$\overline{\partial}$ Problems in Random Matrix Theory

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Abstract

A $\overline{\partial}$ problem is a kind of generalization of a Riemann-Hilbert problem. We will describe how some $\overline{\partial}$ problems arise in the context of the orthogonal polynomial approach to random matrix theory.

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Circular Ensembles and Universality

Consider the group $U(N)$ of $N \times N$ unitary matrices $\mathbf{U}$ equipped with a probability measure of the form

$$dp(\mathbf{U}) = \frac{1}{Z_N} e^{-N\mathrm{Tr}V(\arg(\mathbf{U}))} dH_N(\mathbf{U}) \,,$$

where dH_N denotes Haar measure and Z_N is a normalization constant (partition function). This is the circular ensemble with weight e^{-NV}.

Spectral theorem: diagonalize $\mathbf{U}$ and integrate out the eigenvector variables (Haar measure again). What remains is the joint law for the eigenvalues $\{z_n = e^{i\theta_n}\}_{n=1}^N$:

$$dp(\theta_1, \ldots, \theta_N) = \frac{1}{Z'_N} \cdot \prod_{m<n} |e^{i\theta_m} - e^{i\theta_n}|^2 \cdot \prod_{n=1}^{N} e^{-NV(\theta_n)} d\theta_n \,.$$

Circular Ensembles and Universality

Correlation functions are expressed in terms of orthogonal polynomials:

$$R_N^{(n)}(\theta_1, \ldots, \theta_n) = \det(K_N(\theta_j, \theta_k))_{j,k=1,\ldots,n} \, ,$$

where

$$K_N(\theta, \phi) = \sum_{n=0}^{N-1} p_n(e^{i\theta})\overline{p_n(e^{i\phi})} e^{-NV(\theta)/2} e^{-NV(\phi)/2} \, ,$$

and $p_n(z) = \gamma_{n,n} z^n + \cdots$ satisfies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} p_n(e^{i\theta})\overline{p_m(e^{i\theta})} e^{-NV(\theta)} \, d\theta = \delta_{n,m} \, .$$

Christoffel-Darboux formula:

$$K_N(\theta, \phi) = \frac{p_N^*(e^{i\theta})\overline{p_N^*(e^{i\phi})} - p_N(e^{i\theta})\overline{p_N(e^{i\phi})}}{1 - e^{i(\theta - \phi)}} \, ,$$

where if $p(z) = c_0 + c_1 z + \cdots + c_n z^n$, then $p^*(z) := \overline{c_0} z^n + \overline{c_1} z^{n-1} + \cdots + \overline{c_n}$.

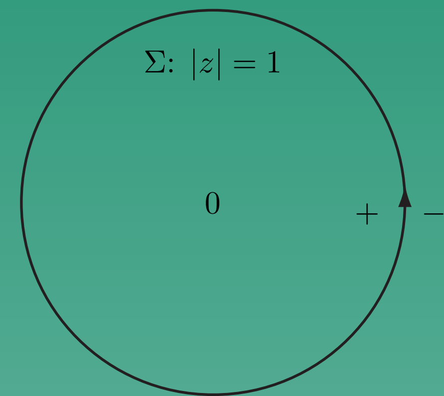
Circle Polynomials: Riemann-Hilbert Problem

Let $e^{-V(\theta)}$ be a weight on the unit circle. Seek $\mathbf{M}^n(z)$, a 2×2 matrix, with the following properties:

Analyticity: $\mathbf{M}^n(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$, taking continuous boundary values $\mathbf{M}^n_\pm(z)$ for $|z| = 1$.

Jump Condition: The boundary values are related by

$$\mathbf{M}^n_+(e^{i\theta}) = \mathbf{M}^n_-(e^{i\theta}) \begin{bmatrix} 1 & e^{-V(\theta)}e^{-in\theta} \\ 0 & 1 \end{bmatrix} .$$



Normalization: The matrix $\mathbf{M}^n(z)$ satisfies

$$\lim_{z \to \infty} \mathbf{M}^n(z) \begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix} = \mathbb{I} .$$

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3. Because $M^n_{12+}(e^{i\theta}) = M^n_{11}(e^{i\theta})e^{-NV(\theta)}e^{-in\theta} + M^n_{12-}(e^{i\theta})$, by the Plemelj formula,

$$M^n_{12}(z) = \frac{1}{2\pi i}\oint \frac{M^n_{11}(s)e^{-NV(\arg(s))}s^{-n}}{s-z}\,ds + e(z)$$

where $e(z)$ is an entire function.

