Integrable Nonlinear Waves and Singular Asymptotics

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Outline

Focusing NLS and N-Solitons

The focusing Nonlinear Schrödinger equation is:

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This solution is a model for the effect of variation of amplitude on pulse propagation in nonlinear fiber optics. It will serve in this talk as a basic example of the theory of the semiclassical limit for the focusing NLS equation.

Linear Algebra Algorithm for the N-Soliton

There is a general scheme for solving initial-value problems for the focusing NLS equation (Zakharov & Shabat, 1972). For the N-soliton this method reduces to linear algebra (Satsuma & Yajima, 1974), and $\psi_N(x,t)$ can be obtained in closed-form.

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Introduce an N-vector $\mathbf{f}$ with elements

$$f_k := e^{i(\lambda_{N,k}x+\lambda_{N,k}^2 t)}\left[\frac{2i}{\gamma_{N,k}}\cdot\frac{\displaystyle\prod_{n=0}^{N-1}(\lambda_{N,k}-\lambda_{N,n}^*)}{\displaystyle\prod_{\substack{n=0\\n\neq k}}^{N-1}(\lambda_{N,k}-\lambda_{N,n})}\right]^{1/2}, \qquad k=0,\dots,N-1\,.$$

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Then build the Hermitian matrix $\mathbf{B}$ with elements

$$B_{jk} := -\frac{1}{2i} \cdot \frac{f_j f_k^*}{\lambda_{N,j} - \lambda_{N,k}^*}\,, \qquad j,k = 0,\ldots,N-1\,,$$

and let $\mathbf{K} := \mathbf{B}^*\mathbf{B}$ ($*$ means conjugation, without transposition).

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Then, the formula

$$\psi_N(x,t) = \mathbf{f}^\dagger \mathbf{c}\,,$$

($\dagger$ means conjugate-transpose) gives the N-soliton solution.

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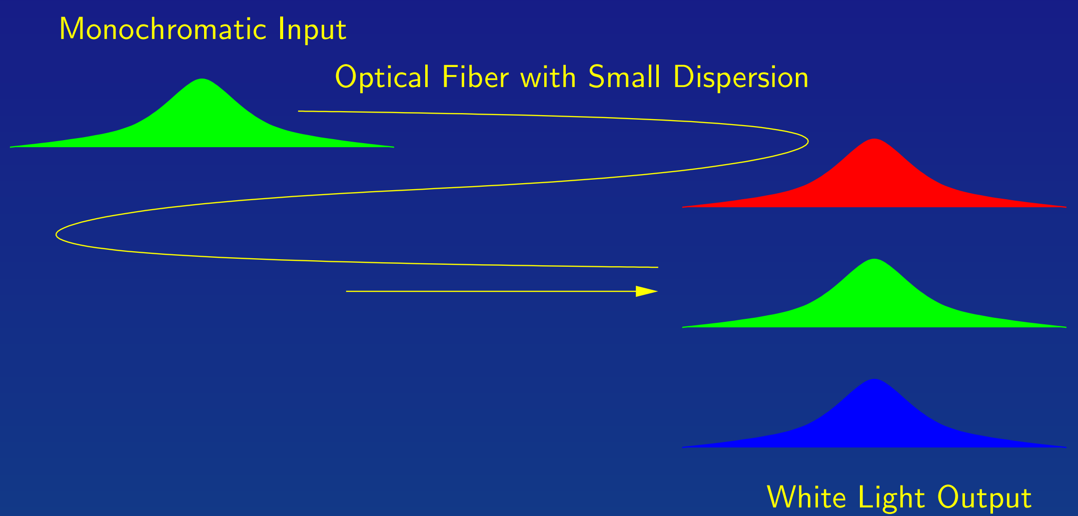
$$\psi_N = \frac{1}{\epsilon}\tilde{\psi}\,, \qquad t = \epsilon\tilde{t}\,, \qquad \epsilon = \frac{A}{N}\,,$$

and dropping tildes, the renormalized N-soliton satisfies:

$$i\epsilon\frac{\partial\psi}{\partial t} + \frac{\epsilon^2}{2}\frac{\partial^2\psi}{\partial x^2} + |\psi|^2\psi = 0\,, \qquad \psi(x,0) = A\,\mathrm{sech}(x)\,.$$

This problem is connected with the theory of pulse propagation in optical fibers with weak anomalous dispersion.

Application: Supercontinuum Generation



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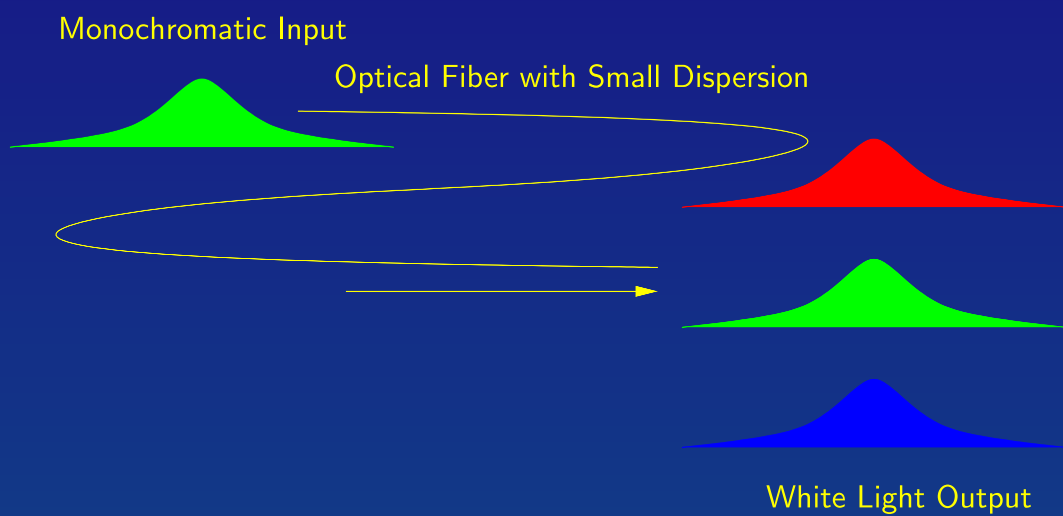

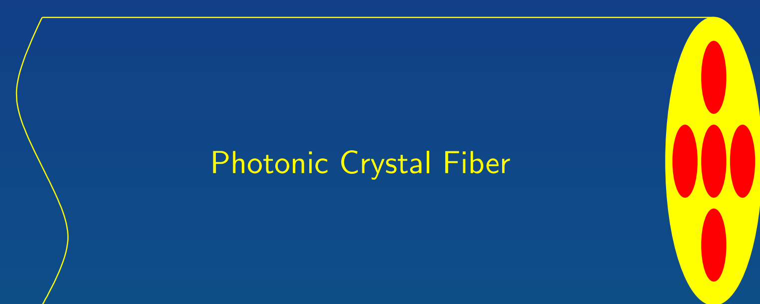

Computation in the Semiclassical Limit

Generally, we would like to be able to analyze solutions of initial-value problems like

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- Direct numerical simulation very difficult:
  - ⋆ The problem is "stiff" when ϵ is small.
  - ⋆ Worse yet, the limiting problem is ill-posed: setting $\rho := |\psi|^2$ and $u := \epsilon\Im(\log(\psi)_x)$, the initial-value problem becomes exactly,

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0\,,\qquad \frac{\partial u}{\partial t}-\frac{\partial\rho}{\partial x}+u\frac{\partial u}{\partial x}=\frac{\epsilon^2}{2}\frac{\partial}{\partial x}\left(\frac{1}{2\rho}\frac{\partial^2\rho}{\partial x^2}-\left[\frac{1}{2\rho}\frac{\partial\rho}{\partial x}\right]^2\right)$$

  with initial data $\rho(x,0)=A(x)^2$ and $u(x,0)=S'(x)$. Setting $\epsilon=0$ yields a Cauchy problem for an elliptic system. Reason: enhanced modulational instability.

Computation in the Semiclassical Limit

- Use of N-soliton formula in reflectionless cases (*e.g.* $A(x) = A\mathrm{sech}(x)$ and $S(x) \equiv 0$ with $\epsilon = A/N$):
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However difficult they are, numerical experiments for small ϵ have suggested remarkable structure:

- Quasiperiodic microstructure on scales $\Delta x \sim \Delta t \sim \epsilon$.
- Asymptotically sharp "caustic curves" separate different phases.

Large-N Analysis by Numerical Linear Algebra

References:

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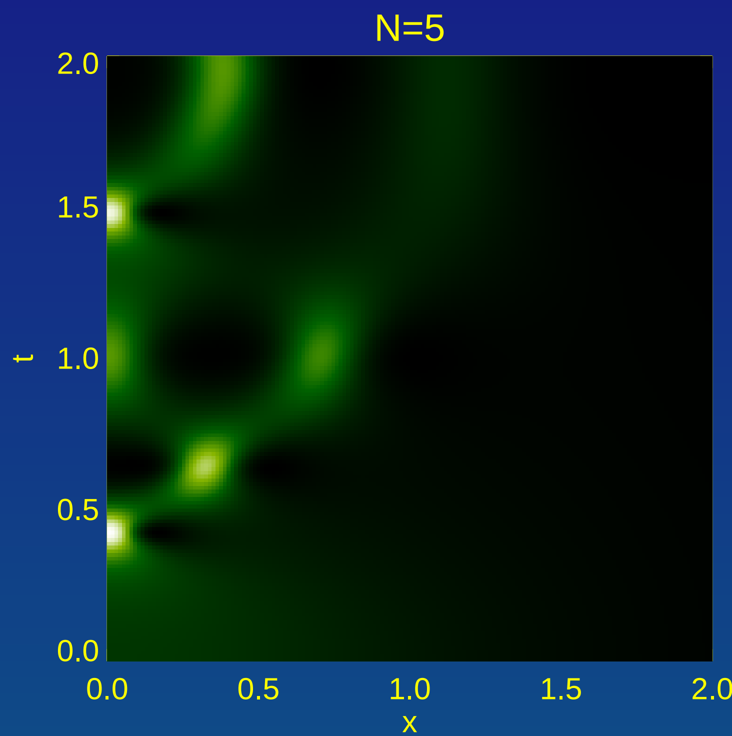

161×161 grid in $[0,2]^2$. Calculations done with 100 decimal digits.

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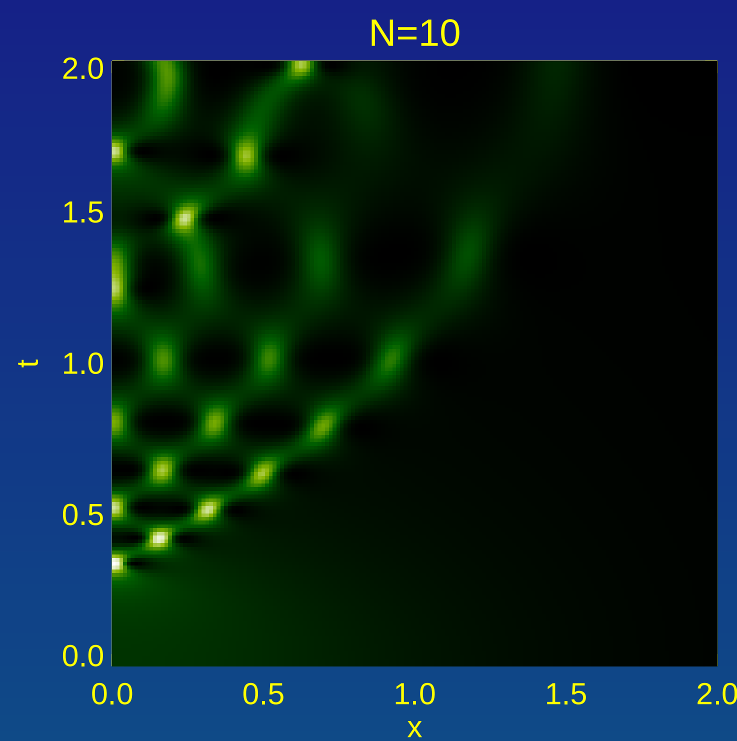

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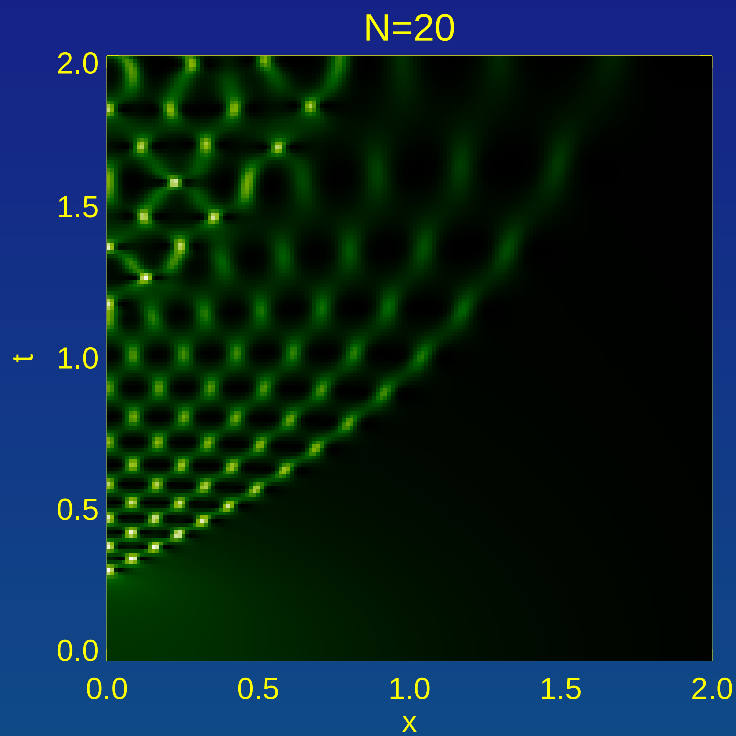

