pasta-Ulam model, with exponential spring forces, and an isospectral flow for the *Jacobi matrix*

$$\mathbf{L} := \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & b_{n-1} & a_n & b_n & & \\ & & b_n & a_{n+1} & b_{n+1} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \quad a_n := \frac{1}{2}\frac{dq_n}{dt}\,, \quad b_n := \frac{1}{2}e^{(q_{n+1}-q_n)/2}\,.$$

  It is also important in Hermitian random matrix theory and the theory of real orthogonal polynomials.

The Zoo of Integrable Systems

Differential-difference equations:

- The Toda lattice equations: $\dfrac{d^2q_n}{dt^2} = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1})\,, \qquad V(\Delta) := e^{\Delta}.$
  This is a special case of the Fermi-Pasta-Ulam model, with exponential spring forces, and an isospectral flow for the *Jacobi matrix*

$$\mathbf{L} := \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & b_{n-1} & a_n & b_n & & \\ & & b_n & a_{n+1} & b_{n+1} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}\,, \quad a_n := \frac{1}{2}\frac{dq_n}{dt}\,, \quad b_n := \frac{1}{2}e^{(q_{n+1}-q_n)/2}\,.$$

  It is also important in Hermitian random matrix theory and the theory of real orthogonal polynomials.
- The Ablowitz-Ladik equations:

$$i\frac{d\psi_n}{dt} + (1 \pm |\psi_n|^2)(\psi_{n+1} + \psi_{n-1}) = 0\,.$$

  This may be viewed as a spatial discretization of the NLS equation. It is also important in unitary random matrix theory and the theory of orthogonal polynomials on the unit circle.

The Zoo of Integrable Systems

The Painlevé transcendents for $w = w(z)$:

PI: $w'' = 6w^2 + z\,.$ PII: $w'' = 2w^3 + zw + a\,.$

PIII: $$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{aw^2 + b}{z} + cw^3 + \frac{d}{w}\,.$$

PIV: $$w'' = \frac{(w')^2}{2w} + \frac{3w^3}{2} + 4zw^2 + 2(z^2 - a)w + \frac{b}{w}\,.$$

PV: $$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(aw + \frac{b}{w}\right) + \frac{cw}{z} + \frac{dw(w+1)}{w-1}\,.$$

PVI: $$w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' + \frac{w(w-1)(w-2)}{z^2(z-1)^2}\left[a + \frac{bz}{w^2} + \frac{c(z-1)}{(w-1)^2} + \frac{dz(z-1)}{(w-z)^2}\right]\,.$$

Conclusions

A soliton is . . .

- A particular solution of an integrable equation that has “particle-like” properties:
  - ⋆ Solitons “survive” interactions despite lack of a simple superposition principle.
  - ⋆ Spectral interpretation as an eigenvalue means that the soliton is always there, even if it is not obvious in the field.

Conclusions

A soliton is . . .

- A particular solution of an integrable equation that has "particle-like" properties:
  - ⋆ Solitons "survive" interactions despite lack of a simple superposition principle.
  - ⋆ Spectral interpretation as an eigenvalue means that the soliton is always there, even if it is not obvious in the field.

- A coherent structure in nature modeled by integrable equations:
  - ⋆ Scott-Russell's great solitary wave.
  - ⋆ Coherent packets of phonons obstructing equipartition in solids.
  - ⋆ Bits in digital optical fiber transmission systems.
  - ⋆ Fantastic internal waves in the atmosphere.

Conclusions

A soliton is . . .

- A particular solution of an integrable equation that has “particle-like” properties:
  - ⋆ Solitons “survive” interactions despite lack of a simple superposition principle.
  - ⋆ Spectral interpretation as an eigenvalue means that the soliton is always there, even if it is not obvious in the field.

- A coherent structure in nature modeled by integrable equations:
  - ⋆ Scott-Russell’s great solitary wave.
  - ⋆ Coherent packets of phonons obstructing equipartition in solids.
  - ⋆ Bits in digital optical fiber transmission systems.
  - ⋆ Fantastic internal waves in the atmosphere.

- An eigenvalue of an isospectral family of operators L.

Conclusions

A soliton is . . .

- A particular solution of an integrable equation that has "particle-like" properties:
  - ⋆ Solitons "survive" interactions despite lack of a simple superposition principle.
  - ⋆ Spectral interpretation as an eigenvalue means that the soliton is always there, even if it is not obvious in the field.

- A coherent structure in nature modeled by integrable equations:
  - ⋆ Scott-Russell's great solitary wave.
  - ⋆ Coherent packets of phonons obstructing equipartition in solids.
  - ⋆ Bits in digital optical fiber transmission systems.
  - ⋆ Fantastic internal waves in the atmosphere.

- An eigenvalue of an isospectral family of operators L. Thank You!