The Semiclassical Modified Nonlinear Schrödinger Equation: Facts and Artifacts

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Outline

Background: The Nonlinear Schrödinger Equation

Let $\epsilon > 0$ be a parameter. The nonlinear Schrödinger (NLS) equation is:

$$i\epsilon\frac{\partial\phi}{\partial t} + \frac{\epsilon^2}{2}\frac{\partial^2\phi}{\partial x^2} + \kappa|\phi|^2\phi = 0\,.$$

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- *weak* if $\epsilon > 0$ is small,
- *anomalous* if $\kappa = 1$,
- *normal* if $\kappa = -1$.

Modulational Stability/Instability

Both flavors of the NLS equation have exact *plane wave* solutions:

$$\phi = \phi_0(x,t) = Ae^{i(kx-\omega t)/\epsilon}, \qquad \omega = \frac{1}{2}k^2 - \kappa|A|^2 .$$

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$$p(x,t) = (a_\pm + ib_\pm)e^{i\Delta x/\epsilon}e^{\sigma_\pm t/\epsilon}, \qquad \sigma_\pm := -ik\Delta \pm \frac{\Delta}{2}\sqrt{4\kappa A^2 - \Delta^2} .$$

Here Δ is a relative wavenumber. The dichotomy of $\kappa = \pm 1$ is now clear:

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Here Δ is a relative wavenumber. The dichotomy of $\kappa = \pm 1$ is now clear:

- $\kappa = -1$: $\Re\{\sigma_\pm\} = 0$, $\forall\Delta$, k, and A. Unconditional modulational stability.
- $\kappa = 1$: $\Re\{\sigma_\pm\} \neq 0$ if $\Delta^2 < 4A^2$. Instability of each plane wave to "sideband" perturbations, and hence unconditional modulational instability.

Semiclassical Behavior

The modulational instability of the focusing NLS equation is enhanced when $\epsilon > 0$ is small:

- The width of the band of unstable wavenumbers is inversely proportional to ϵ. Stable perturbations correspond only to waves of length $O(\epsilon)$.
- The exponential growth rate of the most unstable mode scales like ϵ^{-1}.

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This fact, and recent interest in applications to “dispersion shifted” photonic crystal optical fibers, motivates the study of the *semiclassical* Cauchy problem for the NLS equation: set initial data in the form $\phi(x,0) = A(x)e^{iS(x)/\epsilon}$ and analyze the solution $\phi(x,t)$ in the limit $\epsilon \downarrow 0$. In particular, look for differences between focusing and defocusing cases.

Semiclassical Behavior: Modulation Equations

An old approach to Schrödinger equations originally advocated by Madelung is a "quantum-corrected" hydrodynamical theory: define

$$\rho(x,t) := |\phi|^2 \quad \text{(density)}, \qquad u(x,t) := \epsilon \Im \frac{\partial}{\partial x} \log(\phi) \quad \text{(velocity)}.$$

Then, the NLS equation becomes

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \\ &\frac{\partial u}{\partial t} - \kappa\frac{\partial \rho}{\partial x} + u\frac{\partial u}{\partial x} = \frac{\epsilon^2}{2}\left(\frac{1}{2\rho}\frac{\partial^2 \rho}{\partial x^2} - \left(\frac{1}{2\rho}\frac{\partial \rho}{\partial x}\right)^2\right) . \end{aligned}$$

Initial data is independent of ϵ: $\rho(x,0) = A(x)^2$ and $u(x,0) = S'(x)$.

Semiclassical Behavior: Modulation Equations

The formal limiting problem as $\epsilon \downarrow 0$ is the Cauchy problem for the system of *modulation equations*:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ u\end{bmatrix} + \begin{bmatrix} u & \rho \\ -\kappa & u\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\rho \\ u\end{bmatrix} = \mathbf{0}\,, \qquad \rho(x,0) = A(x)^2\,, \quad u(x,0) = S'(x)\,.$$

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Hyperbolicity of modulation equations corresponds to modulational stability. Ellipticity corresponds to (asymptotically catastrophic) modulational instability.

Semiclassical Behavior: Rigorous Asymptotic Analysis

By viewing the NLS equation as a singular perturbation of the corresponding system modulation equations, it is possible to prove by PDE techniques that the modulation equations provide an accurate model for the semiclassical dynamics for $0 \leq t \leq T < \infty$, T independent of ϵ:

- E. Grenier (1998) established this result for general defocusing semilinear Schrödinger equations, where T corresponds to the shock time for the limiting (hyperbolic) system.
- P. Gérard (1993) established this result for general subcritical focusing semilinear Schrödinger equations with analytic initial data, where T corresponds to the singularity formation time for the limiting (elliptic) system.

Semiclassical Behavior: Rigorous Asymptotic Analysis

Restricting to the integrable cases (one dimensional, cubic) and using the corresponding machinery allows one to prove these results in a different way, and more significantly, to obtain asymptotics for the solution beyond the time T at which the modulation equations break down.

- The defocusing case was analyzed using the method of Lax and Levermore by S. Jin, D. Levermore, and D. McLaughlin (1998).
- The focusing case was analyzed using the nonclassical steepest descent method of Deift and Zhou by S. Kamvissis, K. McLaughlin, and M (2003). Other solutions not analyzed in this paper were studied using similar techniques by A. Tovbis, S. Venakides, and X. Zhou (2004). Note: analyticity of initial data is essential for this analysis, even though it is not based on the Cauchy-Kovalevskaya solution of the elliptic modulation equations.

Return to outline.

The Modified Nonlinear Schrödinger Equation

For very short pulses, the focusing NLS equation should be corrected:

$$i\epsilon\frac{\partial\phi}{\partial t}+\frac{\epsilon^2}{2}\frac{\partial^2\phi}{\partial x^2}+|\phi|^2\phi=-i\alpha\epsilon\frac{\partial}{\partial x}(|\phi|^2\phi)+\alpha'\epsilon\frac{\partial}{\partial x}(|\phi|^2)\cdot\phi+i\alpha''\epsilon^3\frac{\partial^3\phi}{\partial x^3}\,.$$

- $\alpha\geq 0$: Nonlinear dispersion.
- $\alpha'\geq 0$: Raman scattering.
- $\alpha''\in\mathbb{R}$: Higher-order linear dispersion.

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- $\alpha'' \in \mathbb{R}$: Higher-order linear dispersion.

Generally the correction terms break the integrability. However the special case of

$$i\epsilon\frac{\partial\phi}{\partial t} + \frac{\epsilon^2}{2}\frac{\partial^2\phi}{\partial x^2} + |\phi|^2\phi + i\alpha\epsilon\frac{\partial}{\partial x}(|\phi|^2\phi) = 0$$

remains integrable but by different machinery for $\alpha > 0$ than for $\alpha = 0$. This equation is the *modified nonlinear Schrödinger* (MNLS) equation.

Lax Pair and Riemann-Hilbert Problem

For $k \in \mathbb{C}$ and a complex-valued function $\phi = \phi(x,t)$, let

$$\mathbf{L} := \begin{bmatrix} \Lambda & 2ik\phi \\ 2ik\phi^* & -\Lambda \end{bmatrix}, \qquad \Lambda := -\frac{2i}{\alpha}\left(k^2 - \frac{1}{4}\right),$$

$$\mathbf{B} := \begin{bmatrix} i\Lambda^2 + 2ik^2|\phi|^2 & -2k\Lambda\phi - k\epsilon\phi_x - 2i\alpha k|\phi|^2\phi \\ -2k\Lambda\phi^* + k\epsilon\phi_x^* - 2i\alpha k|\phi|^2\phi^* & -i\Lambda^2 - 2ik^2|\phi|^2 \end{bmatrix}.$$

Then the simultaneous linear equations (*Lax pair*)

$$\epsilon\frac{\partial \mathbf{v}}{\partial x} = \mathbf{L}\mathbf{v} \quad \text{and} \quad \epsilon\frac{\partial \mathbf{v}}{\partial t} = \mathbf{B}\mathbf{v}$$

are compatible if and only if the *zero curvature condition*

$$\epsilon\frac{\partial \mathbf{L}}{\partial t} - \epsilon\frac{\partial \mathbf{B}}{\partial x} + [\mathbf{L}, \mathbf{B}] = \mathbf{0}$$

holds, a condition equivalent to the MNLS equation for $\phi(x,t)$.