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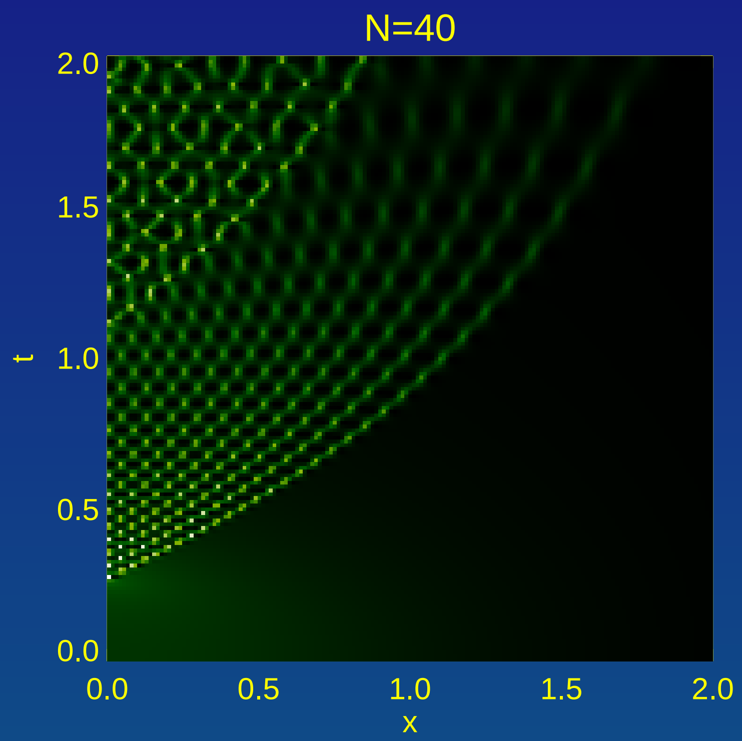

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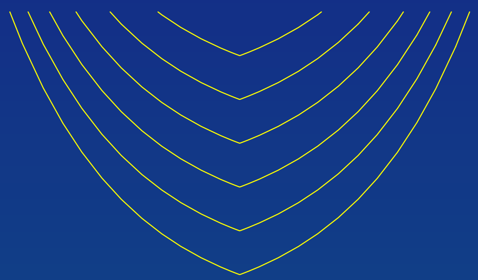

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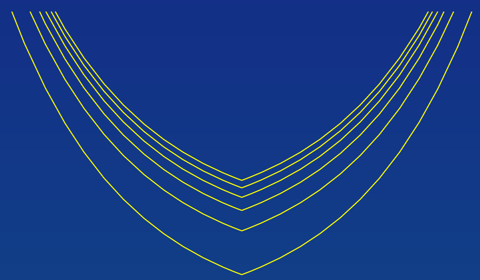

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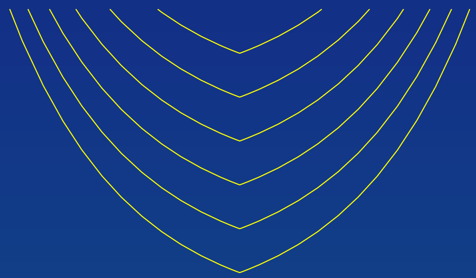

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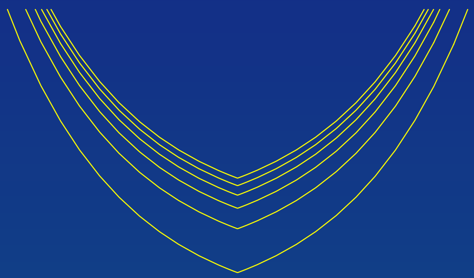

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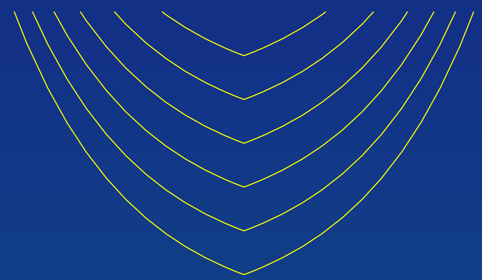

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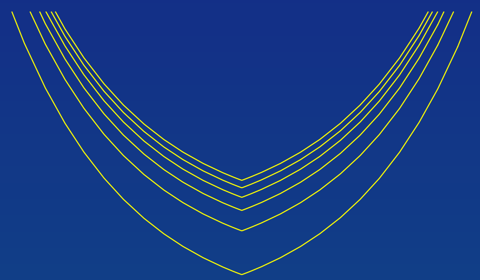

- Can the locations of the caustics be predicted?
- Can the rescaled N-soliton be described asymptotically in between the caustics?

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A method that works for analogous calculations in the Korteweg-de Vries (Lax & Levermore, 1983) and defocusing NLS (Jin, Levermore, & D. McLaughlin, 1999) contexts: maybe only certain terms in $\det(\mathbb{I} + \mathbf{K})$ are important. Expand the Fredholm determinant in principal minors:

$$\det(\mathbb{I} + \mathbf{K}) = 1 + \sum_{S \subset \{0,\dots,N-1\}} \det(\mathbf{K}_S) .$$

Elements of $\mathbf{K}_S$ taken from rows and columns of $\mathbf{K}$ with indices in S. Minors $\det(\mathbf{K}_S)$ are explicitly computed. If they are all positive, the sum is dominated by the largest term. This leads to a variational characterization of the limit.

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This method fails in the focusing case due to sign-indefinite minors. Cancellation. No dominant term.

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$$c_k(x,t) := \frac{1}{\gamma_{N,k}} \operatorname*{Res}_{\lambda_{N,k}} W(\lambda)\,, \qquad W(\lambda) := e^{2i(\lambda x+\lambda^2 t)/\epsilon} \prod_{n=0}^{N-1} \frac{\lambda-\lambda_{N,n}^*}{\lambda-\lambda_{N,n}}\,.$$

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The Riemann-Hilbert problem is: seek a 2×2 matrix $\mathbf{m}(\lambda)$, $\lambda\in\mathbb{C}$, with the following properties:

Riemann-Hilbert Problem

1. Rationality: $\mathbf{m}(\lambda)$ is a rational function of λ with simple poles confined to $\{\lambda_{N,n}, \lambda_{N,n}^*\}_{n=0}^{N-1}$ such that for $k = 0, \ldots, N-1$:

$$\operatorname*{Res}_{\lambda_{N,k}} \mathbf{m}(\lambda) = \lim_{\lambda \to \lambda_{N,k}} \mathbf{m}(\lambda) \begin{bmatrix} 0 & 0 \\ c_k(x,t) & 0 \end{bmatrix},$$

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From the solution of this Riemann-Hilbert problem,

$$\psi(x,t) = 2i \lim_{\lambda \to \infty} \lambda m_{12}(\lambda).$$

Principles of Riemann-Hilbert Analysis

One could solve for $\mathbf{m}(\lambda)$ by using a partial-fractions ansatz:

$$\mathbf{m}(\lambda) = \mathbb{I} + \sum_{k=0}^{N-1} \frac{\mathbf{a}_k}{\lambda - \lambda_{N,k}} + \sum_{k=0}^{N-1} \frac{\mathbf{b}_k}{\lambda - \lambda_{N,k}^*},$$

which leads directly to a linear algebra problem governing the matrix elements of the coefficients $\mathbf{a}_k$ and $\mathbf{b}_k$. This is essentially the same problem already considered. So that would probably be a bad idea.

Principles of Riemann-Hilbert Analysis

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which leads directly to a linear algebra problem governing the matrix elements of the coefficients $\mathbf{a}_k$ and $\mathbf{b}_k$. This is essentially the same problem already considered. So that would probably be a bad idea.

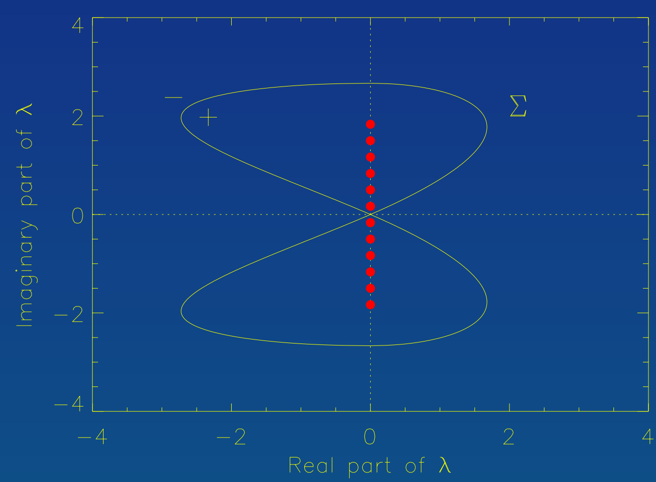
Instead, follow a general procedure that can be viewed as a two-step process:

Principles of Riemann-Hilbert Analysis: Step 1

Remove the poles. Interpolate residues inside a contour Σ. We get a new unknown $\mathbf{M}(\lambda)$ defined for $\Im(\lambda) > 0$ by:

$$\mathbf{M}(\lambda) := \mathbf{m}(\lambda) \begin{bmatrix} 1 & 0 \\ iW(\lambda)e^{\pi(\lambda - iA)/\epsilon} & 1 \end{bmatrix}, \quad \text{inside } \Sigma ,$$

and outside of Σ, $\mathbf{M}(\lambda) := \mathbf{m}(\lambda)$. Then for $\Im(\lambda) < 0$, set $\mathbf{M}(\lambda) := \sigma_2 \mathbf{M}(\lambda^*)^* \sigma_2$.



Ref: Kamvissis, K. McLaughlin, & M, *Ann. Math. Stud.*, 2003

Principles of Riemann-Hilbert Analysis: Step 1

Then, $\mathbf{M}(\lambda)$ has no poles at all. Instead it has a jump discontinuity across $\Sigma \cup \Sigma^*$:

$$\mathbf{M}_+(\lambda) = \mathbf{M}_-(\lambda) \begin{bmatrix} 1 & 0 \\ iS(\lambda)e^{F(\lambda)/\epsilon} & 1 \end{bmatrix}, \quad \lambda \in \Sigma \, .$$

where $F(\lambda)$ is a function with a branch cut in the interval $[-iA, iA]$:

$$\begin{aligned} F(\lambda) = & 2i(\lambda x + \lambda^2 t) + \pi(\lambda - iA) + 2i\lambda \log(-i\lambda) \\ & - i(\lambda - iA)\log(-i(\lambda - iA)) - i(\lambda + iA)\log(-i(\lambda + iA)) \, . \end{aligned}$$

The branch cut may be thought of as a "continuum limit" of the poles that used to be present.

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The branch cut may be thought of as a "continuum limit" of the poles that used to be present. Some notes:

- $S(\lambda) \approx 1$ if ϵ is small and Σ avoids the poles of $\mathbf{m}(\lambda)$.
- The freedom of choice of Σ is like choice of contour in steepest descent for integrals. But watch out if you want Σ to cross over the poles!

Ref: Kamvissis, K. McLaughlin, & M, *Ann. Math. Stud.*, 2003

Principles of Riemann-Hilbert Analysis: Step 2

Stabilize the problem. Consider $g(\lambda)$ analytic in $\mathbb{C} \setminus \Sigma \cup \Sigma^*$ with $g(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow \infty$ and $g(\lambda) + g(\lambda^*)^* = 0$. Then a new unknown $\mathbf{N}(\lambda)$ is defined by

$$\mathbf{N}(\lambda) := \mathbf{M}(\lambda) \begin{bmatrix} e^{-g(\lambda)/\epsilon} & 0 \\ 0 & e^{g(\lambda)/\epsilon} \end{bmatrix} .$$

The jump condition for $\mathbf{N}(\lambda)$ on Σ is now of the form

$$\mathbf{N}_+(\lambda) = \mathbf{N}_-(\lambda) \begin{bmatrix} e^{i\theta(\lambda)/\epsilon} & 0 \\ iS(\lambda)e^{\phi(\lambda)/\epsilon} & e^{-i\theta(\lambda)/\epsilon} \end{bmatrix} , \qquad \lambda \in \Sigma ,$$

where $\theta(\lambda) := i(g_+(\lambda) - g_-(\lambda))$ and $\phi(\lambda) := F(\lambda) - g_+(\lambda) - g_-(\lambda)$.

Fundamental principle: try to pick the scalar $g(\lambda)$ and the contour Σ to make the jump matrix as simple as possible in the limit $\epsilon \longrightarrow 0$. The identity matrix is the best possible target. Piecewise constant is also good.

