Universality Classes for Semiclassical Eigenvalue Problems

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Abstract

We will discuss some generalized eigenvalue problems (in which the eigenvalue does not necessarily enter linearly) in a semiclassical scaling (where derivatives are multiplied by small coefficients). Such problems arise frequently in the asymptotic analysis of nonlinear problems solvable by an inverse-scattering transform. We will show how the asymptotics of the discrete spectrum leads to the idea of universality classes of potentials and describe how this idea can be used to approximate the discrete spectrum with quantitiative error estimates.

Motivation: Semiclassical Limits for Integrable PDE

Consider, for ϵ positive, the following Cauchy problem (FNLS):

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0\,, \qquad x \in \mathbb{R}\,, \quad t > 0\,,$$

$$\psi(x,0) = A(x)e^{iS(x)/\epsilon}\,.$$

We suppose that $A(\cdot) > 0$, $A(\pm\infty) = 0$, $S(\cdot) \in \mathbb{R}$, and $S'(\pm\infty) = u_\pm$.

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Here is another Cauchy problem (SG):

$$\epsilon^2 u_{tt} - \epsilon^2 u_{xx} + \sin(u) = 0\,, \qquad x \in \mathbb{R}\,, \quad t > 0\,,$$

$$u(x,0) = f(x)\,, \quad \epsilon u_t(x,0) = g(x)\,.$$

We suppose that $f(\cdot) \in \mathbb{R}$, $g(\cdot) \in \mathbb{R}$, $f(\pm\infty) = 2\pi N_\pm$, and $g(\pm\infty) = 0$.

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Both (FNLS) and (SG) may be analyzed for small ϵ because their (unique) solutions may be expressed for all ϵ via an inverse-scattering transform. There are two key steps in such a solution procedure:

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The subject of this talk is step 1. What can we say about the spectral data for general initial conditions? What can be made quantitatively accurate for small ϵ? How accurate?

Generalized Eigenvalue Problems

The eigenvalue problem for (FNLS) is

$$\epsilon\frac{da}{dx} = -i\lambda a + A(x)e^{iS(x)/\epsilon}b$$
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Here $\lambda \in \mathbb{C}$ is the eigenvalue parameter. The generalized eigenvalue problem for (SG) is

$$4i\epsilon\frac{da}{dx} = (z - z^{-1})\cos(\tfrac{1}{2}f(x))a - (z + z^{-1})\sin(\tfrac{1}{2}f(x))b - ig(x)b$$
$$4i\epsilon\frac{db}{dx} = -(z + z^{-1})\sin(\tfrac{1}{2}f(x))a + ig(x)a - (z - z^{-1})\cos(\tfrac{1}{2}f(x))b\,.$$

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- (M. Klaus and J. K. Shaw, 2000) Consider the (FNLS) eigenvalue problem. If $S(\cdot) \equiv 0$ and $A \in L^1(\mathbb{R})$ is a function with a single critical point (a local max) then the discrete spectrum lies on the imaginary axis of the λ-plane, and the number of eigenvalues is the integer part of a specific multiple of $\|A\|_1/\epsilon$.

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- (J. Bronski and M. Johnson, 2007) Consider the (SG) generalized eigenvalue problem. If $g(\cdot) \equiv 0$ and $\sin(\frac{1}{2}f) \in L^1(\mathbb{R})$ is a function with a single critical point (a local max) then the discrete spectrum lies on the unit circle of the z-plane, and the number of eigenvalues is the integer part of a specific multiple of $\|\sin(\frac{1}{2}f)\|_1/\epsilon$.

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In both cases, the analysis is carried out by rewriting the eigenvalue problem as a hypergeometric equation, and using Euler integral representations of solutions to construct the spectral data (including eigenvalues, proportionality constants, and reflection coefficients) in terms of gamma functions.

Universality Classes

The key idea of this talk is that the family of Klaus-Shaw potentials for the (FNLS) spectral problem and the family of Bronski-Johnson potentials for the (SG) spectral problem could be viewed as universality classes of which the Satsuma-Yajima potential and the Buckingham-M. potential are, respectively, completely understood representatives.