Lax Pair and Riemann-Hilbert Problem

Given initial data $\phi = \phi(x, 0)$ rapidly decreasing as $|x| \to \infty$, one considers $\Im\{k^2\} = 0$ and finds *Jost matrices* $\mathbf{J}_\pm(x; k)$ satisfying

$$\epsilon \frac{d\mathbf{J}_\pm}{dx} = \mathbf{L}\mathbf{J}_\pm\,, \qquad \lim_{x\to\pm\infty} \mathbf{J}_\pm(x; k) e^{-\Lambda x \sigma_3/\epsilon} = \mathbb{I}\,.$$

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The *scattering matrix* $\mathbf{S}(k)$ is defined by

$$\mathbf{S}(k) := \mathbf{J}_-(x; k)^{-1} \mathbf{J}_+(x; k) \, , \qquad \Im\{k^2\} = 0 \, .$$

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Under suitable conditions on ϕ, $\mathbf{S}(k)$ is continuous with $\mathbf{S}(0) = \mathbb{I}$, and satisfies the symmetries

$$\mathbf{S}(-k) = \sigma_3 \mathbf{S}(k) \sigma_3 \qquad \text{and} \qquad \mathbf{S}(k^*) = \sigma_2 \mathbf{S}(k)^* \sigma_2 \,.$$

Lax Pair and Riemann-Hilbert Problem

Continuous spectral data: The *reflection coefficient* and *jump matrix* are defined as

$$r(k) := -\frac{S_{12}(k)}{S_{22}(k)}, \qquad \mathbf{V}_0(k) := \begin{bmatrix} 1 \pm |r(k)|^2 & r(k) \\ \pm r(k)^* & 1 \end{bmatrix}, \qquad \pm k^2 > 0 .$$

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Discrete spectral data: $S_{11}(k)$ has an analytic continuation to $\Im\{k^2\} < 0$, where its zeros (assumed simple) are *eigenvalues* k_j. For each eigenvalue k_j there is a proportionality constant γ_j such that

$$\mathbf{j}_+^{(1)}(x;k_j) = \gamma_j \mathbf{j}_-^{(2)}(x;k_j)\,.$$

Set

$$c_j^0 := \frac{\gamma_j}{S'_{11}(k_j)}\,.$$

Let $D := \{k_1, \ldots, k_N\} \cup \{k_1^*, \ldots, k_N^*\}$.

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Riemann-Hilbert problem: Seek a 2×2 matrix-valued function $\mathbf{M}(k; x, t)$ of $k \in \mathbb{C}$ with $(x, t) \in \mathbb{R}^2$ with the following properties:

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Analyticity: $\mathbf{M}(k; x, t)$ is analytic for $\Im\{k^2\} \neq 0$ and $k \notin D$ and takes continuous boundary values on the axes $\Im\{k^2\} = 0$ from each of the four sectors of analyticity. Moreover, $\mathbf{M}(k; x, t)$ is uniformly bounded for large k.

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Jump Condition: Letting $\mathbf{M}_\pm(k; x, t)$ denote the boundary value taken from the region where $\pm\Im\{k^2\} < 0$, the boundary values are related by

$$\mathbf{M}_+(k; x, t) = \mathbf{M}_-(k; x, t)e^{(\Lambda x + i\Lambda^2 t)\sigma_3/\epsilon}\mathbf{V}_0(k)e^{-(\Lambda x + i\Lambda^2 t)\sigma_3/\epsilon}, \qquad \pm k^2 > 0\,.$$

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Riemann-Hilbert problem: Seek a 2×2 matrix-valued function $\mathbf{M}(k; x, t)$ of $k \in \mathbb{C}$ with $(x, t) \in \mathbb{R}^2$ with the following properties:

Singularities: $\mathbf{M}(k; x, t)$ has simple poles at the points of D. If $k_j \in D$ with $\Im\{k_j\} > 0$ and $\Re\{k_j\} < 0$, then with $c_j(x, t) := c_j^0 e^{-2(\Lambda_j x + i\Lambda_j^2 t)/\epsilon}$, $\Lambda_j := \Lambda(k_j)$:

$$\operatorname*{Res}_{k=\pm k_j} \mathbf{M}(k; x, t) = \lim_{k \to \pm k_j} \mathbf{M}(k; x, t) \begin{bmatrix} 0 & 0 \\ c_j(x, t) & 0 \end{bmatrix}$$

$$\operatorname*{Res}_{k=\pm k_j^*} \mathbf{M}(k; x, t) = \lim_{k \to \pm k_j^*} \mathbf{M}(k; x, t) \begin{bmatrix} 0 & -c_j(x, t)^* \\ 0 & 0 \end{bmatrix}$$

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Normalization at the Origin: The matrix $\mathbf{M}(k; x, t)$ is normalized in the sense that

$$\lim_{k \to 0} \mathbf{M}(k; x, t) = \mathbb{I}\,.$$

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From the solution of this problem,

$$\phi(x, t) := \lim_{k \to \infty} \frac{2k}{\alpha} \frac{M_{12}(k; x, t)}{M_{22}(k; x, t)}$$

solves the Cauchy problem for the MNLS equation.

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However, it can be shown (see Kaup & Newell, 1978) that

$$\mathbf{N}(z; x, t) := k^{\sigma_3/2}\mathbf{M}(k; x, t)k^{-\sigma_3/2}$$

is a function only of $z = -k^2$. Consequently, (the first row of) $\mathbf{N}(z; x, t)$ satisfies a Riemann-Hilbert problem with a jump discontinuity only on the real z-axis, and with half the number of poles, arranged in complex-conjugate pairs, with no further symmetry.

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This "de-symmetrized" formulation of the Riemann-Hilbert problem is more like that for the focusing NLS equation. It is better suited to semiclassical analysis with a "g-function" because the genus of the microstructure will be correctly predicted.

Formal Semiclassical Limit

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$$\rho(x,t) := |\phi|^2 \qquad \text{and} \qquad u(x,t) := \epsilon \Im \frac{\partial}{\partial x} \log(\phi)\,,$$

the initial-value problem for the MNLS equation becomes, exactly,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[u\rho + \frac{3\alpha}{2}\rho^2\right] = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}u^2 - \rho + \alpha u \rho\right] = \frac{\epsilon^2}{2}\frac{\partial}{\partial x}\frac{(\rho^{1/2})_{xx}}{\rho^{1/2}}\,,$$

with initial data independent of ϵ: $\rho(x,0) = A(x)^2$ and $u(x,0) = S'(x)$.

Formal Semiclassical Limit: Modulation Equations

Setting $\epsilon = 0$ yields a Cauchy problem for the quasilinear system of modulation equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ u\end{bmatrix} + \begin{bmatrix}u + 3\alpha\rho & \rho \\ \alpha u - 1 & u - \alpha\rho\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\rho \\ u\end{bmatrix} = \mathbf{0}\,.$$

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This system is

- Elliptic if $\alpha^2\rho + \alpha u - 1 < 0$.
- Hyperbolic if $\alpha^2\rho + \alpha u - 1 > 0$.

Therefore, modulational stability can be recovered with a focusing nonlinearity if $\alpha > 0$ is sufficiently large, and if $u > 0$ in the tails of ϕ. In particular, since $\rho > 0$, the condition $u > 1/\alpha$ is sufficient (but not necessary) for hyperbolicity.

Formal Semiclassical Limit: Lack of Galilean Invariance

The fact that a sufficiently large velocity $u > 1/\alpha$ makes the modulation equations hyperbolic might have been expected, because the MNLS equation is not invariant under the group of Galilean velocity boosts.