Principles of Riemann-Hilbert Analysis: Target

A near-identity jump matrix is too much to hope for in this situation. So instead, we shall try to choose $g(\lambda)$ and the contour Σ so that Σ splits into two kinds of intervals:

- Bands: in which $\phi(\lambda)$ is an imaginary constant, and $\theta(\lambda)$ is real decreasing in the positive direction.
- Gaps: in which $\theta(\lambda)$ is a real constant, and $\Re(\phi(\lambda)) < 0$.

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Then (skipping many steps) $\mathbf{N}(\lambda)$ can be approximately built (for small ϵ) from Riemann Θ functions of the hyperelliptic surface with branch points at the band endpoints.

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Then (skipping many steps) $\mathbf{N}(\lambda)$ can be approximately built (for small ϵ) from Riemann Θ functions of the hyperelliptic surface with branch points at the band endpoints. There are two important implications:

- The semiclassical asymptotics of $\psi(x,t)$ in between the caustic curves are precisely described by modulated multiphase waves written in terms of Θ, where the number of phases is related to the genus of the surface.
- Caustic curves in the (x,t)-plane are genus transitions.

Ref: Kamvissis, K. McLaughlin, & M, *Ann. Math. Stud.*, 2003

Finding $g(\lambda)$ and the Contour Σ

Analysis of caustics boils down to finding $g(\lambda)$ and the contour Σ, and determining the way the number of bands and gaps varies with x and t.

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In some problems (*e.g.* zero-dispersion Korteweg-de Vries, orthogonal polynomials) the conditions that determine $g(\lambda)$ are encoded in a convex variational principle. Analogy: electrostatic equilibrium in an external field. Genus corresponds to number of components of support of equilibrium charge distribution.

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For focusing NLS, there is a variational formulation for $g(\lambda)$, but the problem is nonconvex. Instead, proceed by ansatz.

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3. Enforce on this formula the band/gap conditions on θ and ϕ. This has two effects:
   - It implicitly determines the endpoints as functions of x and t.
   - It determines certain inequalities that must be satisfied if the correct genus (for some fixed x and t) is indeed G.

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Finding $g(\lambda)$ and the Contour Σ

The problem of finding $g(\lambda)$ can be approached computationally. By contrast with direct numerical simulation of NLS for small ϵ, or use of the linear algebra algorithm for large N, the conditions determining g are independent of any large or small parameter.

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- Finding the endpoints from the nonlinear equations that implicitly define them as functions of x and t amounts to root-finding.

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- Once the endpoints are known, the band intervals of Σ are determined by solving ODEs to determine paths in the complex plane connecting pairs of endpoints.
- The (x, t)-plane can then be explored to search for genus transitions (caustics). Pick x and t and shade the region of the complex λ-plane where the inequality $\Re(\phi) < 0$ allows the gaps to reside. Topological changes in this region can indicate caustics.

Return to outline.

Ref: Kamvissis, K. McLaughlin, & M, *Ann. Math. Stud.*, 2003

The Genus Zero Region

The assumption of $G = 0$ can be proven to hold for $|t|$ sufficiently small but independent of ϵ. The analysis exactly at $t = 0$ requires a modification of this technique based on two simultaneous interpolants of residues in step 1 (M, 2002).

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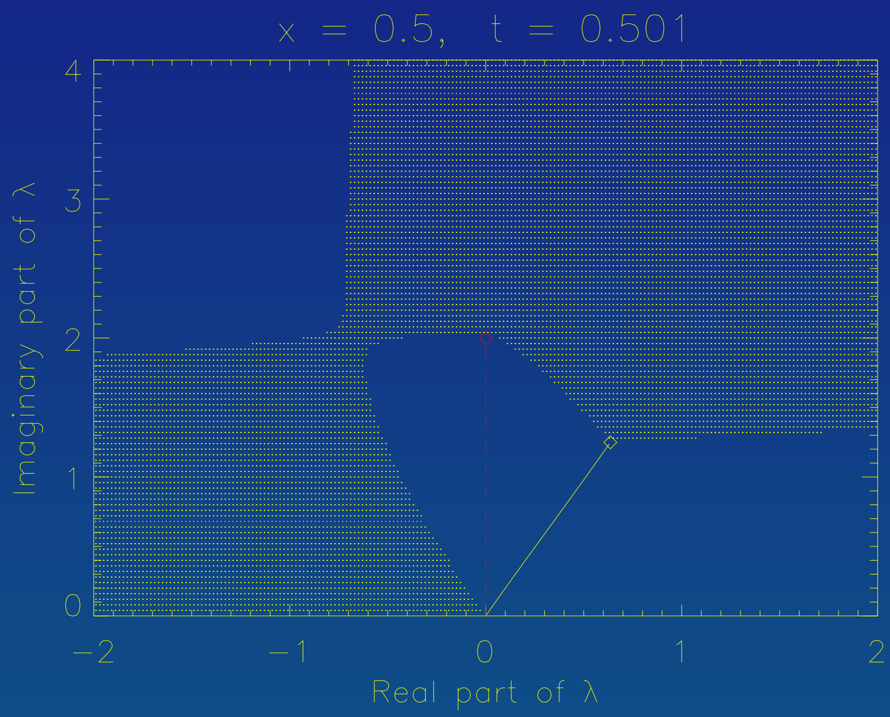
A topological change ("pinch-off") of the region where $\Re(\phi) < 0$ leads to the birth of a new band on Σ (and by symmetry a new band on Σ^*). The genus transition at the primary caustic is from $G = 0$ to $G = 2$.

Ref: Kamvissis, K. McLaughlin, & M, *Ann. Math. Stud.*, 2003

The Genus Zero Region: Primary Caustic

The phase transition upon crossing the primary caustic. Shaded region: $\Re(\phi) < 0$. Solid curve: band. Complete Σ with a gap around poles (red) lying in shaded region.



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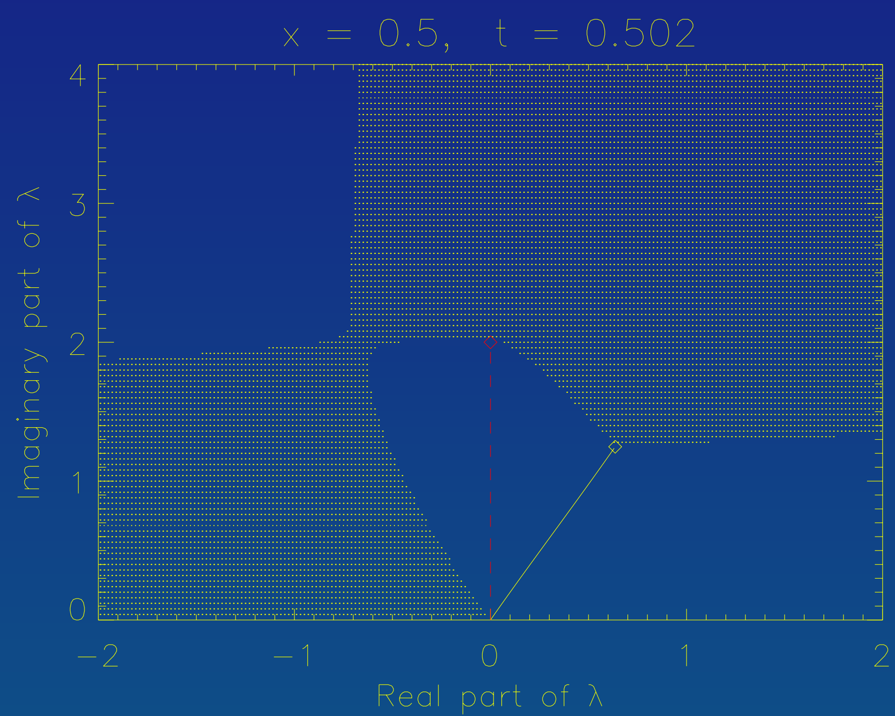

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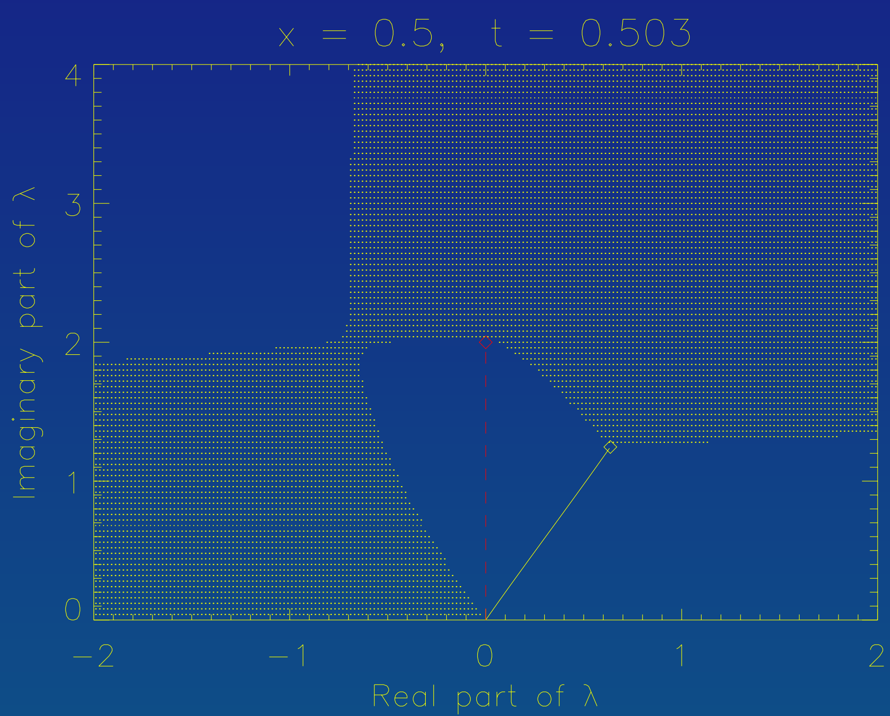

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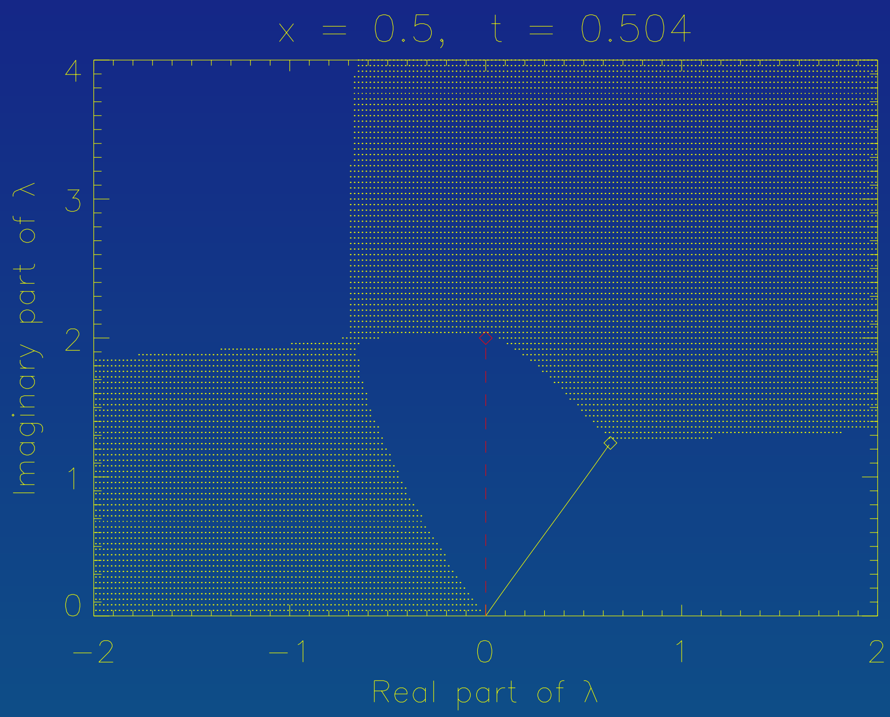

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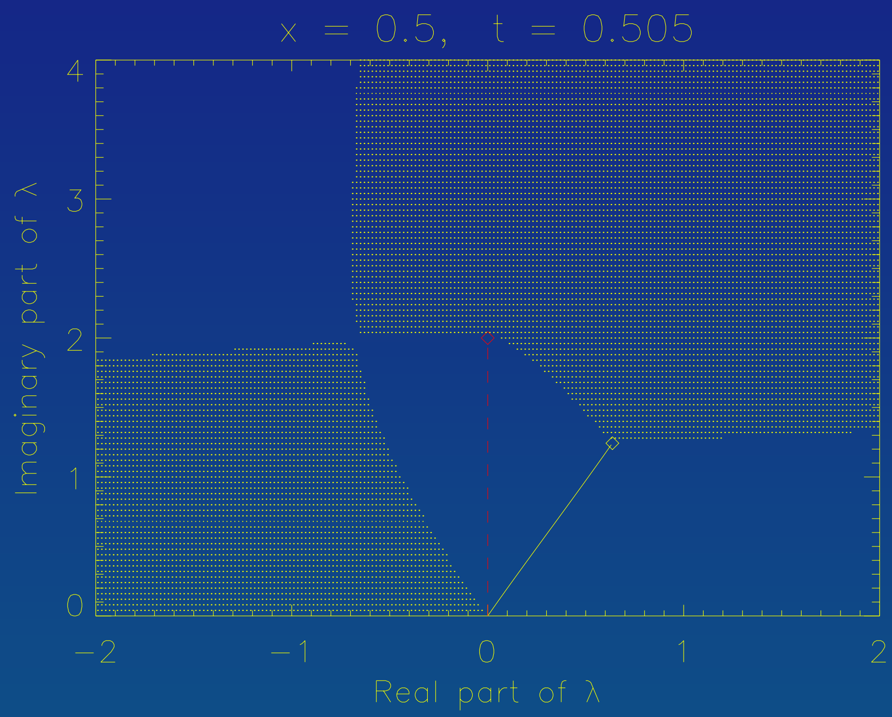