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$$\frac{1}{2\pi}\int_{-\pi}^{\pi} M^n_{11}(e^{i\theta})e^{-im\theta}e^{-NV(\theta)}\,d\theta = 0\,, \qquad m = 0, 1, 2, \ldots, n-1\,.$$

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This result identifies $M^n_{11}(z)$ with $\pi_n(z)$, the monic orthogonal polynomial of degree n.

Analytic Weights: Steepest Descent Asymptotics

The simplest case is to take $N = 1$ and let $n \to \infty$. Make the substitution

$$\mathbf{N}^n(z) := \begin{cases} \mathbf{M}^n(z)\,, & |z| < 1\,, \\ \mathbf{M}^n(z) z^{-n\sigma_3}\,, & |z| > 1\,. \end{cases}$$

This removes the non-identity asymptotics for large z and the jump condition for $\mathbf{N}^n(z)$ becomes:

$$\mathbf{N}^n_+(e^{i\theta}) = \mathbf{N}^n_-(e^{i\theta}) \begin{bmatrix} e^{in\theta} & e^{-V(\theta)} \\ 0 & e^{-in\theta} \end{bmatrix}\,, \qquad z \in \Sigma\,.$$

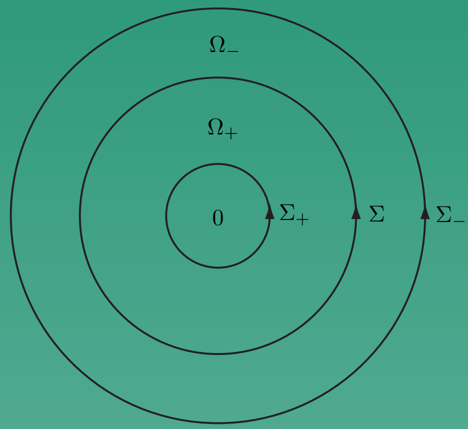
Then, note the factorization:

$$\begin{bmatrix} e^{in\theta} & e^{-V(\theta)} \\ 0 & e^{-in\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^{-in\theta}e^{V(\theta)} & 1 \end{bmatrix} \begin{bmatrix} 0 & e^{-V(\theta)} \\ -e^{V(\theta)} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ e^{in\theta}e^{V(\theta)} & 1 \end{bmatrix}\,.$$

Analytic Weights: Steepest Descent Asymptotics

When V is analytic, we may create a new piecewise-analytic unknown as follows:

$$\mathbf{P}^n(z) := \begin{cases} \mathbf{N}^n(z) \begin{bmatrix} 1 & 0 \\ -z^n e^{V(-i\log(z))} & 1 \end{bmatrix}, & z \in \Omega_+ \,, \\ \mathbf{N}^n(z) \begin{bmatrix} 1 & 0 \\ z^{-n} e^{V(-i\log(z))} & 1 \end{bmatrix}, & z \in \Omega_- \,, \\ \mathbf{N}^n(z) \,, & \text{otherwise} \,. \end{cases}$$



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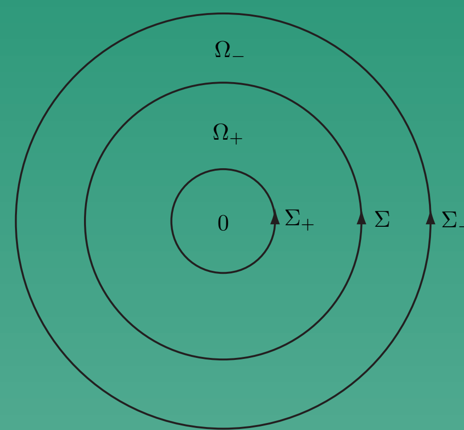