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Set $\phi(x,t) = e^{i(c\xi + c^2\tau)/\epsilon}\psi(\xi,\tau)$ with $\xi = x - ct$ and $\tau = t$. If $\phi(x,t)$ satisfies the MNLS equation, then $\psi(\xi,\tau)$ satisfies

$$i\epsilon\frac{\partial\psi}{\partial\tau} + \frac{\epsilon^2}{2}\frac{\partial^2\psi}{\partial\xi^2} + (1-\alpha c)|\psi|^2\psi + i\alpha\epsilon\frac{\partial}{\partial\xi}(|\psi|^2\psi) = 0\,.$$

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If $c > 1/\alpha$, this equation looks like a perturbation of the modulationally stable *defocusing* NLS equation rather than the modulationally unstable focusing NLS equation.

Formal Semiclassical Limit: Connection with Focusing NLS

Suppose that $\alpha^2\rho + \alpha u - 1 < 0$ (unstable case) and that $\rho > 0$, defining an open domain $D_-(\alpha) \subset \mathbb{R}^2$. Consider the map F_- taking $(\rho, u) \in D_-(\alpha)$ to $(\hat{\rho}, \hat{u}) \in \mathbb{R}^2$:

$$\hat{\rho} := -\rho \cdot (\alpha^2\rho + \alpha u - 1)$$

$$\hat{u} := u + 2\alpha\rho\,.$$

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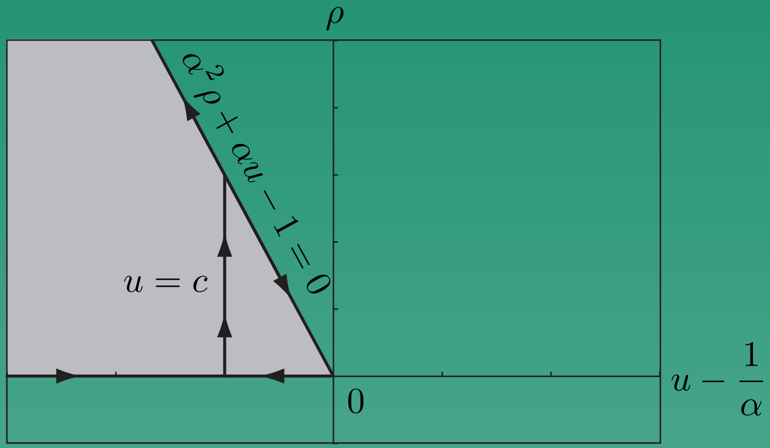
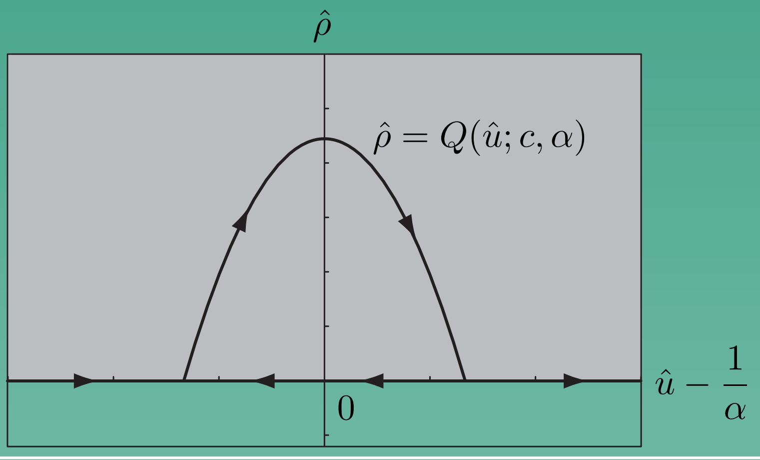
F_- is one-to-one and maps $D_-(\alpha)$ onto the upper half-plane $\hat{\rho} > 0$:





Formal Semiclassical Limit: Connection with Focusing NLS

It is a direct matter to check that if $(\rho, u) \in D_{-}(\alpha)$ and satisfy the (elliptic) MNLS modulation equations, then

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{\rho}\\ \hat{u}\end{bmatrix} + \begin{bmatrix}\hat{u} & \hat{\rho}\\ -1 & \hat{u}\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\hat{\rho}\\ \hat{u}\end{bmatrix} = \mathbf{0}\,.$$

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This is exactly the system of modulation equations for the focusing NLS equation. The semiclassical dynamics of the MNLS equation on the modulationally unstable sector of its phase space is equivalent to the semiclassical dynamics of the focusing NLS equation.

Formal Semiclassical Limit: Connection with Defocusing NLS

Suppose instead that $\alpha^2\rho + \alpha u - 1 > 0$ (stable case) and that $\rho > 0$, defining an open domain $D_+(\alpha) \subset \mathbb{R}^2$. Consider the map F_+ taking $(\rho, u) \in D_+(\alpha)$ to $(\hat{\rho}, \hat{u}) \in \mathbb{R}^2$:

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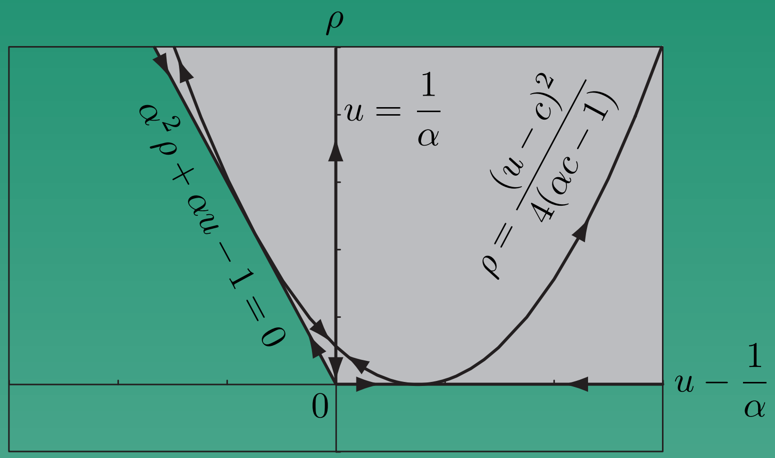
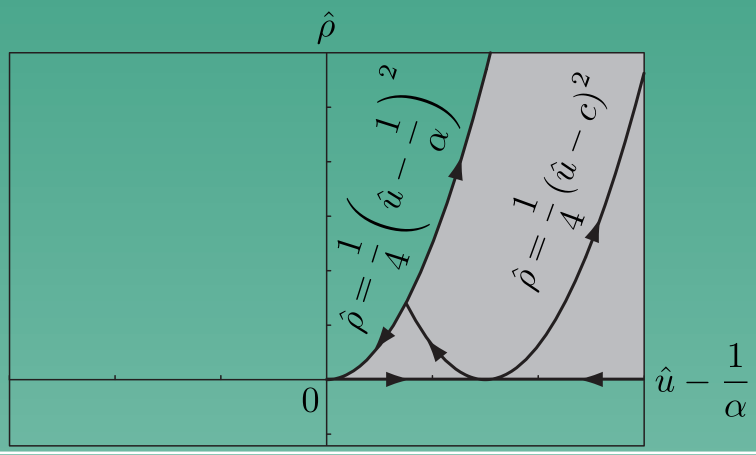
Unlike F_-, F_+ is generally two-to-one and has a smaller range:





Formal Semiclassical Limit: Connection with Defocusing NLS

It is a direct matter to check that if $(\rho, u) \in D_+(\alpha)$ and satisfy the (hyperbolic) MNLS modulation equations, then

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This is exactly the system of modulation equations for the defocusing NLS equation. The semiclassical dynamics of the MNLS equation on the modulationally stable sector of its phase space is equivalent (modulo issues related to the noninvertibility of the map F_+) to the semiclassical dynamics of the defocusing NLS equation.

Bounds on the Discrete Spectrum

The first step in a rigorous semiclassical analysis of the MNLS equation via inverse-scattering techniques is to study the spectral problem

$$\epsilon\frac{d\mathbf{v}}{dx} = \mathbf{L}\mathbf{v}\,, \quad \mathbf{L} := \begin{bmatrix} \Lambda & 2ik\phi \\ 2ik\phi^* & -\Lambda \end{bmatrix}\,, \quad \Lambda := -\frac{2i}{\alpha}\left(k^2 - \frac{1}{4}\right)\,,$$

where $\phi = A(x)e^{iS(x)/\epsilon}$ is the initial data for the Cauchy problem.