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The Genus Zero Region: Supercontinuum Generation

In the genus zero region, the rescaled N-soliton $(\psi(x,0) = A\,\mathrm{sech}(x))$ is

$$\psi(x,t) = b_0(x,t)e^{-i\kappa_0(x,t)/\epsilon} + O(\epsilon)\,, \qquad \frac{\partial \kappa_0}{\partial x} = 2a_0(x,t)\,,$$

where the endpoint is $\lambda_0 = a_0 + ib_0$, determined implicitly from

$$a_0^2 = t^2 b_0^4 \frac{A^2 - b_0^2 + t^2 b_0^4}{A^2 + t^2 b_0^4}\,, \quad \text{and} \quad x = -2ta_0 + \Re\left(\mathrm{arcsinh}\left(\frac{a_0 + iA}{b_0}\right)\right)\,.$$

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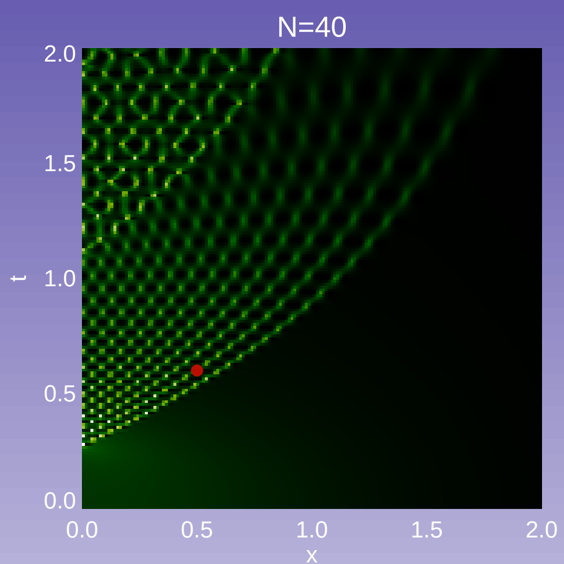
Calculate the Fourier transform of $b_0 e^{-i\kappa_0/\epsilon}$ at the (large) frequency $\omega = \Omega/\epsilon$ with Ω held fixed in the limit $\epsilon \longrightarrow 0$ by stationary phase. Stationary phase points only exist for $|\Omega| < M(t)$, for some $M(t)$ independent of ϵ. Thus the power spectrum is broad and flat (on average; there is a "ripple") with width $|\omega| \sim \epsilon^{-1}$. Note: spectral broadening consistent with supercontinuum generation occurs before wave breaking (primary caustic).

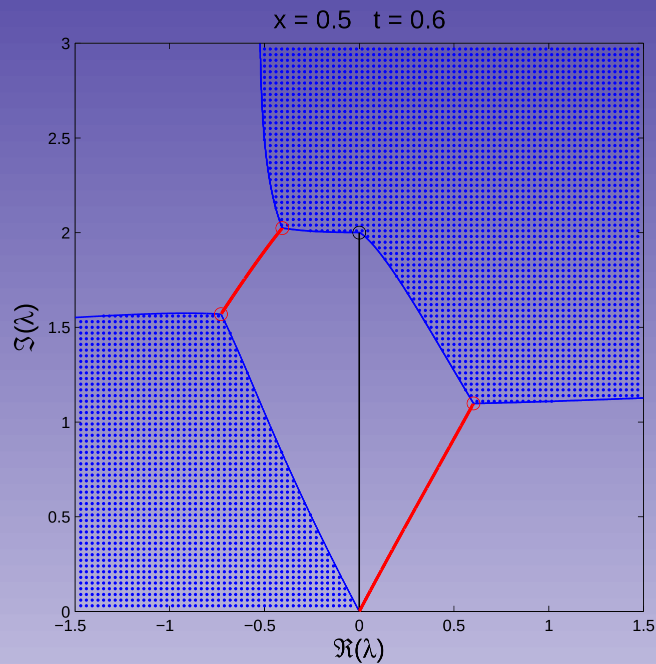
Return to outline.

Refs: Akhmanov, Sukhorukov, & Khokhlov, 1965; Lyng & M, *CPAM*, 2006

The Genus Two Region

Numerical construction of the genus $G = 2$ ansatz beyond the primary caustic.





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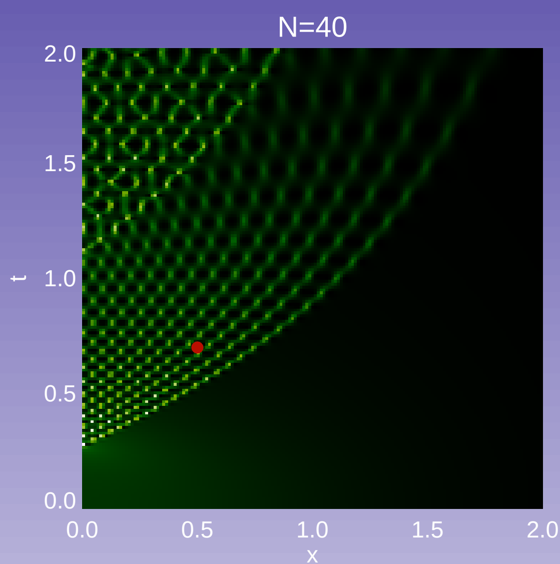

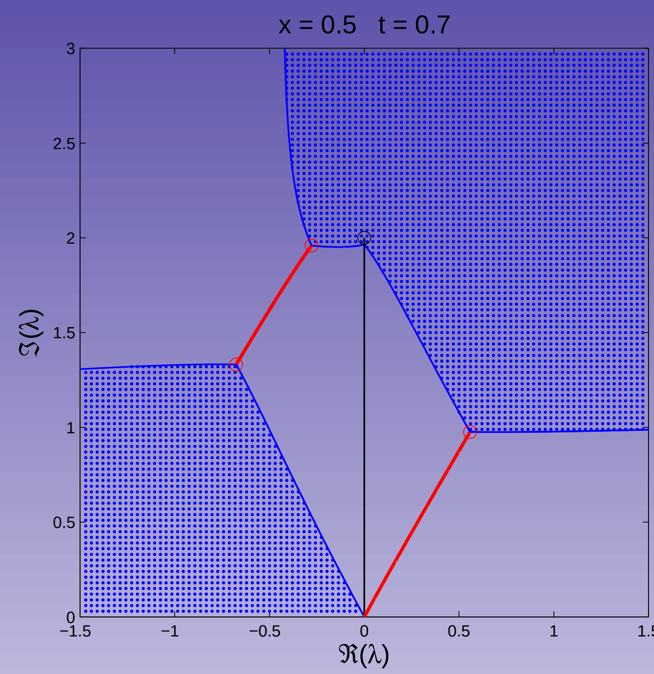

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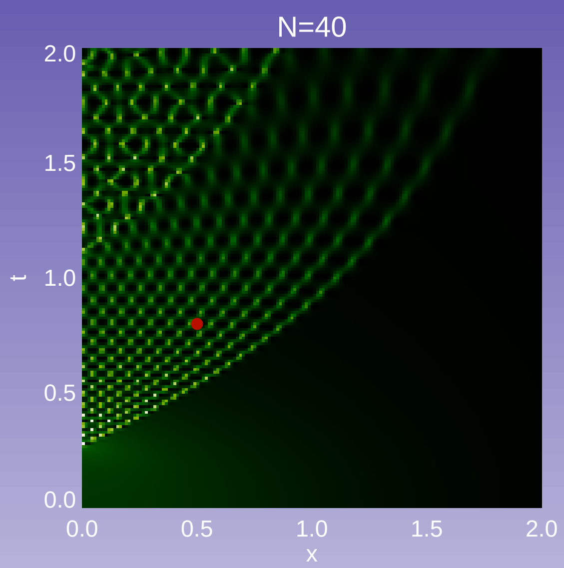

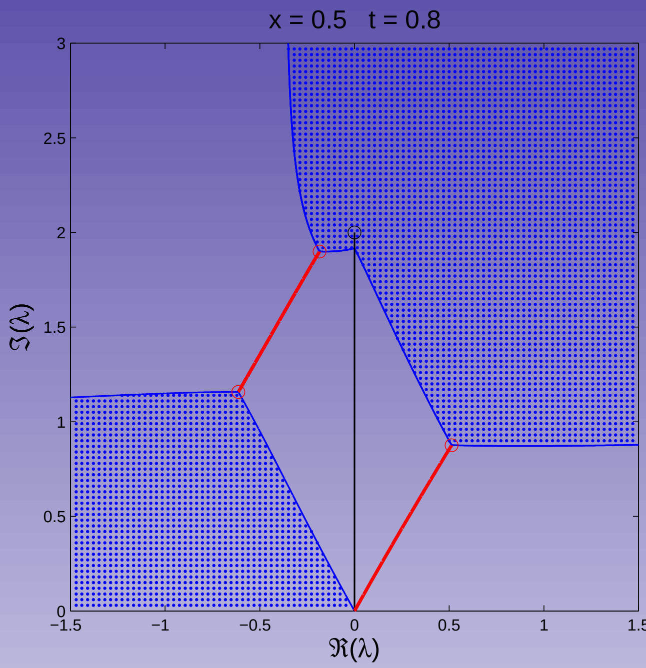

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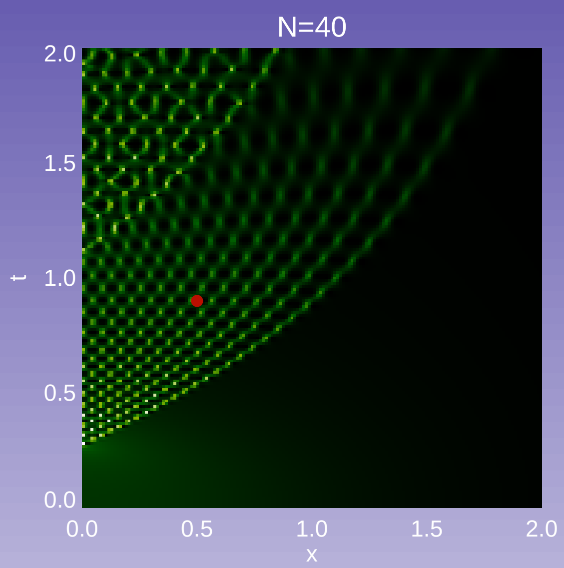

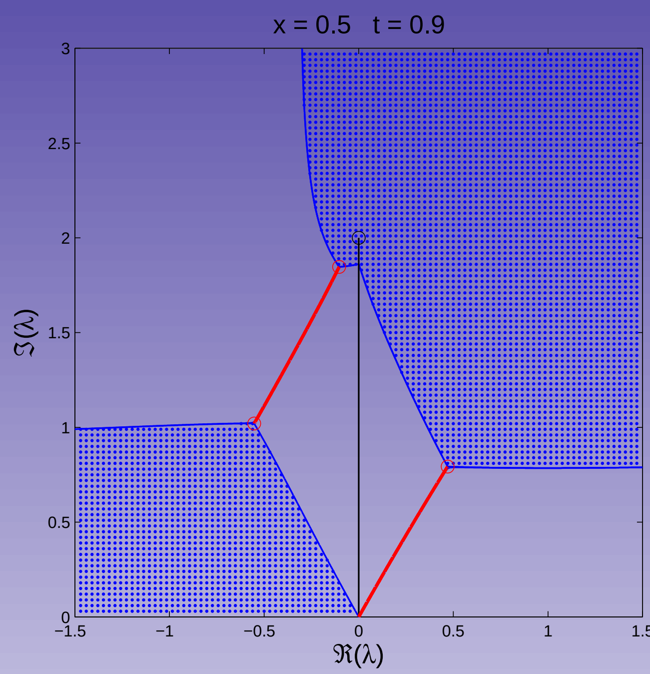

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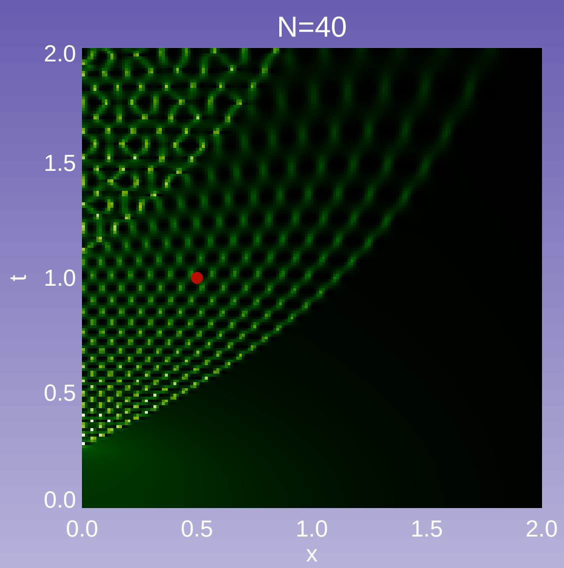