Then the jump condition for $\mathbf{P}^n(z)$ is exponentially negligible for large n except:

$$\mathbf{P}^n_+(e^{i\theta}) = \mathbf{P}^n_-(e^{i\theta}) \begin{bmatrix} 0 & e^{-V(\theta)} \\ -e^{V(\theta)} & 0 \end{bmatrix}, \qquad z = e^{i\theta} \in \Sigma .$$

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Normalization: The matrix $\dot{\mathbf{P}}(z)$ satisfies $\lim_{z\to\infty} \dot{\mathbf{P}}(z) = \mathbb{I}$.

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This problem has a unique explicit solution in terms of the Szegő function $S(z)$:

$$\dot{\mathbf{P}}(z) := \begin{cases} \begin{bmatrix} S(z) & 0 \\ 0 & S(z)^{-1} \end{bmatrix}, & |z| > 1 \\ \begin{bmatrix} 0 & S(z) \\ -S(z)^{-1} & 0 \end{bmatrix}, & |z| < 1, \end{cases} \qquad S(z) := \exp\left(-\frac{1}{2\pi i}\oint_\Sigma \frac{V(\arg(s))\,ds}{s-z}\right).$$

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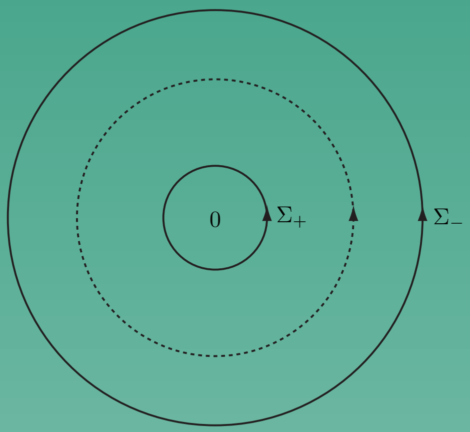
It remains to control the errors. Compare $\mathbf{P}(z)$ with $\dot{\mathbf{P}}(z)$. Define the discrepancy: $\mathbf{H}^n(z) := \mathbf{P}(z)\dot{\mathbf{P}}(z)^{-1}$. This matrix is analytic except on $\Sigma_{\pm}$.



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The discrepancy matrix $\mathbf{H}^n(z)$ satisfies a "small-norm" Riemann-Hilbert problem for n large: seek $\mathbf{H}^n(z)$, 2×2, with the following properties:

Analyticity: $\mathbf{H}^n(z)$ is analytic for $z \in \mathbb{C} \setminus (\Sigma_+ \cup \Sigma_-)$, and takes continuous boundary values $\mathbf{H}^n_\pm(z)$ on these contours.

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Riemann-Hilbert problems are equivalent to systems of singular integral equations (Cauchy kernels) on the system of jump contours. The integral equations for small-norm problems can be solved by Neumann series. This yields: $\mathbf{H}^n(z) \approx \mathbb{I}$, with error terms given by an asymptotic series.

Nonanalytic Weights: $\overline{\partial}$ Steepest Descent Method

If V is not analytic, however smooth, this technique fails. We need an alternative to analytic continuation for extending a smooth function from the unit circle.

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If V is not analytic, however smooth, this technique fails. We need an alternative to analytic continuation for extending a smooth function from the unit circle.

Let $x = r\cos\theta$ and $y = r\sin\theta$ where $z = x + iy$. Here is a formula for an "almost analytic extension" of $V(\theta)$:

$$E_m V(r,\theta) := \sum_{p=0}^{m-1} \frac{V^{(p)}(\theta)}{p!}(-i\log(r))^p\,.$$

Note that Σ is characterized by $r = 1$, or equivalently $\log(r) = 0$. Therefore $E_m V(1,\theta) = V(\theta)$ so we have indeed defined an extension of V from the unit circle.