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- Any information that further confines the discrete spectrum, especially in the limit $\epsilon \downarrow 0$, is essential for semiclassical analysis.
- We generalize an argument of Deift, Venakides, and Zhou for the Zakharov-Shabat eigenvalue problem to the present context.

Bounds on the Discrete Spectrum

A WKB approach for small ϵ: set $\mathbf{w} := e^{-(iS(x)/(2\epsilon))\sigma_3}\mathbf{v}$ to remove oscillations from the coefficients:

$$2\alpha\epsilon\frac{d\mathbf{w}}{dx} = i\mathbf{M}\mathbf{w}\,, \qquad \mathbf{M} := \begin{bmatrix} -4k^2 + 1 - \alpha S'(x) & 4\alpha k A(x) \\ 4\alpha k A(x) & 4k^2 - 1 + \alpha S'(x) \end{bmatrix}.$$

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Then expand (formally) $\mathbf{w} = e^{i\sigma/(2\alpha\epsilon)}(\mathbf{w}_o + \epsilon\mathbf{w}_1 + \cdots)$. At leading order,

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Eigenvalues of $\mathbf{M}$ are $\pm\omega$ where

$$\omega(x;k) := \left[16\alpha^2k^2A(x)^2 + (4k^2-1+\alpha S'(x))^2\right]^{1/2}.$$

Bounds on the Discrete Spectrum

Turning points in WKB are values of $x \in \mathbb{R}$ for which the eigenvalues of $\mathbf{M}$ degenerate. For most $k \in \mathbb{C}$ there are no turning points at all. In this case, $\omega(x;k)$ is well-defined for $x \in \mathbb{R}$ by continuity and a choice of branch. The exceptional values of $k \in \mathbb{C}$ with $\Im\{k^2\} \neq 0$ lie on the *turning point curve* $\mathcal{T}$ defined parametrically by

$$\Im\{k\} = s_1 \frac{\alpha}{2} A(x)\,, \quad \Re\{k\} = s_2 \frac{1}{2}\sqrt{1 - \alpha S'(x) - \alpha^2 A(x)^2}\,,$$

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for modulationally unstable $x \in \mathbb{R}$, where s_j are independent signs. Define also

$$q(x;k) := \frac{2\alpha k A(x)}{\omega(x;k)} \cdot \frac{d}{dx} \log\left(\frac{A(x)}{\omega(x;k) + 4k^2 - 1 + \alpha S'(x)}\right)\,,$$

and set

$$L_k := \sup_{x\in\mathbb{R}} \left|\frac{d}{dx}\left(\frac{1}{\Im\{\omega(x;k)\}}\right)\right| + 2\sup_{x\in\mathbb{R}}\left|\frac{\Re\{q(x;k)\}}{\Im\{\omega(x;k)\}}\right|\,.$$

Bounds on the Discrete Spectrum

Theorem 1. *Let $A : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a uniformly Lipschitz function of class $L^1(\mathbb{R})$ and let $S' : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitz with $S''(\cdot)$ of class $L^1(\mathbb{R})$. Let k be a fixed complex number with $\Im\{k^2\} \neq 0$ and $k \notin \mathcal{T}$. The following statements hold:*

(a) If k is an eigenvalue, then $|\Im\{k\}| \leq \dfrac{\alpha}{2} \sup_{x \in \mathbb{R}} A(x)$.

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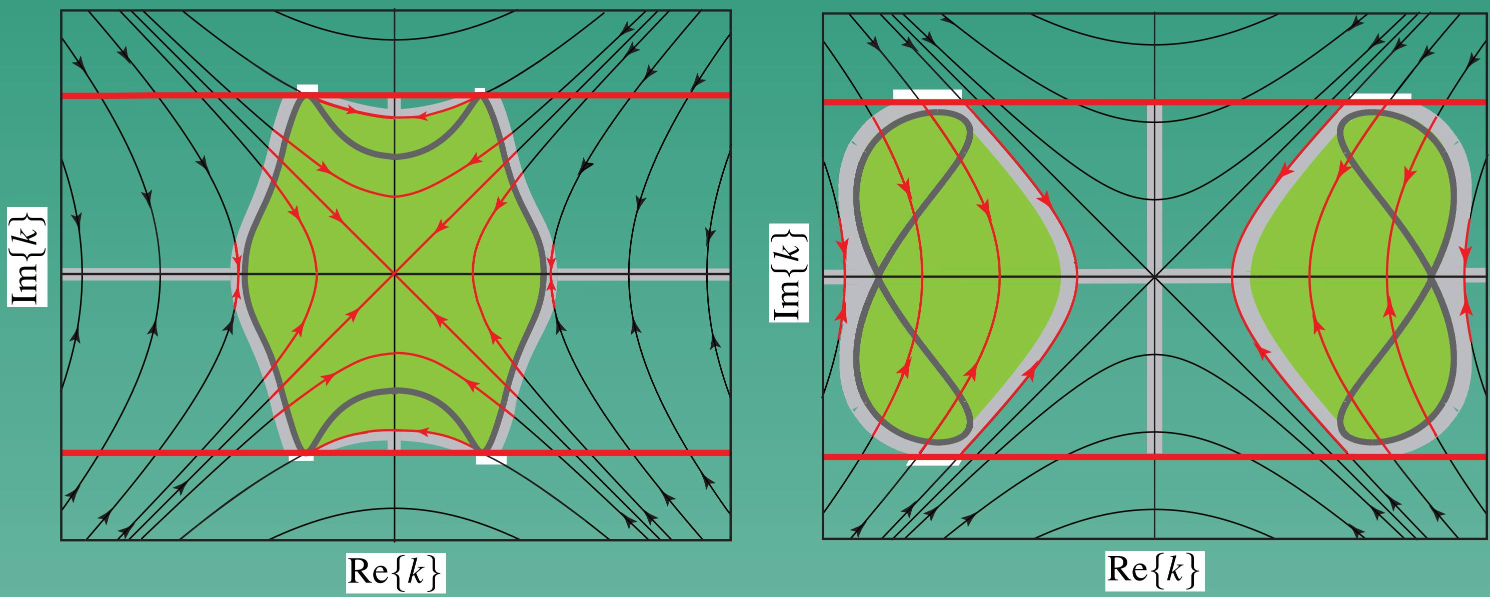
We use part (b) in the following way: we seek conditions on $k \in \mathbb{C}$ for which we can prove that $L_k < +\infty$. Such k cannot be eigenvalues for any values of ϵ sufficiently small.

Note that for the class of potentials under consideration, $L_k < +\infty$ if $\Im\{\omega(x; k)\}$ does not vanish for any $x \in \mathbb{R}$.

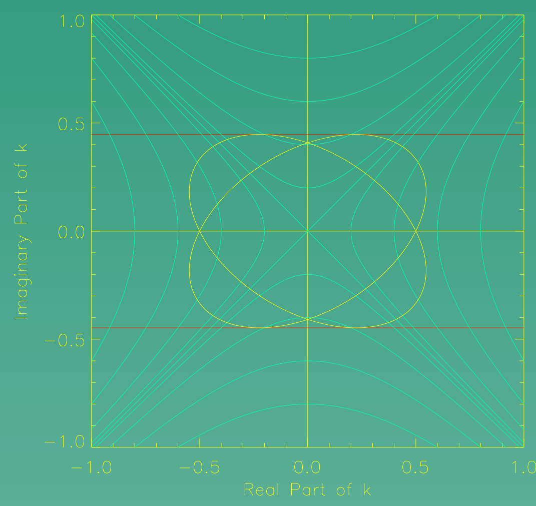
Bounds on the Discrete Spectrum

The condition $\Im\{\omega(x;k)\} \neq 0$ for all $x \in \mathbb{R}$ has a simple geometrical interpretation: as $\epsilon \downarrow 0$ all eigenvalues lie in the "hyperbolic shadow" of the turning point curve $\mathcal{T}$:



Bounds on the Discrete Spectrum

Example: $A(x) = \text{sech}(x)$ and $S'(x) = \text{sech}(x)\tanh(x)$, $\alpha = 0.894$.