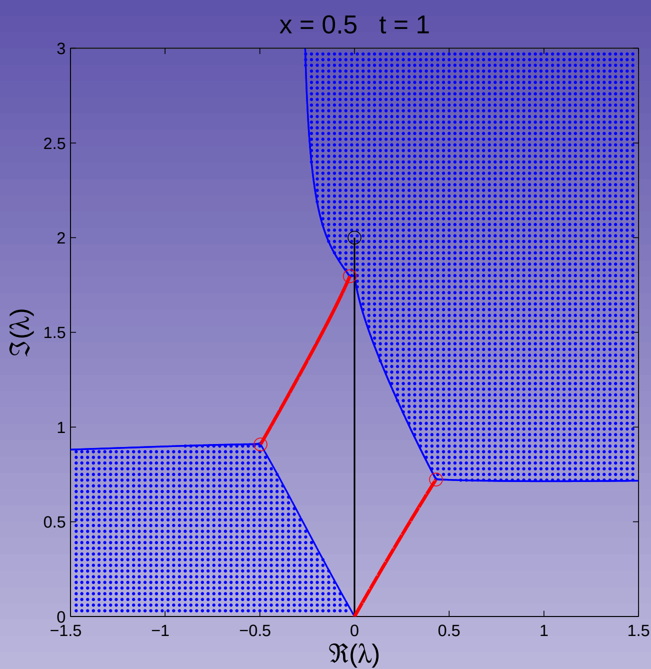

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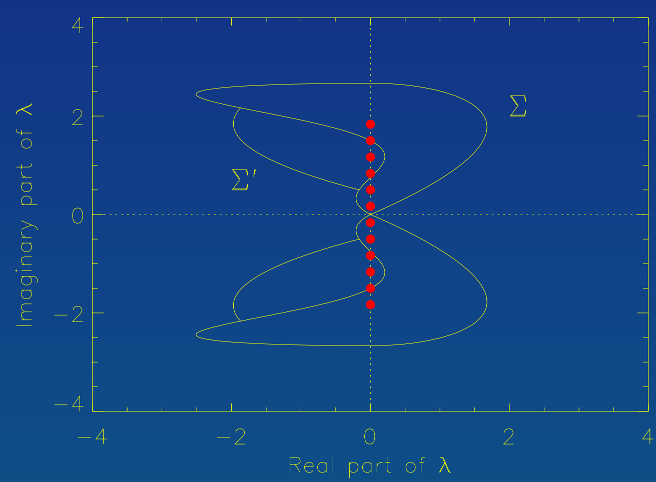

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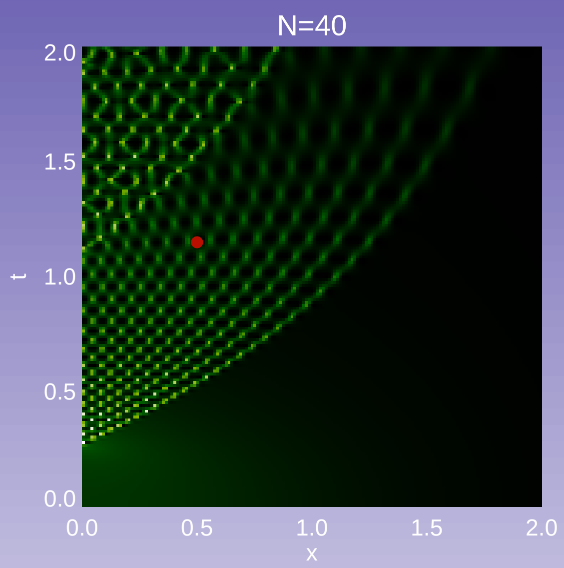
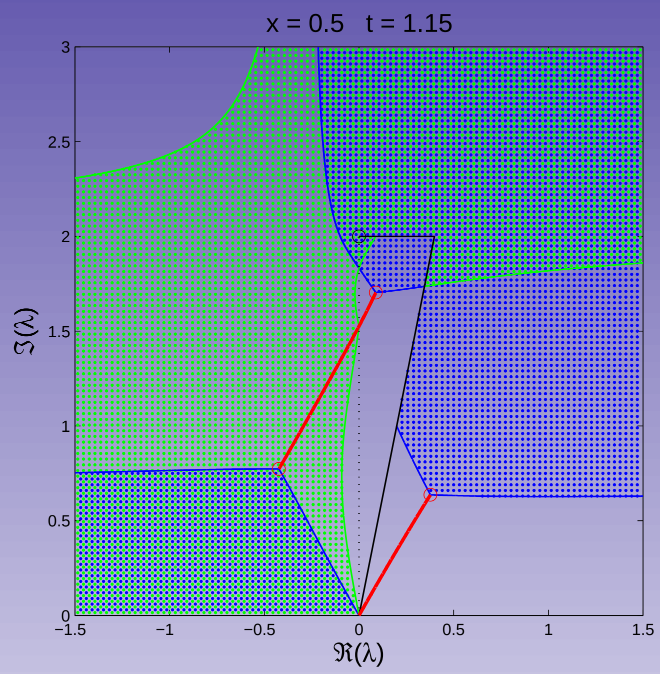
The new contour arc Σ' is "all gap". However, its inequality is different from the inequality $\Re(\phi) < 0$ that has to hold in gaps of Σ. We shade the new inequality in green.





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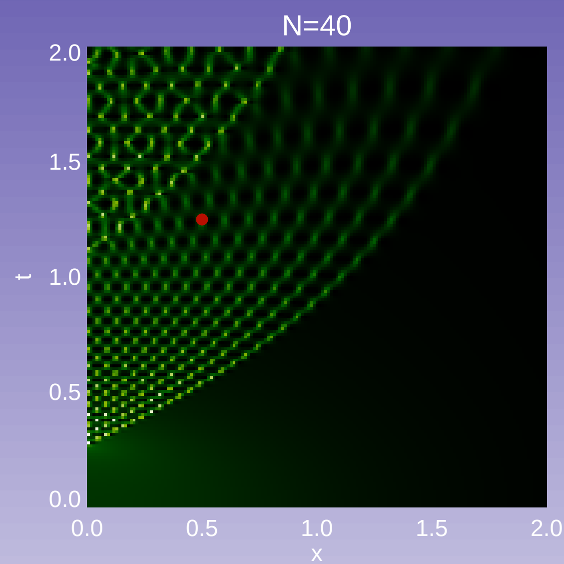

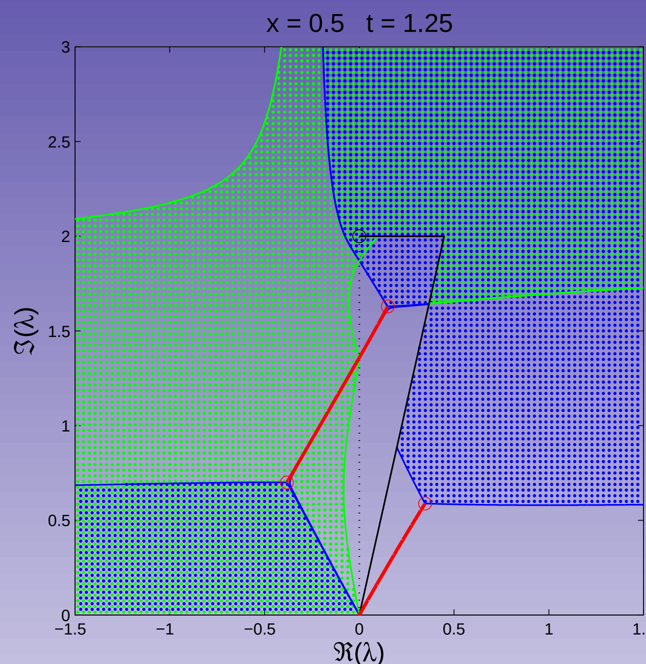

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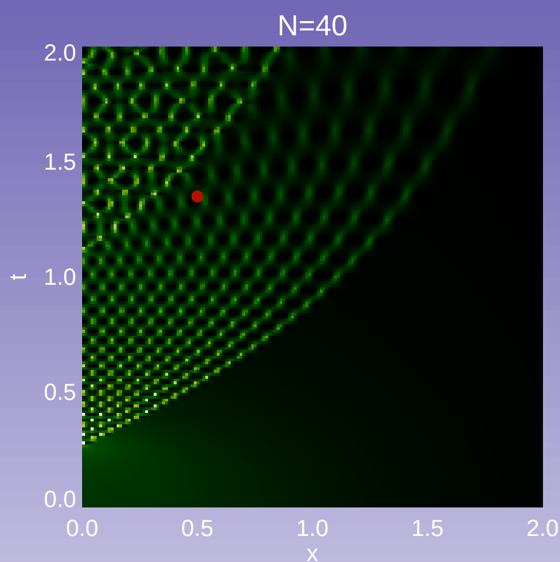

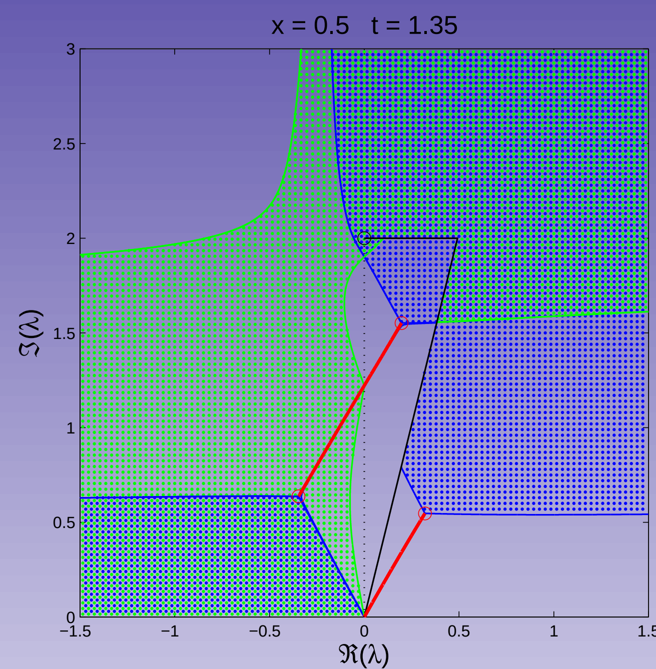

Ref: Lyng & M, *CPAM*, 2006

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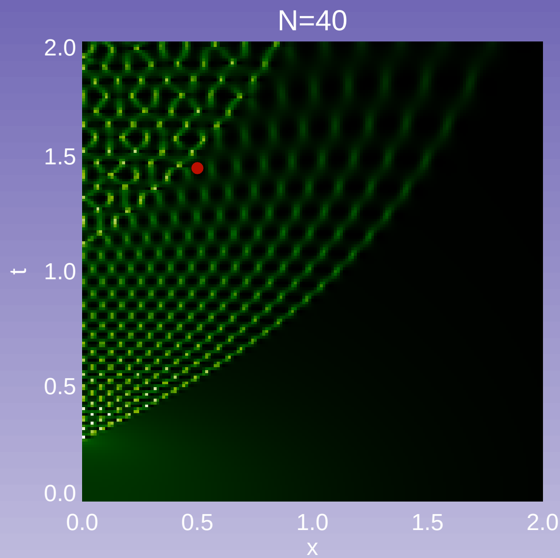

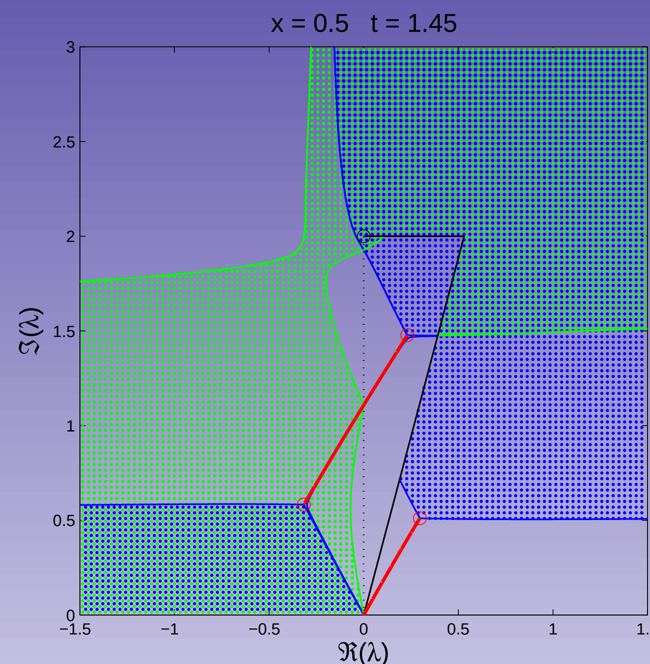

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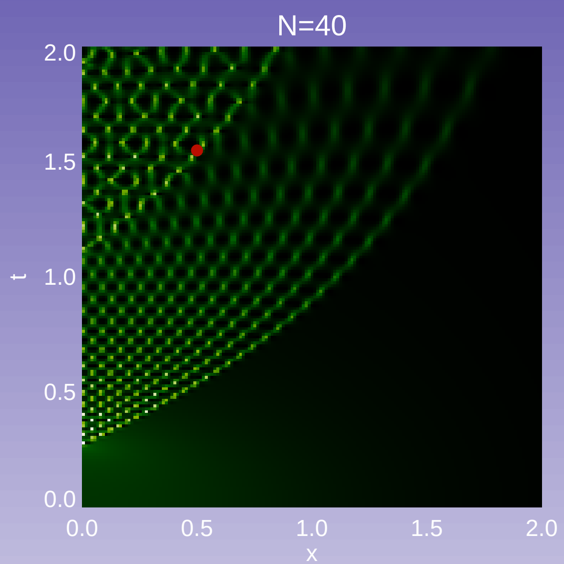