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Nonanalytic Weights: $\overline{\partial}$ Steepest Descent Method

What about "near analyticity"? Analytic functions f are characterized by the Cauchy-Riemann equations $\overline{\partial} f = 0$ where

$$\overline{\partial} := \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) .$$

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Applying $\overline{\partial}$ to $E_m V(r,\theta)$:

$$\overline{\partial} E_m V(r,\theta) = \frac{ie^{i\theta}}{2r(m-1)!} V^{(m)}(\theta)(-i\log(r))^{m-1} \qquad \text{(sum telescopes).}$$

This is not zero, but it vanishes to order $m-1$ as $r \to 1$.

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This is not zero, but it vanishes to order $m-1$ as $r \to 1$.

(If V is analytic, then the infinite series $E_\infty V(r,\theta)$ converges uniformly for r in a neighborhood of $r = 1$ and $\overline{\partial} E_\infty V(r,\theta) = 0$; in other words, $E_\infty V(r,\theta)$ is a series representation of the analytic continuation of V.)

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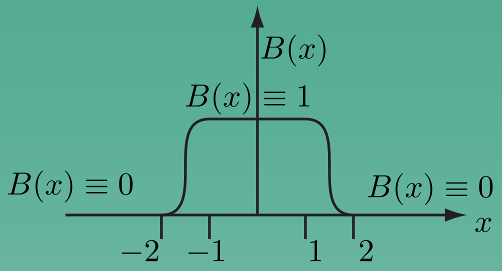
We use an extension $E_m V$ of V to make use of the factorization of the jump matrix for $\mathbf{N}^n(z)$:

$$\mathbf{P}^n_m(r,\theta) := \begin{cases} \mathbf{N}^n(re^{i\theta}) \begin{bmatrix} 1 & 0 \\ (re^{i\theta})^{-n} B(\log(r)) e^{E_m V(r,\theta)} & 1 \end{bmatrix}, & r > 1 \\ \\ \mathbf{N}^n(re^{i\theta}) \begin{bmatrix} 1 & 0 \\ -(re^{i\theta})^{n} B(\log(r)) e^{E_m V(r,\theta)} & 1 \end{bmatrix}, & r < 1\,. \end{cases}$$

Here $B(\cdot)$ is a C^∞ "bump function":



Nonanalytic Weights: $\overline{\partial}$ Steepest Descent Method

We may expect that $\mathbf{P}^n_m(r,\theta) \approx \dot{\mathbf{P}}(z)$ in some sense. To analyze, define the discrepancy by $\mathbf{H}^n_m(r,\theta) := \mathbf{P}^n_m(r,\theta)\dot{\mathbf{P}}(re^{i\theta})^{-1}$.

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The discrepancy $\mathbf{H}_m^n(r,\theta)$ satisfies another kind of problem, a $\overline{\partial}$ problem: seek $\mathbf{H}_m^n(r,\theta)$, 2×2, with the following properties:

Smoothness: $\mathbf{H}_m^n(r,\theta)$ is a Lipschitz continuous function on the whole polar plane.

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$$\overline{\partial}\mathbf{H}_m^n(r,\theta) = \mathbf{H}_m^n(r,\theta)\mathbf{W}_m^n(r,\theta)$$

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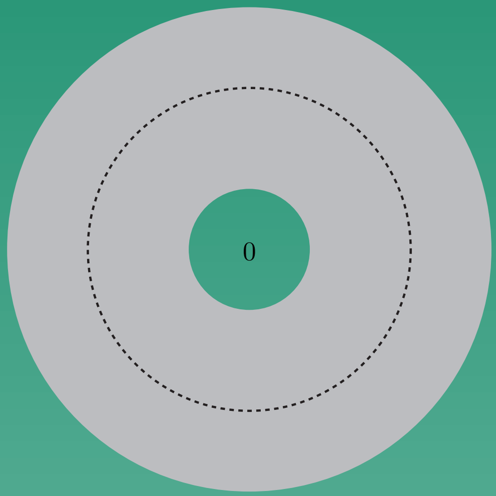
The matrix $\mathbf{W}_m^n(r,\theta)$ is nonzero only in the annulus $|\log(r)| < 2$.