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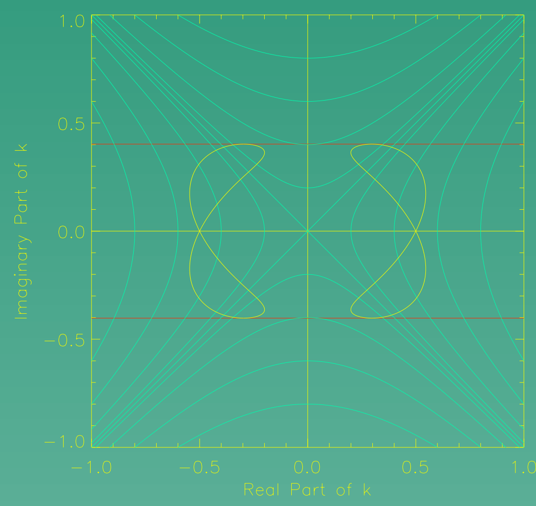
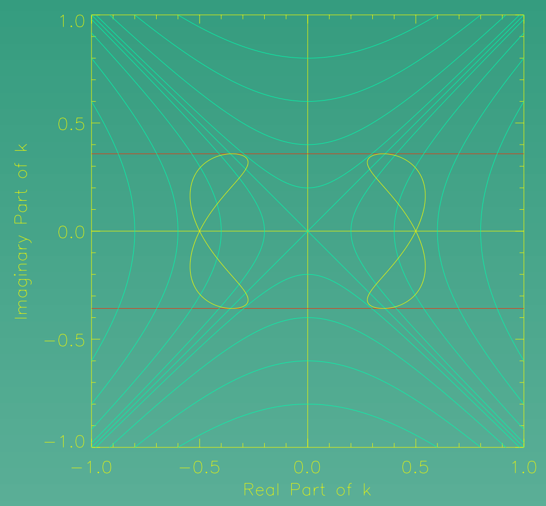
Example: $A(x) = \operatorname{sech}(x)$ and $S'(x) = \operatorname{sech}(x)\tanh(x)$, $\alpha = 0.805$.



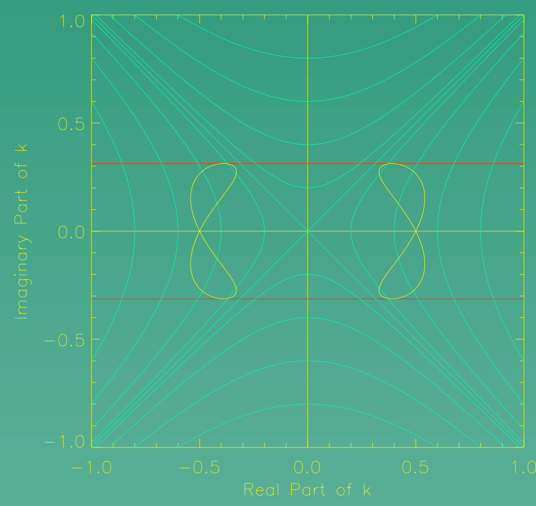
Bounds on the Discrete Spectrum

Example: $A(x) = \text{sech}(x)$ and $S'(x) = \text{sech}(x)\tanh(x)$, $\alpha = 0.716$.



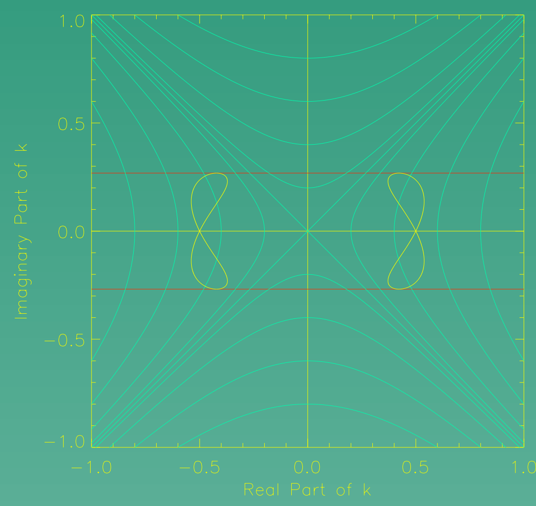
Bounds on the Discrete Spectrum

Example: $A(x) = \operatorname{sech}(x)$ and $S'(x) = \operatorname{sech}(x)\tanh(x)$, $\alpha = 0.626$.



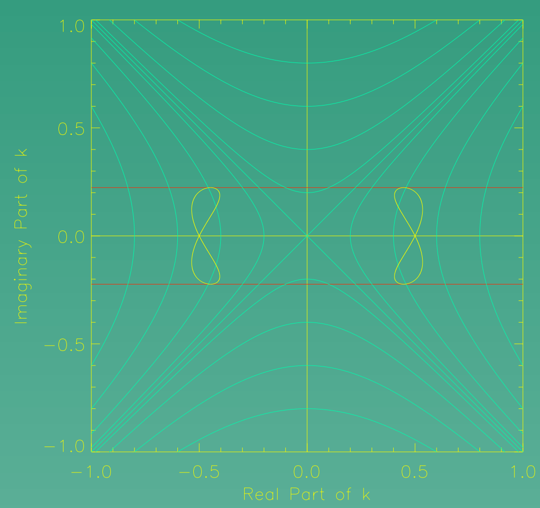
Bounds on the Discrete Spectrum

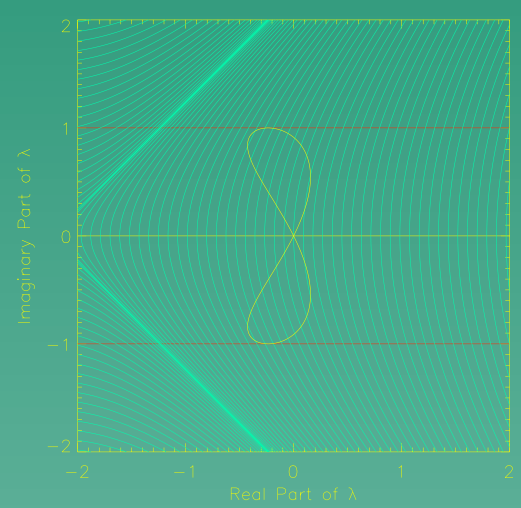
Example: $A(x) = \operatorname{sech}(x)$ and $S'(x) = \operatorname{sech}(x)\tanh(x)$, $\alpha = 0.537$.



Bounds on the Discrete Spectrum

Example: $A(x) = \mathrm{sech}(x)$ and $S'(x) = \mathrm{sech}(x)\tanh(x)$, $\alpha = 0.447$.

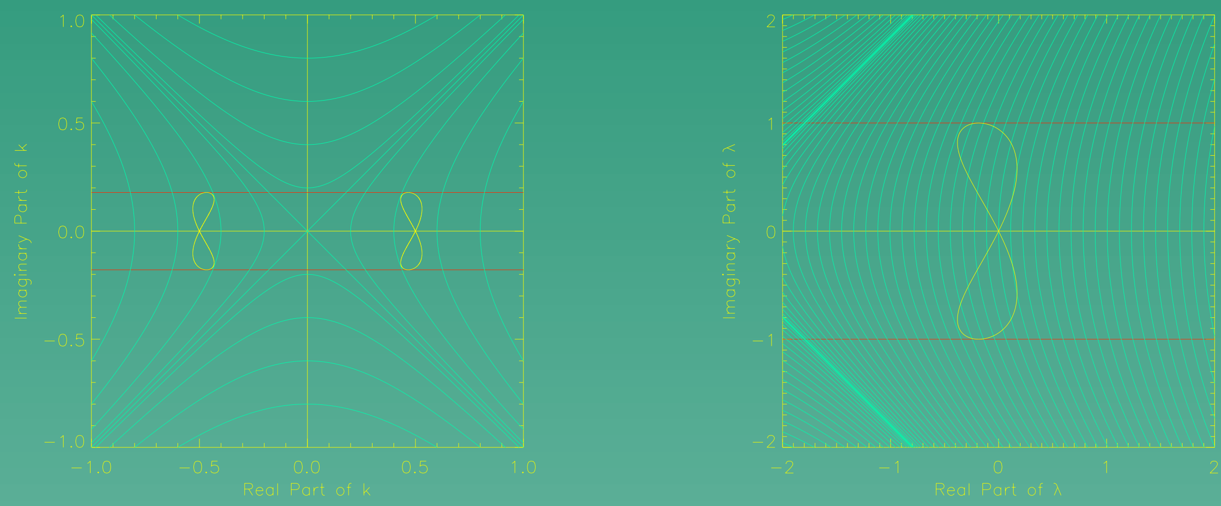




Note: $\lambda = (2k-1)/\alpha$ is the spectral parameter for focusing NLS ($\alpha = 0$).