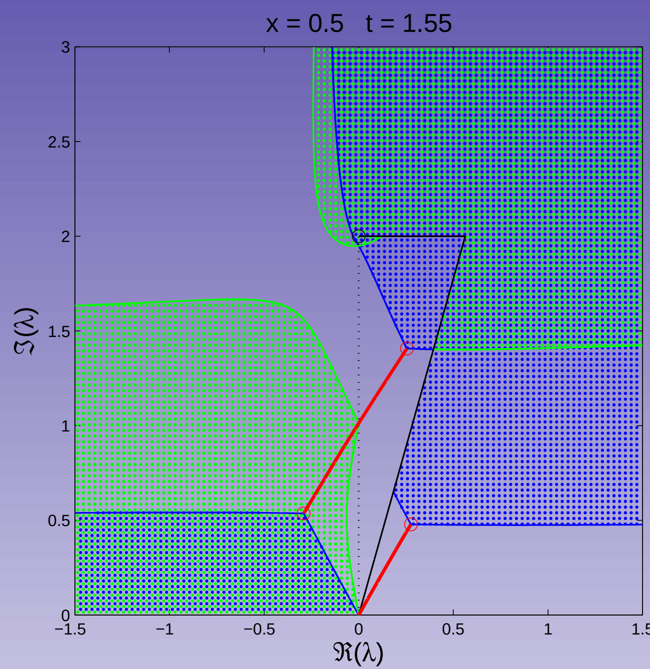

The Genus Two Region: Secondary Caustic

It has now been proved that

- The "pinched" genus two configuration corresponds to a degenerate genus four configuration with a closed-up band on Σ'.
- As t is increased through the secondary caustic, the closed band on Σ' opens up. The dynamics immediately beyond the secondary caustic are described by Riemann Θ functions of genus four.
- The ten equations implicitly governing the endpoints have a different form than had the new band opened up on Σ. Nonetheless, the endpoints satisfy the same set of Whitham (modulation) equations:

$$\frac{\partial \lambda_k}{\partial t} + c_k(\lambda_0, \ldots, \lambda_5)\frac{\partial \lambda_k}{\partial x} = 0\,, \qquad k = 1, \ldots, 5\,.$$

Ref: Lyng & M, *CPAM*, 2006

The Genus Two Region: Secondary Caustic

An important point: a natural approach one might adopt from the beginning of the problem is to slightly modify the initial data so that for each ϵ sufficiently small the poles making up the spectral data are "condensed" into a continuous branch cut, *i.e.* so that $S(\lambda) \longrightarrow 1$ exactly.

This turns out to be a small modification right up until the secondary caustic, but any theory that does not take into account the discrete nature of the spectrum will fail to correctly capture the secondary caustic.

Return to outline.

Ref: Lyng & M, *CPAM*, 2006

Generalizations

Going beyond the special initial data $\psi(x,0) = A\,\mathrm{sech}(x)$ in the semiclassical analysis of the focusing NLS equation requires asymptotic information about the spectrum of the nonselfadjoint Zakharov-Shabat problem:

$$\epsilon \frac{d\mathbf{u}}{dx} = \begin{bmatrix} -i\lambda & \psi(x,0) \\ -\psi(x,0)^* & i\lambda \end{bmatrix} \mathbf{u}\,.$$

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State of current knowledge:

1. Tovbis & Venakides (2000) have generalized the hypergeometric analysis of Satsuma & Yajima to potentials having the form $\psi(x,0) = A\,\mathrm{sech}(x)e^{iS(x)/\epsilon}$ where $S'(x) = \mu\tanh(x)$, which have a "fast phase". Inverse scattering analysis by Tovbis, Venakides, & Zhou (2004,2006).
2. Klaus & Shaw (2002) have proved that if $\psi(x,0)$ is real and "single-hump", then the discrete spectrum is purely imaginary and the number of eigenvalues is directly proportional to the L^1-norm.

Generalizations

3. Bronski (1996) used numerics to show that for certain potentials of the form $\psi(x,0) = A(x)e^{iS(x)/\epsilon}$ with A and S analytic, the discrete spectrum apparently accumulates with some regularity on certain curves in the complex plane. The curves were tied to analytic turning point theory in (M, 2001), and this theory is being strengthened in current work of Servat and Tovbis.
4. Rigorous, eigenvalue-by-eigenvalue asymptotics, as well as spectrally uniform asymptotics of the reflection coefficient for Klaus-Shaw potentials are the subject of current work by Bronski, K. McLaughlin, and M.
5. The ultimate goal for this problem is to understand the semiclassical dynamics for nonanalytic initial data.

The Semiclassical Modified Nonlinear Schrödinger Equation

Generalization of the focusing nonlinear Schrödinger equation: let $\alpha \geq 0$, and consider the initial-value problem

$$i\epsilon\frac{\partial\psi}{\partial t} + \frac{\epsilon^2}{2}\frac{\partial^2\psi}{\partial x^2} + |\psi|^2\psi + i\alpha\epsilon\frac{\partial}{\partial x}(|\psi|^2\psi) = 0\,, \qquad \psi(x,0) = A(x)e^{iS(x)/\epsilon}\,.$$

In recent work with Jeffery DiFranco, we are beginning a study of this problem in the $\epsilon \longrightarrow 0$ limit.

Modulation Equations

One of the main reasons for our interest in this problem is summarized by the following calculation.

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$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[u\rho + \frac{3\alpha}{2}\rho^2\right] = 0\,, \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}u^2 - \rho + \alpha u \rho\right] = \frac{\epsilon^2}{2}\frac{\partial}{\partial x}\frac{(\rho^{1/2})_{xx}}{\rho^{1/2}}\,,$$

with initial data $\rho(x,0) = A(x)^2$ and $u(x,0) = S'(x)$.

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- Elliptic if $\alpha^2\rho + \alpha u - 1 < 0$.
- Hyperbolic if $\alpha^2\rho + \alpha u - 1 > 0$.

Therefore, modulational stability can be recovered with a focusing nonlinearity if $\alpha > 0$ is sufficiently large, and if $u > 0$ in the tails of ψ.

Scattering Problem I: Bounds on Discrete Spectrum

The relevant spectral problem for inverse scattering is

$$\epsilon\frac{d\mathbf{v}}{dx} = \begin{bmatrix} -2i(k^2 - 1/4)/\alpha & 2k\psi \\ -2k\psi^* & 2i(k^2 - 1/4)/\alpha \end{bmatrix} \mathbf{v} \, .$$

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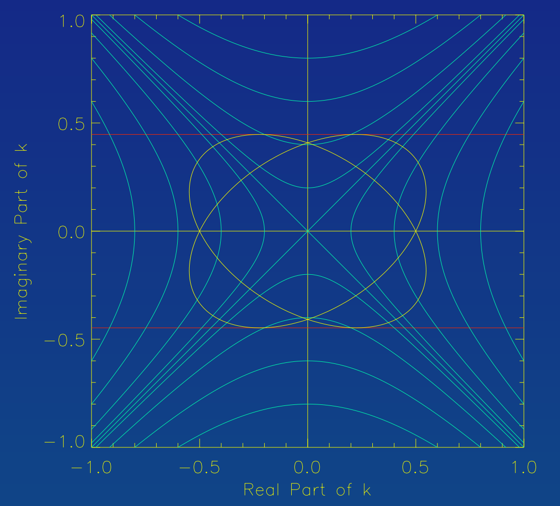
Suppose the initial data is of the form $\psi(x) = A(x)e^{iS(x)/\epsilon}$. Then, the discrete eigenvalues k are confined to the strip $|\Im(k)| \le \sup_{x\in\mathbb{R}} A(x)$, and furthermore, as $\epsilon \longrightarrow 0$ they must lie in the "hyperbolic shadow" of the so-called *turning point curve* in the k-plane given parametrically by the equations:

$$\Im(k)^2 = \frac{\alpha^2}{4}A(x)^2\,, \qquad \Re(k)^2 = \frac{1}{4}\left(1-\alpha S'(x)-\alpha^2 A(x)^2\right)\,, \qquad x\in\mathbb{R}\,.$$

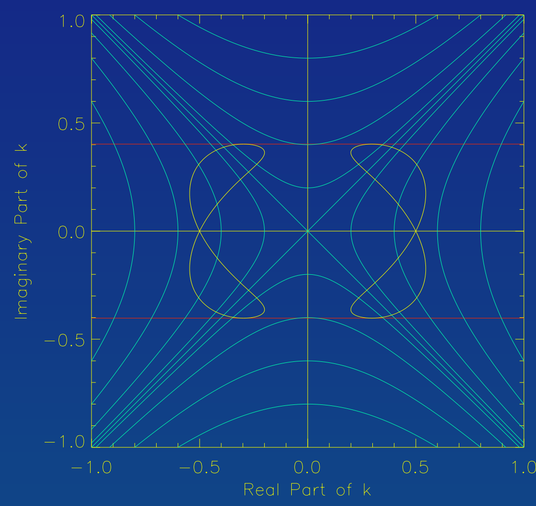
Scattering Problem I: Bounds on Discrete Spectrum

Example: $A(x) = \operatorname{sech}(x)$ and $S'(x) = \operatorname{sech}(x)\tanh(x)$, $\alpha = 0.894$.



Scattering Problem I: Bounds on Discrete Spectrum

Example: $A(x) = \operatorname{sech}(x)$ and $S'(x) = \operatorname{sech}(x)\tanh(x)$, $\alpha = 0.805$.



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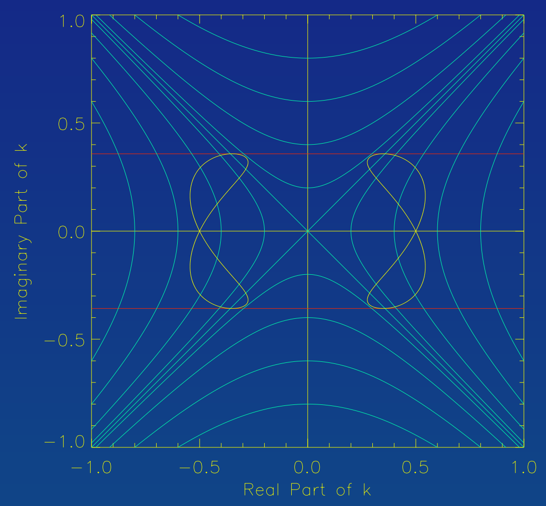
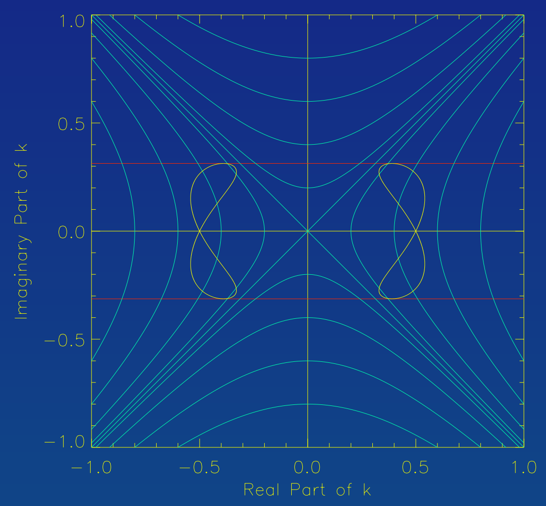
Example: $A(x) = \operatorname{sech}(x)$ and $S'(x) = \operatorname{sech}(x)\tanh(x)$, $\alpha = 0.716$.



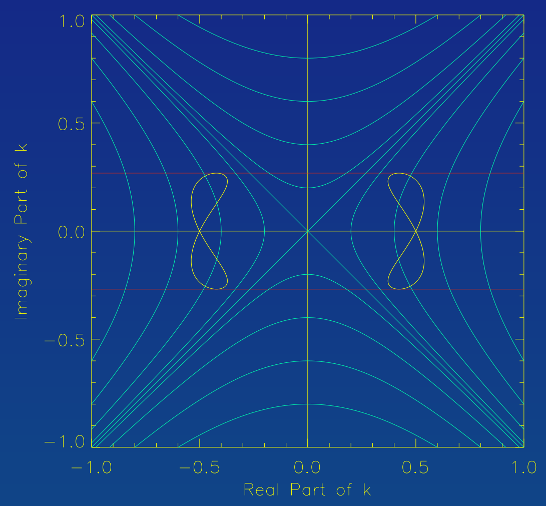
Scattering Problem I: Bounds on Discrete Spectrum

Example: $A(x) = \operatorname{sech}(x)$ and $S'(x) = \operatorname{sech}(x)\tanh(x)$, $\alpha = 0.626$.