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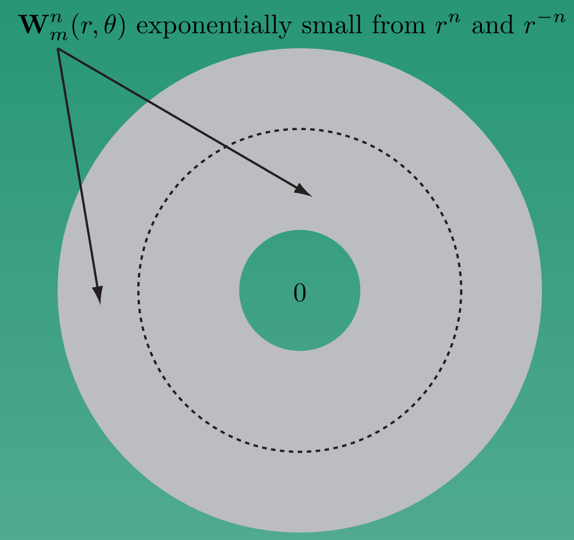

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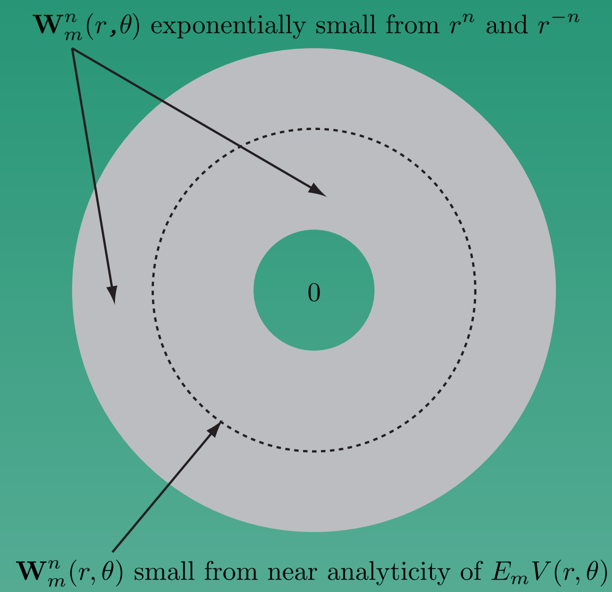

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Nonanalytic Weights: $\overline{\partial}$ Steepest Descent Method

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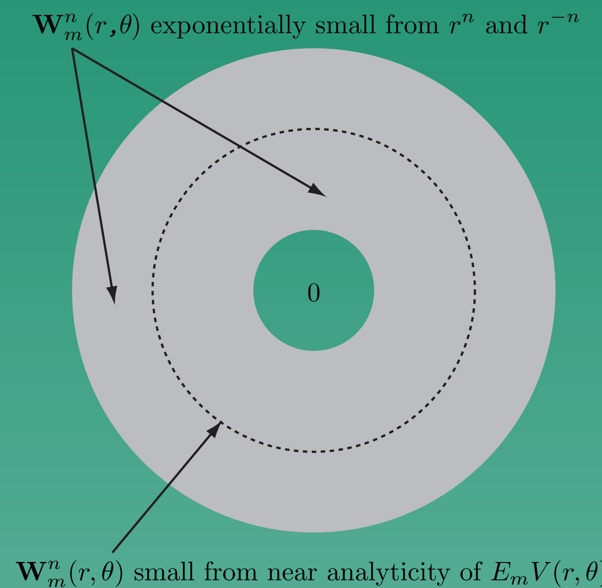

This makes the $\overline{\partial}$ problem for $\mathbf{H}_m^n(r,\theta)$ a kind of small-norm problem that can be analyzed with great precision, more easily than small-norm Riemann-Hilbert problems due to local integrability of the Cauchy kernel on the plane.