Bounds on the Discrete Spectrum

Example: $A(x) = \operatorname{sech}(x)$ and $S'(x) = \operatorname{sech}(x)\tanh(x)$, $\alpha = 0.358$.



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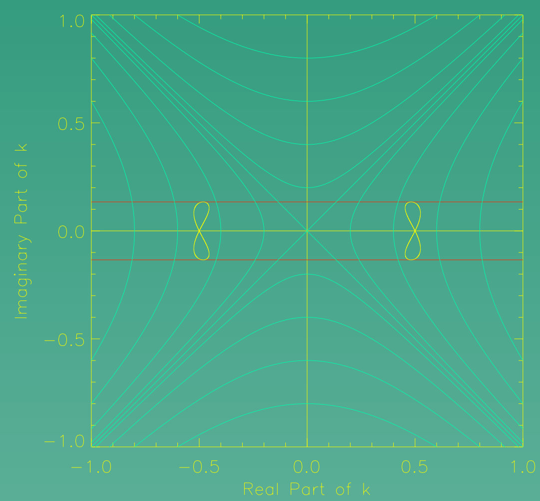
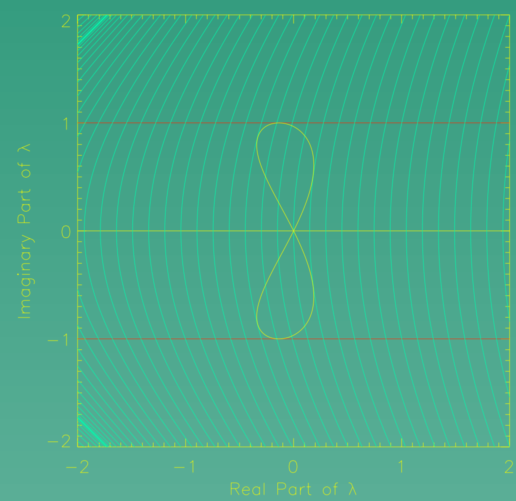
Example: $A(x) = \operatorname{sech}(x)$ and $S'(x) = \operatorname{sech}(x)\tanh(x)$, $\alpha = 0.268$.





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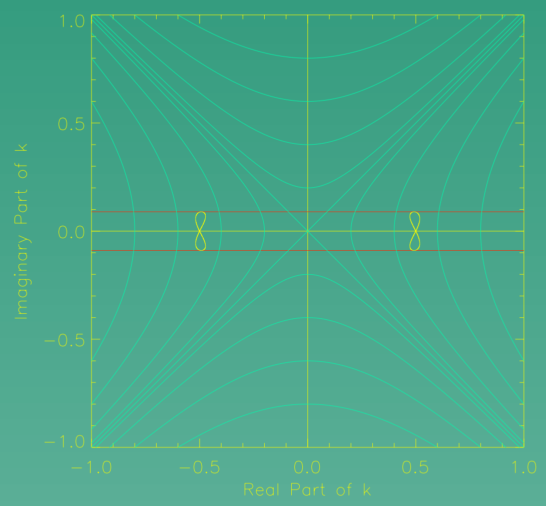

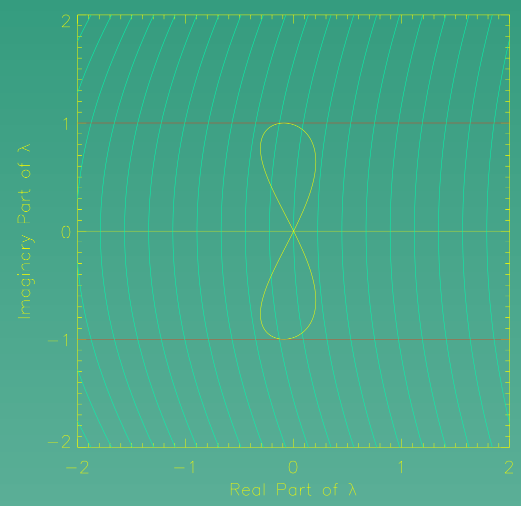

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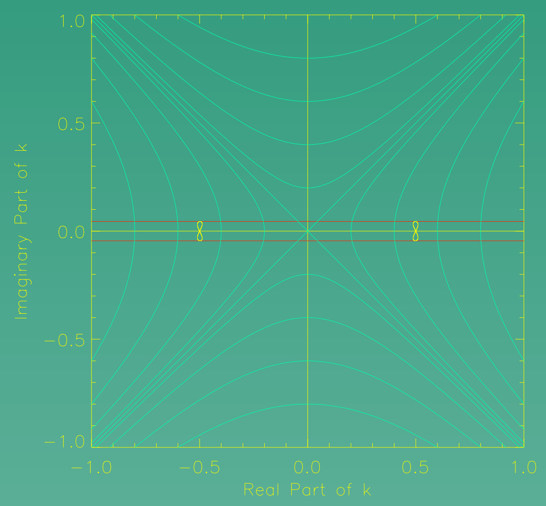

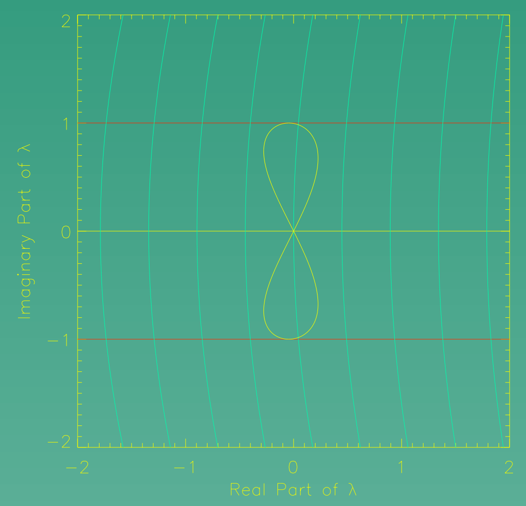

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Hypergeometric Potentials

Some history:

- In 1973, Satsuma and Yajima showed that for potentials of the form $\phi(x) = \nu \operatorname{sech}(x)$ in the nonselfadjoint Zakharov-Shabat spectral problem (appropriate for focusing NLS):

$$\epsilon \frac{d\mathbf{v}}{dx} = \begin{bmatrix} -i\lambda & \phi \\ -\phi^* & i\lambda \end{bmatrix} \mathbf{v}$$

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- In 2000, Tovbis and Venakides generalized this result to potentials of the form $\phi(x) = \nu\,\mathrm{sech}(x)e^{iS(x)/\epsilon}$ where $S'(x) = \mu\tanh(x)$ and ν and μ are independent real parameters.

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It is easy to see that the Tovbis-Venakides analysis also goes through virtually unchanged if $S'(x) = \mu \tanh(x) + \delta$ for any $\delta \in \mathbb{R}$.

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  - ⋆ if $|\mu| < (\alpha\delta - 1)/\alpha$, then the modulation equations are hyperbolic for all x at $t = 0$, while
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  - if $|\mu| > (\alpha\delta - 1)/\alpha$, then there exist $x \in \mathbb{R}$ for which the modulation equations are elliptic at $t = 0$.
- If $\alpha\delta < 1$, then regardless of the value of $\mu \in \mathbb{R}$ there exist $x \in \mathbb{R}$ for which the modulation equations are elliptic at $t = 0$. In this case,
  - if $|\mu| < 2\nu\sqrt{1 - \alpha\delta}$ then there are $\sim \epsilon^{-1}$ discrete eigenvalues, while
  - if $|\mu| > 2\nu\sqrt{1 - \alpha\delta}$ then there are no discrete eigenvalues.