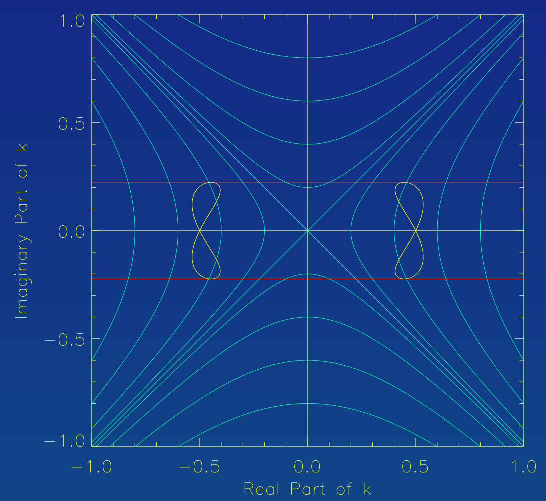
Scattering Problem I: Bounds on Discrete Spectrum

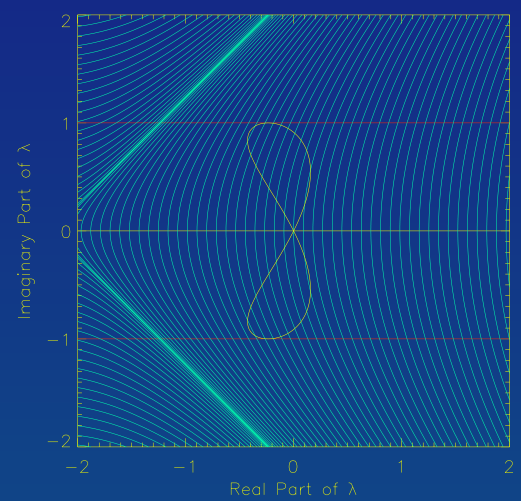
Example: $A(x) = \mathrm{sech}(x)$ and $S'(x) = \mathrm{sech}(x)\tanh(x)$, $\alpha = 0.537$.



Scattering Problem I: Bounds on Discrete Spectrum

Example: $A(x) = \mathrm{sech}(x)$ and $S'(x) = \mathrm{sech}(x)\tanh(x)$, $\alpha = 0.447$.





Note: $\lambda = (2k-1)/\alpha$ is the spectral parameter for focusing NLS ($\alpha = 0$).

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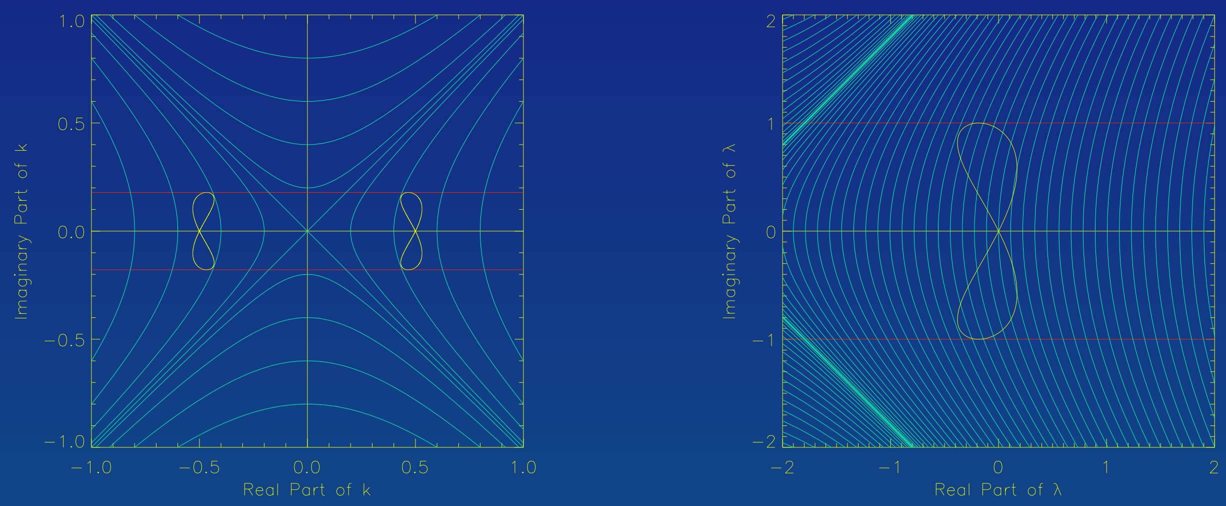
Example: $A(x) = \mathrm{sech}(x)$ and $S'(x) = \mathrm{sech}(x)\tanh(x)$, $\alpha = 0.358$.



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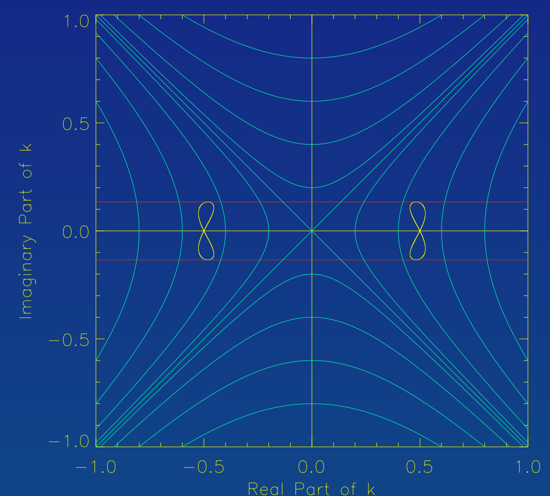

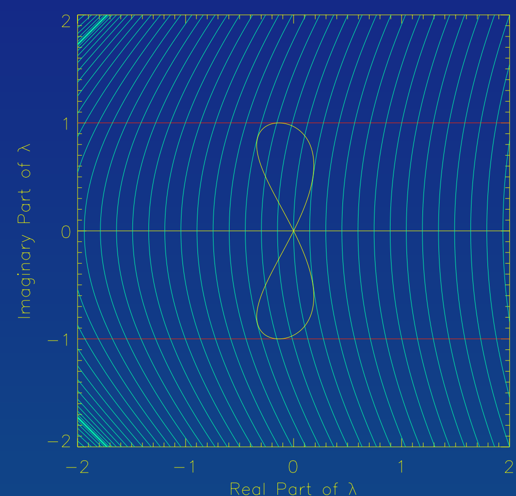

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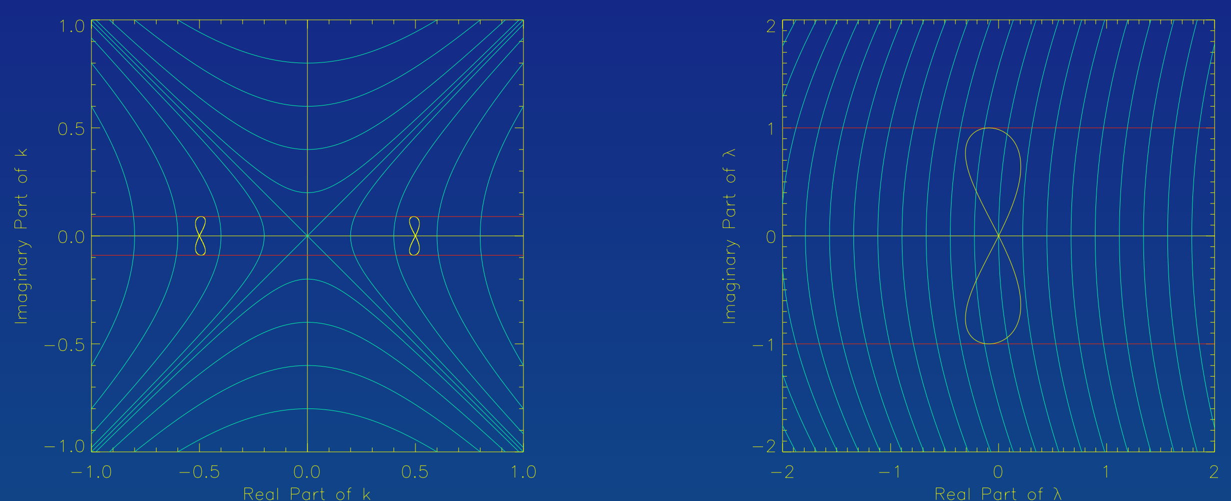

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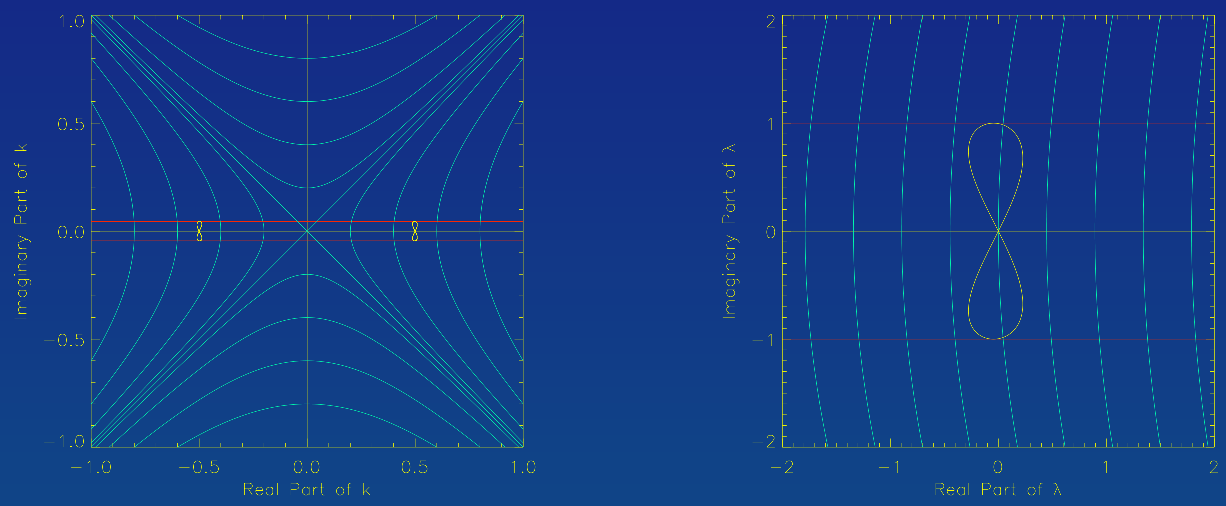

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Scattering Problem II: Hypergeometric Potentials

Some history:

- In 1973, Satsuma and Yajima showed that for potentials of the form $\psi(x) = A\text{sech}(x)$ in the nonselfadjoint Zakharov-Shabat spectral problem (appropriate for focusing NLS):

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It is easy to see that the Tovbis-Venakides result also holds if $S'(x) = \mu \tanh(x) + \delta$ for any $\delta \in \mathbb{R}$.

Scattering Problem II: Hypergeometric Potentials

We have found that potentials of the Tovbis-Venakides class are hypergeometric also for the spectral problem of the modified focusing NLS, for arbitrary $\epsilon > 0$ and $\alpha > 0$.

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- If $\alpha\delta > 1$, then there are no discrete eigenvalues. In this case,
  - ⋆ if $|\mu| < (\alpha\delta - 1)/\alpha$, then the modulation equations are hyperbolic for all x at $t = 0$, while
  - ⋆ if $|\mu| > (\alpha\delta - 1)/\alpha$, then there exist $x \in \mathbb{R}$ for which the modulation equations are elliptic at $t = 0$.

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- If $\alpha\delta < 1$, then regardless of the value of $\mu \in \mathbb{R}$ there exist $x \in \mathbb{R}$ for which the modulation equations are elliptic at $t = 0$. In this case,
  - ⋆ if $|\mu| < 2A\sqrt{1 - \alpha\delta}$ then there are $\sim \epsilon^{-1}$ discrete eigenvalues, while
  - ⋆ if $|\mu| > 2A\sqrt{1 - \alpha\delta}$ then there are no discrete eigenvalues.

The Semiclassical Sine-Gordon Equation

Consider the initial-value problem

$$\epsilon^2\left[\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}\right]+\sin(u)=0\,,\qquad u(x,0)=f(x)\,,\quad \epsilon\frac{\partial u}{\partial t}(x,0)=g(x)\,.$$

In recent work with Robert Buckingham, we are beginning a study of this problem in the $\epsilon \longrightarrow 0$ limit.