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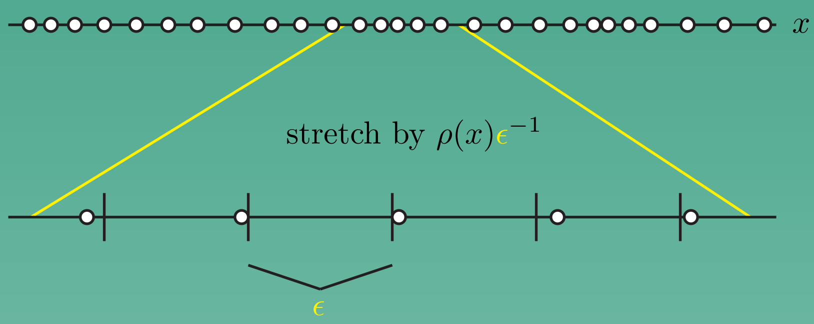
Here by universality we mean: upon appropriate rescaling of the spectrum near any point, the local spacings of eigenvalues converge with great precision as $\epsilon \downarrow 0$ to the corresponding equal spacings of the exactly solvable model. In the semiclassical limit $\epsilon \downarrow 0$, the discrete spectrum of all Klaus-Shaw potentials looks (locally) the same, and that of all Bronski-Johnson potentials looks (locally) the same.



Analysis of the Eigenvalue Problem: Langer Transformation

A way to approach proving universality is to introduce a gauge transformation to try to convert the spectral problem into a perturbation of that corresponding to the exactly solvable potential. Consider, for example, the (FNLS) spectral problem with a Klaus-Shaw potential:

$$\epsilon\frac{d}{dx}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} s & A(x) \\ -A(x) & -s \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}.$$

Here $s = -i\lambda \in \mathbb{R}_+$ for all eigenvalues. Introducing a gauge transformation and a change of independent variable by

$$x = x(y)\,, \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{M}(y,s)\begin{bmatrix} a' \\ b' \end{bmatrix}$$

for some invertible matrix $\mathbf{M}(y,s)$, we obtain

$$\epsilon\frac{d}{dy}\begin{bmatrix} a' \\ b' \end{bmatrix} = \left\{\frac{dx}{dy}\mathbf{M}^{-1}\begin{bmatrix} s & A(x) \\ -A(x) & -s \end{bmatrix}\mathbf{M} - \epsilon\mathbf{M}^{-1}\frac{d\mathbf{M}}{dy}\right\}\begin{bmatrix} a' \\ b' \end{bmatrix}.$$

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We may in fact write the spectral problem for (FNLS) in the form

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Strategy: try to choose $x(y)$ and $\mathbf{M}$ to make the coefficient matrix an ϵ-perturbation of that for the Satsuma-Yajima potential, with some other eigenvalue parameter r:

$$\frac{dx}{dy}\mathbf{M}^{-1}\begin{bmatrix}s & A(x)\\-A(x) & -s\end{bmatrix}\mathbf{M} = \frac{\|A\|_1}{\pi}\begin{bmatrix}r & \operatorname{sech}(y)\\-\operatorname{sech}(y) & -r\end{bmatrix}.$$

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For this equation to hold, the determinants must be equal:

$$\left(\frac{dx}{dy}\right)^2 (A(x)^2 - s^2) = \left(\frac{\|A\|_1}{\pi}\right)^2 (\mathrm{sech}^2(y) - r^2).$$

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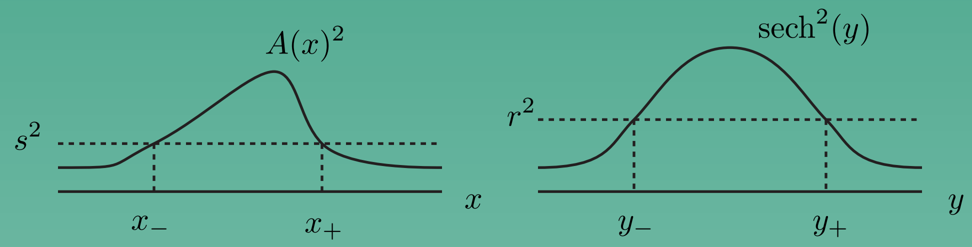
A problem: the factors $A(x)^2 - s^2$ and $\operatorname{sech}^2(y) - r^2$ are not of one sign. But, if $A(x)$ is a Klaus-Shaw potential, there are in each case exactly two turning points whenever $0 < s < \max A$ and $0 < r < 1$.