Hypergeometric Potentials

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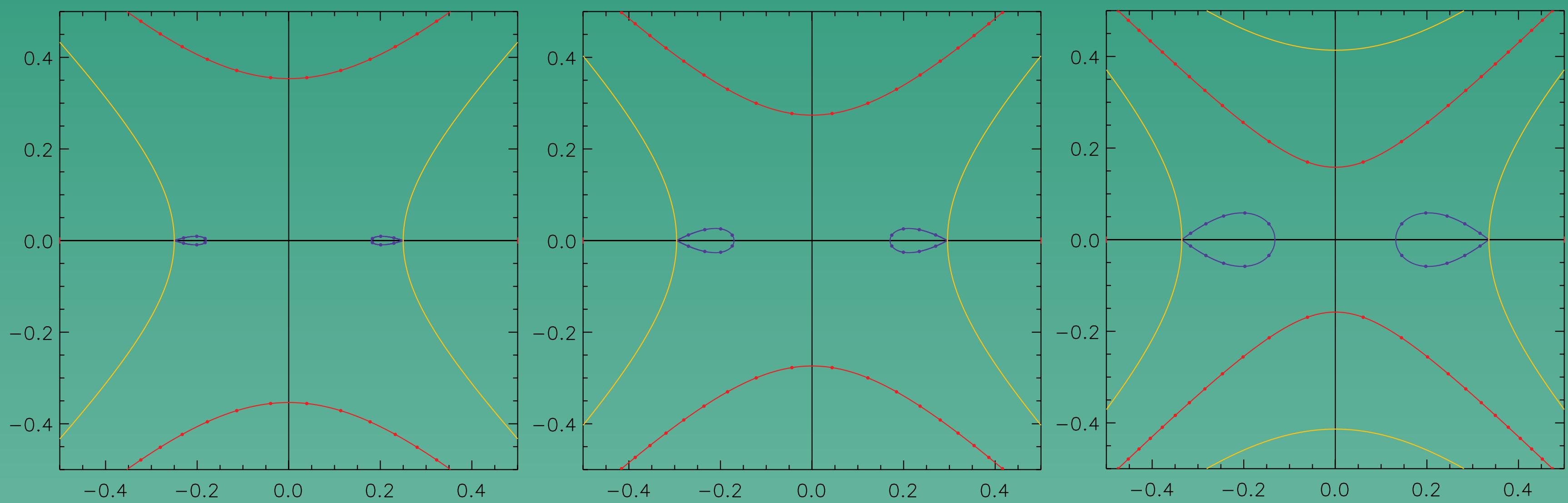
- "Phantom poles": these are simple poles $k = k_m$ that have nothing to do with eigenvalues. Their representatives in the second quadrant are given by

$$\frac{i\mu}{2} + \Omega(k_m) = -\left(m + \frac{1}{2}\right)\epsilon\,,$$

  for $m = 0, 1, 2, \ldots$.

Hypergeometric Potentials

Eigenvalue poles and phantom poles may only interact if $-\alpha^2\nu^2/2 \leq \mu \leq 0$.
Interaction of eigenvalue poles and phantom poles for $\epsilon = 0.075$, $\nu = 0.6846$, $\delta = 0.5$ and $\mu = -0.5$:



$\alpha = 1.5$ $\alpha = 1.3$ $\alpha = 1.1$

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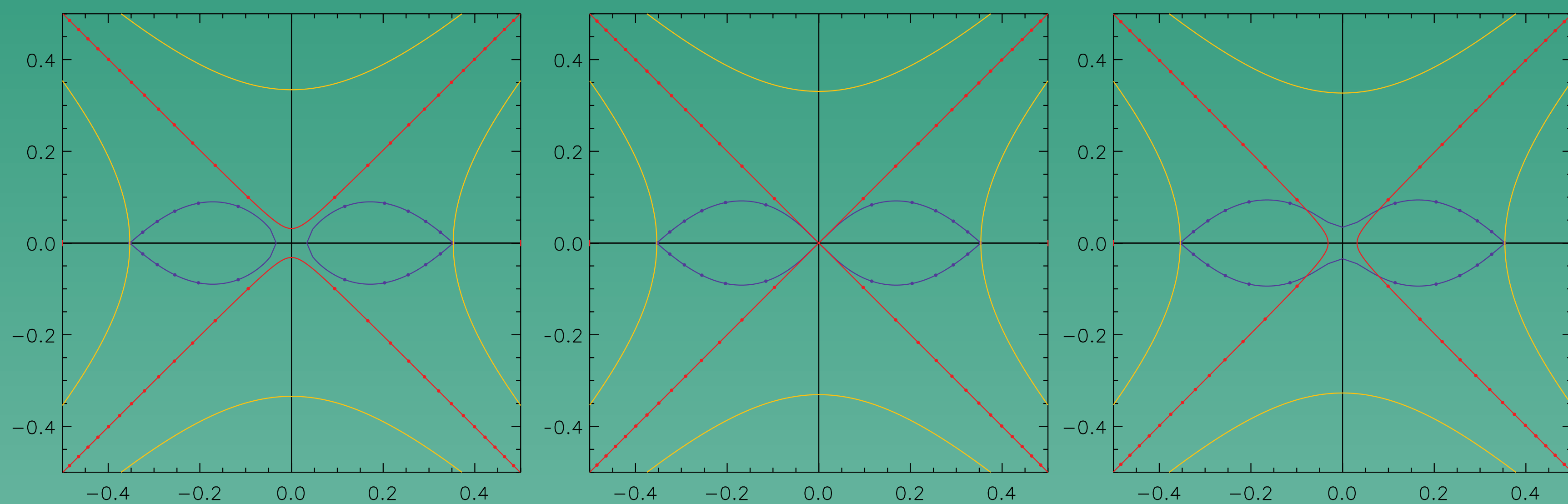

$\alpha = 1.004$ $\alpha = 1$ $\alpha = 0.996$

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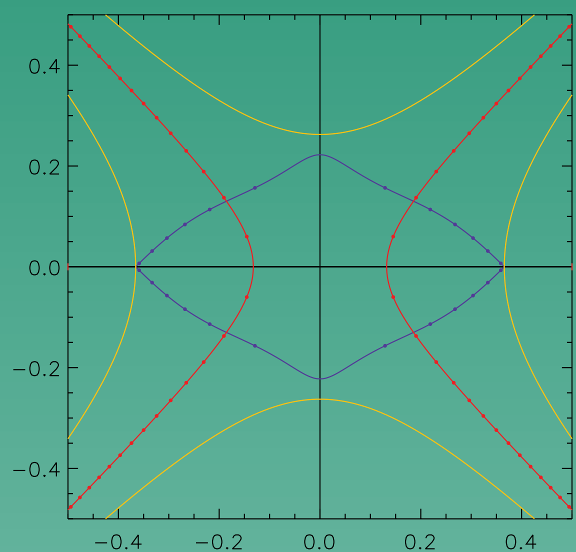
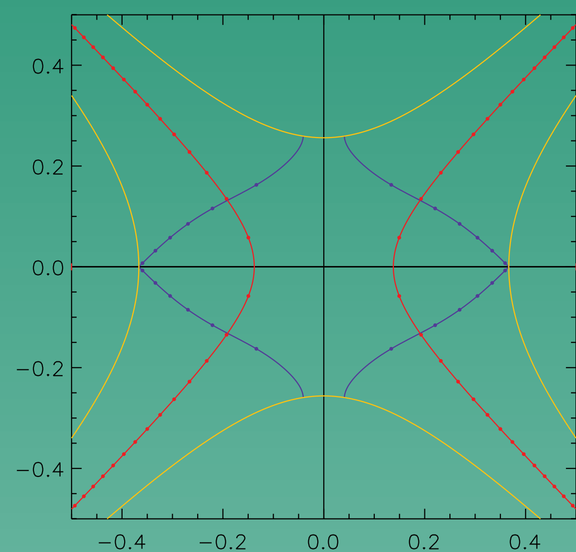