Scattering Problem I: Gauge Transformations

This initial-value problem can be integrated with the help of the spectral problem

$$4\epsilon\frac{d\mathbf{v}}{dx} = \left(-iz\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + iz^{-1}\begin{bmatrix}\cos(u) & \sin(u)\\ \sin(u) & -\cos(u)\end{bmatrix} + \epsilon(u_x+u_t)\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)\mathbf{v}\,.$$

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It might appear that solutions of this problem should have complicated singularities at $z=0$. That this is not the case can be seen more easily by making a certain *gauge transformation*. Following Kaup (1975), consider a rotation of the vector $\mathbf{v}$ by an x-dependent angle of $u(x)/2$:

$$\mathbf{u} := \begin{bmatrix}\cos(u/2) & \sin(u/2)\\ -\sin(u/2) & \cos(u/2)\end{bmatrix}\mathbf{v}\,.$$

Then, the scattering problem becomes

$$4\epsilon\frac{d\mathbf{u}}{dx} = \left(iz^{-1}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - iz\begin{bmatrix}\cos(u) & -\sin(u)\\ -\sin(u) & -\cos(u)\end{bmatrix} - \epsilon(u_x-u_t)\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)\mathbf{u}\,.$$

Scattering Problem I: Gauge Transformations

A gauge rotation by only half as much is useful for other purposes:

$$\mathbf{w} := \begin{bmatrix} \cos(u/4) & \sin(u/4) \\ -\sin(u/4) & \cos(u/4) \end{bmatrix} \mathbf{v} \,.$$

This yields the spectral problem in the *symmetric gauge*:

$$4\epsilon \frac{d\mathbf{w}}{dx} = -iz \begin{bmatrix} \cos(u/2) & -\sin(u/2) \\ -\sin(u/2) & -\cos(u/2) \end{bmatrix} \mathbf{w} + iz^{-1} \begin{bmatrix} \cos(u/2) & \sin(u/2) \\ \sin(u/2) & -\cos(u/2) \end{bmatrix} \mathbf{w} + \epsilon u_t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{w} \,.$$

This spectral problem exhibits symmetry between z and z^{-1}, and u_x does not appear.

Scattering Problem II: Hypergeometric Potentials

An important application of the symmetric gauge is its solution for all $\epsilon > 0$ in the case when $u_t \equiv 0$ and

$$\sin(u/2) = \operatorname{sech}(x) \qquad \text{and} \qquad \cos(u/2) = \tanh(x)$$

in terms of hypergeometric function theory.

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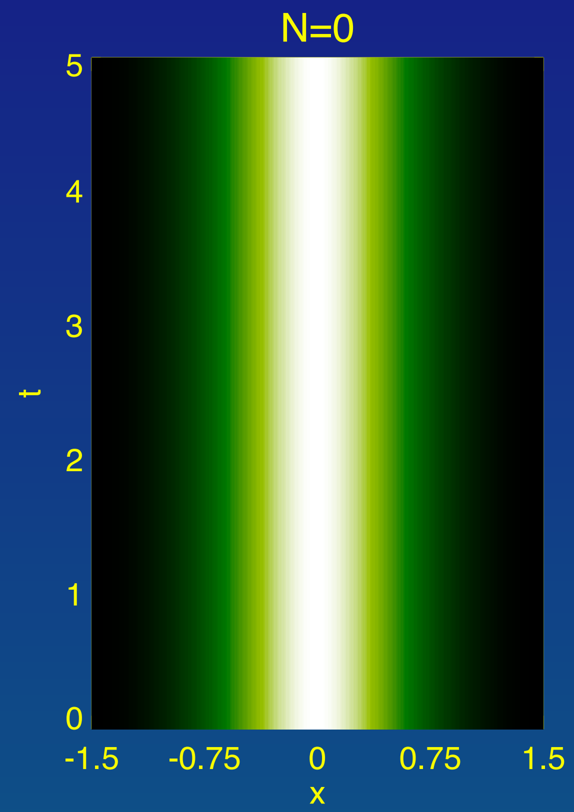
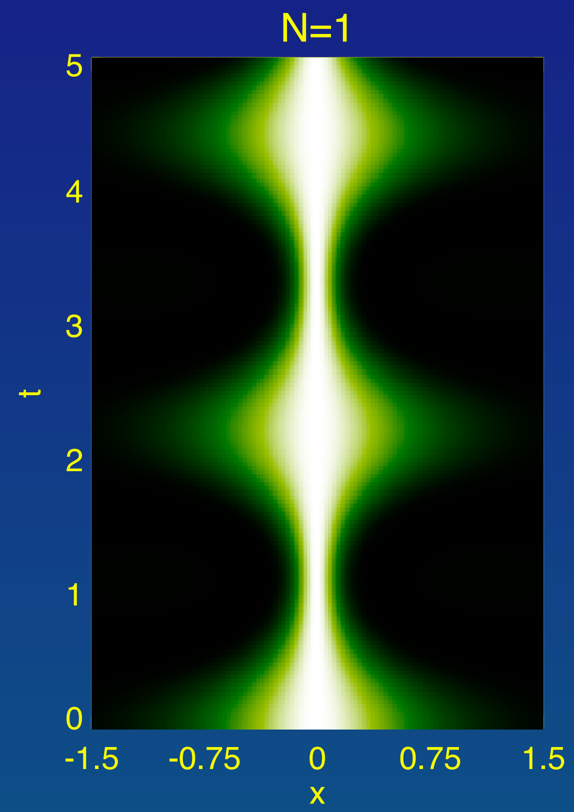
- The ($\sim \epsilon^{-1}$) discrete eigenvalues are all on the unit circle in the z-plane, uniformly distributed with respect to $\Im(z)$.
- The reflection coefficient vanishes identically when $\epsilon = 1/(2N+1)$, for $N = 0, 1, 2, 3, \ldots$. For such ϵ, numerical linear algebra may be used as for the N-soliton of focusing NLS.

Inverse Scattering: Reflectionless Hypergeometric Case



Inverse Scattering: Reflectionless Hypergeometric Case



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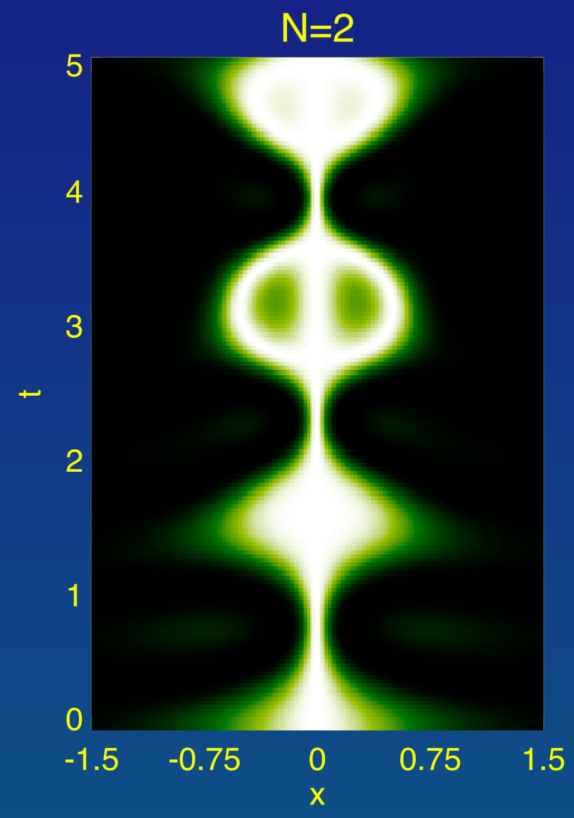

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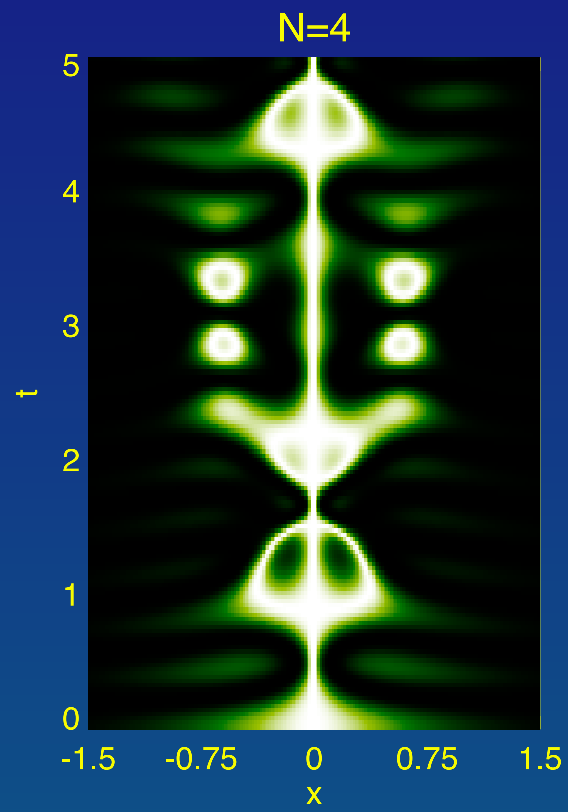

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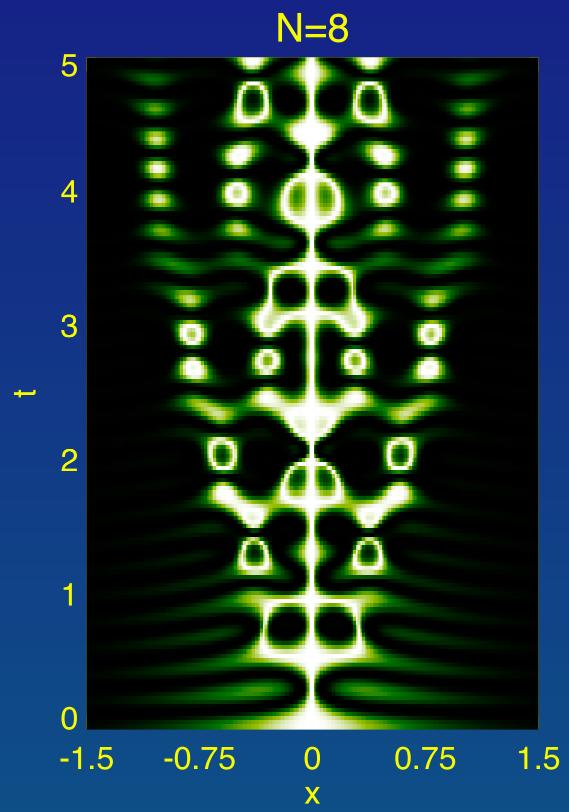

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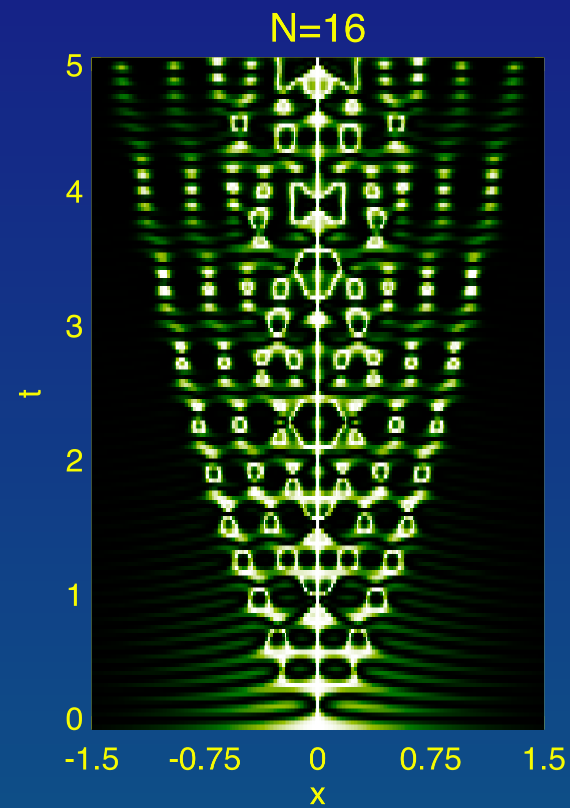

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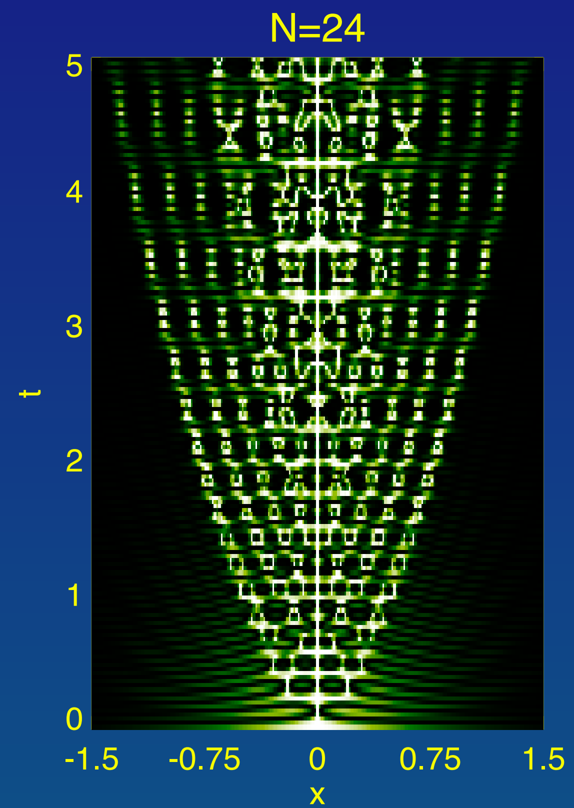

Reflectionless Hypergeometric Case: Observations

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Thus what we have is an example of integrable homoclinic chaos!

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Conclusion

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Thank You!

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