Ref: McLaughlin & M, *IMRP*, 2006

Nongaussian Unitary Ensemble With Convex Exponential Weights

In a similar way as for the circular ensembles, the measure on $N \times N$ Hermitian matrices

$$dp(\mathbf{M}) = \frac{1}{Z_N} e^{-N\operatorname{tr}(V(\mathbf{M}))} d\mathbf{M}\,,\ \ d\mathbf{M} = \text{Lebesgue measure on independent entries}$$

leads to the joint law for the real eigenvalues $x_1 \le \cdots \le x_N$:

$$dp(x_1, \ldots, x_N) = \frac{1}{Z'_N} \cdot \prod_{m<n} (x_n - x_m)^2 \cdot \prod_{n=1}^{N} e^{-NV(x_n)} dx_n\,.$$

The correlation functions have determinantal form
$R_N^{(n)}(x_1, \ldots, x_n) = \det(K_N(x_j, x_k))_{j,k=1,\ldots,n}$ with kernel

$$K_N(x, y) := \sum_{n=0}^{N-1} p_n(x) p_n(y) e^{-NV(x)/2} e^{-NV(y)/2}\,,$$

and $p_n(x)$ is the orthonormal polynomial of degree n for the measure $e^{-NV(x)}\,dx$ on $\mathbb{R}$.

Orthogonal Polynomials on the Real Line: Nonanalytic Weights

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In work in progress with K. McLaughlin, we are extending the $\overline{\partial}$ method to handle support endpoints. Our aim is to establish universality of key limiting kernels (sine kernel in the bulk, Airy kernel at the edge leading to the Tracy-Widom law for the fluctuations of the extreme eigenvalues) describing local eigenvalue statistics, *beyond the analytic class of weights* V.

Normal Matrix Models

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Consider the set $\mathcal{N}_N$ of $N \times N$ complex matrices $\mathbf{M}$ that are *normal*: $[\mathbf{M}, \mathbf{M}^\dagger] = 0$. Let $\mathbf{X} := \frac{1}{2}(\mathbf{M} + \mathbf{M}^\dagger)$ and $\mathbf{Y} := \frac{1}{2i}(\mathbf{M} - \mathbf{M}^\dagger)$ be the Hermitian "real" and "imaginary" parts of $\mathbf{M}$, and let $V(x, y)$ be a real function on $\mathbb{R}^2$ with sufficient growth at infinity. Equip $\mathcal{N}_N$ with the probability distribution

$$dp(\mathbf{M}) = \frac{1}{Z_N} e^{-N\mathrm{tr}(V(\mathbf{X},\mathbf{Y}))}\, d\mu(\mathbf{M})$$

where μ is the measure on $\mathcal{N}_N$ induced by the flat metric on all $N \times N$ complex matrices.

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Normal Matrix Models

Diagonalization of $\mathbf{M}$ and integrating out the eigenvectors yields the joint law for the (generally complex) eigenvalues $z_n = x_n + iy_n$ in the form

$$dp(x_1, y_1, \ldots, x_N, y_N) = \frac{1}{Z'_N} \prod_{j<k} |z_j - z_k|^2 \cdot \prod_{n=1}^{N} e^{-NV(x_n,y_n)} dx_n \, dy_n \, .$$

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Here $p_n(z)$ is the orthonormal polynomial of degree n for the weight $e^{-NV(x,y)}\, dx\, dy$. There is, unfortunately, no Christoffel-Darboux formula to telescope the sum in $K_N(z, w)$. Nonetheless, information about the asymptotic behavior of eigenvalue statistics lies in the large degree behavior of these orthogonal polynomials.

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Twisted $\overline{\partial}$ Problem

The orthogonal polynomials with respect to $e^{-NV(x,y)}\,dx\,dy$ are characterized directly by a slightly modified $\overline{\partial}$ problem. Let n and N be positive integers, with $n \le N$, and seek $\mathbf{M}^n(x,y)$, a 2×2 matrix-valued function on $\mathbb{R}^2$ with the following properties:

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Then, $M_{11}(x,y)$ is the monic orthogonal polynomial of degree n in $z = x + iy$, with respect to the measure $e^{-NV(x,y)}\,dx\,dy$.

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- However, a genuinely two-dimensional analogue of the three-factor factorization and subsequent deformation of Riemann-Hilbert problems is required for this problem. This is the subject of current work.

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Thank You!