$\alpha = 0.930$



$\alpha = 0.924$

Hypergeometric Potentials

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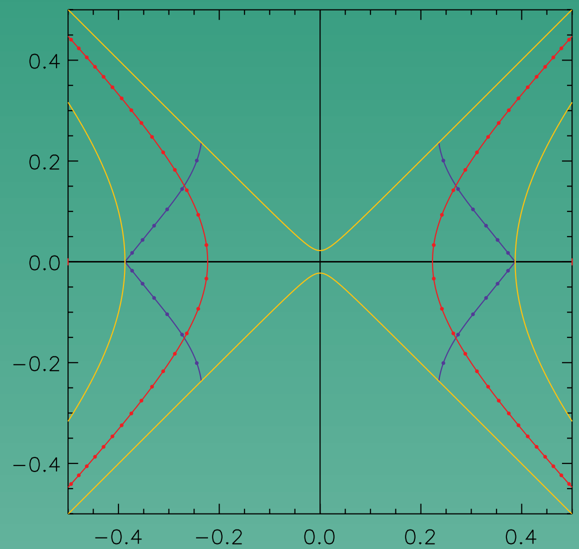
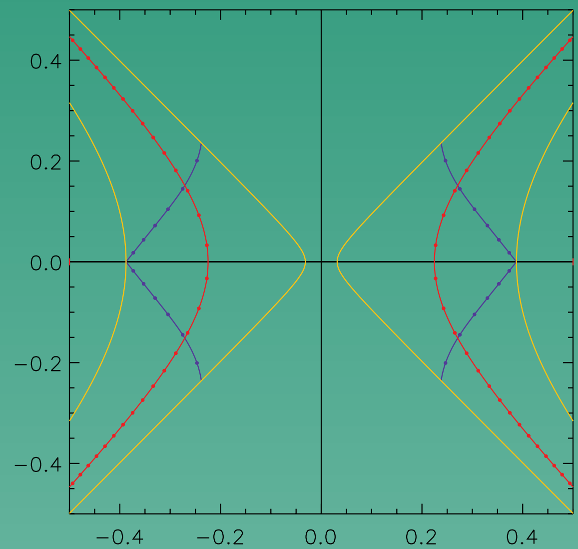


$\alpha = 0.801$



$\alpha = 0.798$

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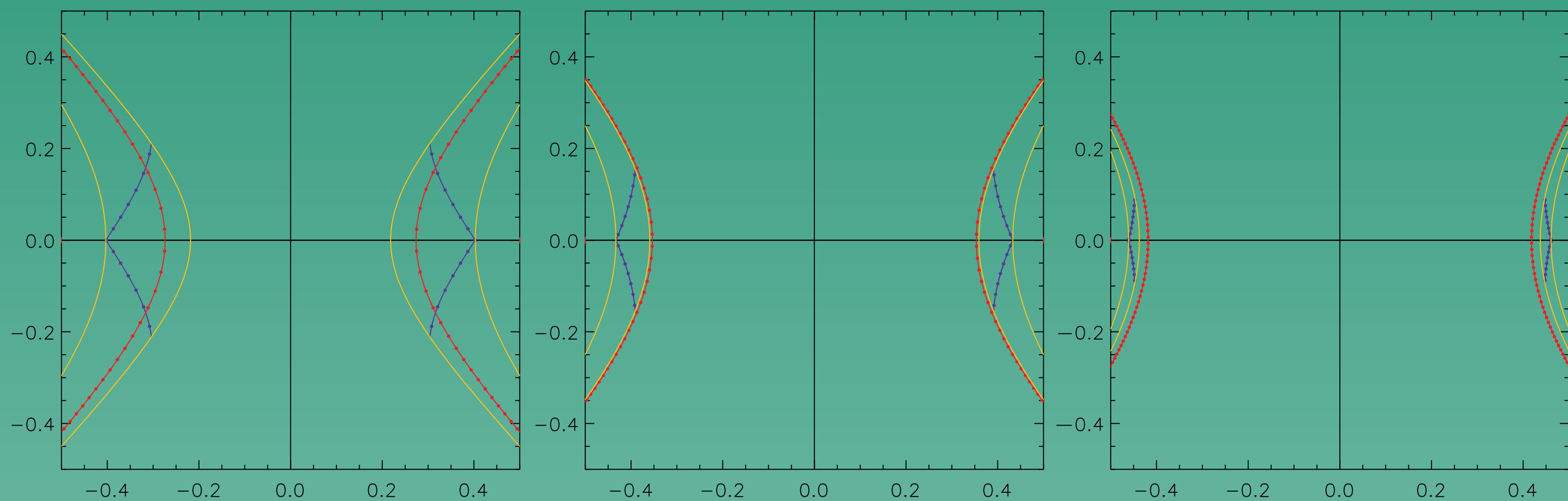
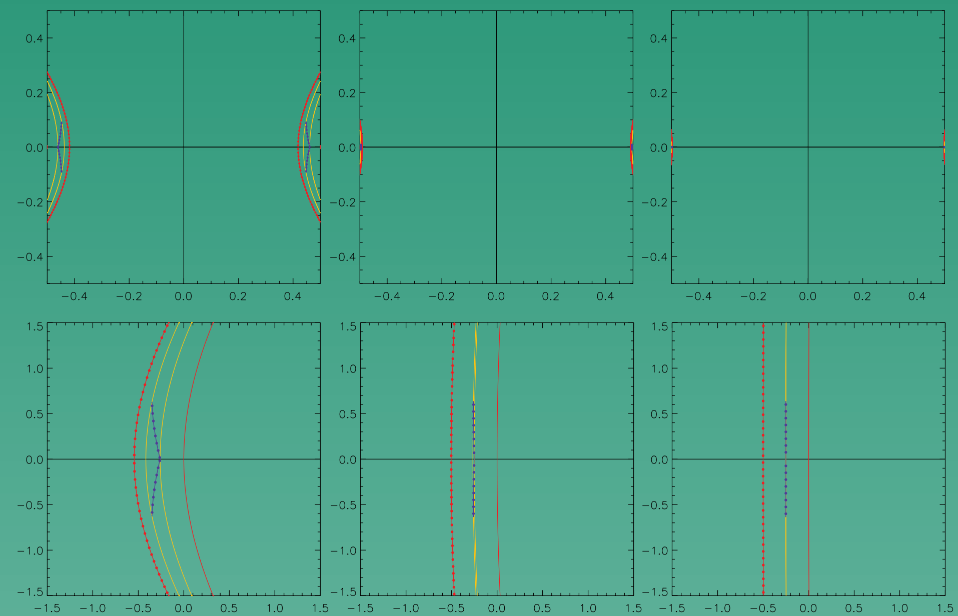


$\alpha = 0.7$ $\alpha = 0.5$ $\alpha = 0.3$

Hypergeometric Potentials

The limiting behavior as $\alpha \longrightarrow 0$ corresponds to the predictions of Tovbis and Venakides for the Zakharov-Shabat problem:



$\alpha = 0.3$ $\alpha = 0.03$ $\alpha = 0.003$

Return to outline.

Ongoing Work and Conclusions

The next phase of the project includes the following subprojects:

- The investigation of a Klaus-Shaw type exact eigenvalue confinement condition. Hypothesis: if $A(x) > 0$ has a single local maximum and if $4\alpha A(x)A'(X) + S''(x) \equiv 0$ (this puts the $\mathcal{T}$ on a single hyperbola), then the eigenvalues are exactly confined to this hyperbola for all $\epsilon > 0$.

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- The rigorous semiclassical asymptotic analysis of the Riemann-Hilbert problem of inverse scattering for the hypergeometric cases, using the nonclassical steepest descent method of Deift and Zhou. Special attention paid to
  - ⋆ implications of interactions of the phantom poles with eigenvalues,
  - ⋆ implications of crossing the modulational stability threshold.

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Return to outline.

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Return to outline.

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Thank You!

Return to outline.