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Only for Klaus-Shaw potentials A can we match up the turning points by relating r and s. For $x_- < x < x_+$ and $y_- < y < y_+$ the differential equation is:

$$\sqrt{A(x)^2 - s^2}\, dx = \frac{\|A\|_1}{\pi}\sqrt{\operatorname{sech}^2(y) - r^2}\, dy$$

so for $y_\pm$ to correspond to $x_\pm$, we must have

$$\int_{x_-}^{x_+} \sqrt{A(x)^2 - s^2}\, dx = \frac{\|A\|_1}{\pi}\int_{y_-}^{y_+} \sqrt{\operatorname{sech}^2(y) - r^2}\, dy = \|A\|_1(1-r)\,.$$

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Note: if we neglect the correction term in the Langer-transformed spectral problem, then we know the eigenvalues exactly (Satsuma-Yajima):
$r = 1 - \mu/2, 1 - 3\mu/2, 1 - 5\mu/2, \ldots,$ where $\mu = \pi\epsilon/\|A\|_1$. Plugging these approximate values into the above relation yields the Bohr-Sommerfeld formula of WKB theory: it ought to give approximations of some kind or other to the true eigenvalues s.

Ingredients in Rigorous Analysis

We don't want to neglect the correction term. We want to estimate its effect.

1. We removed a determinant obstruction by picking $x = x(y)$. Finish the job by writing down the gauge matrix $\mathbf{M}$. Then push the perturbation to formally higher order by modifying $\mathbf{M}$ with a factor of the form $\mathbb{I} + \epsilon\mathbf{B}$.

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2. Write the perturbed equation as a forced problem:

$$\mu\frac{d}{dy}\begin{bmatrix} a' \\ b' \end{bmatrix} - \begin{bmatrix} r & \mathrm{sech}(y) \\ -\mathrm{sech}(y) & -r \end{bmatrix}\begin{bmatrix} a' \\ b' \end{bmatrix} = \mu^2 g(y, r, \mu)\mathbf{R}\begin{bmatrix} a' \\ b' \end{bmatrix}$$

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3. "Solve" for a' and b' by inverting the Satsuma-Yajima operator on the left-hand side by means of "variation of parameters". This requires a basis of solutions of the unperturbed problem, which are hypergeometric functions. This yields an integral equation for a' and b'.

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4. Estimate the hypergeometric resolvent for arbitrarily small μ. One expects these estimates to resemble standard steepest-descent approximations for the Euler integral representations. The issue is to obtain estimates that are uniform with respect to the eigenvalue parameter r, especially near $r = 0$ and $r = 1$ (apparently much harder).

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A wrinkle: the estimates we know how to obtain in step 4 are not sufficiently refined to allow control of the integral equations for large y. Instead, for $y < -1$ (say) we construct a different Langer transformation to the potential e^y instead of $\mathrm{sech}(y)$. The exponential potential problem can also be solved exactly, in terms of Bessel functions. These integrals can be controlled for large y.

Langer Transformations for Nonsemiclassical Problems: Homotopy Method

Langer transformations may provide a method for proving exact spectral confinement theorems like those of Klaus-Shaw and Bronski-Johnson. The key observation is: in both the (FNLS) and (SG) spectral problems, the Langer transformation to the exactly solvable case only exists if the correct monotonicity condition is satisfied! This should have significance for finite ϵ not necessarily small.

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When $h = 0$ the spectrum is known exactly, and is confined to the imaginary axis by exact calculation. One could deduce the Klaus-Shaw theorem if one could show that as h is increased to $h = 1$, the eigenvalues do not collide.

Conclusions

Some ideas to take home:

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Thank You!