APPENDIX 1. QUOTIENTS BY FINITE GROUP ACTIONS AND GROUND FIELD EXTENSIONS OF ALGEBRAIC VARIETIES

We recall in this appendix some basic facts about quotients of quasiprojective schemes by finite group actions, following [SGA1]. As an application, we discuss in the second section some generalities concerning ground field extensions for algebraic varieties.

1. The general construction

Let Y be a scheme of finite type over a field k, and let G be a finite group, acting (on the right) on Y by algebraic automorphisms over k. We denote by σ_g the automorphism corresponding to $g \in G$. A *quotient of* Y *by* G is a morphism $\pi \colon Y \to W$ with the following two properties:

i) π is G-invariant, that is $\pi \circ \sigma_g = \pi$ for every $g \in G$.
ii) π is universal with this property: for every scheme Z over k, and every G-invariant morphism $f \colon Y \to Z$, there is a unique morphism $h \colon W \to Z$ such that $h \circ \pi = f$.

It is clear from this universal property that if a quotient exists, then it is unique, up to a canonical isomorphism. In this case, we write $W = Y/G$.

We start by considering the case when $Y = \operatorname{Spec} A$ is an affine scheme. Note that G acts on A on the left. We show that the induced morphism $\pi \colon \operatorname{Spec} A \to W = \operatorname{Spec} A^G$ is the quotient of Y by G.

Proposition 1.1. *With the above notation, the following hold:*

i) *W is a scheme of finite type over k, and π is a finite, surjective morphism.*
ii) *The fibers of π are precisely the orbits of the G-action on Y.*
iii) *The topology on W is the quotient topology.*
iv) *We have a natural isomorphism $\mathcal{O}_W = \pi_*(\mathcal{O}_Y)^G$.*

Proof. It is clear that $A^G \hookrightarrow A$ is integral: indeed, for every $u \in A$, we have $P(u) = 0$, where $P = \prod_{g \in G}(x - gu) \in A^G[x]$. Since A is finitely generated over k, it follows that there is a finitely generated k-algebra $B \subseteq A^G$ such that A is integral over B, hence finite over B. Since B is Noetherian, it follows that A^G is a finite over B. We conclude that A^G is a finitely generated k-algebra, and the morphism π is finite. Since $A^G \to A$ is injective, it follows that π is surjective.

It is clear that π is G-invariant, hence each orbit is contained in a fiber. Conversely, if P, Q are primes in A such that $P \cap A^G = Q \cap A^G$, then $P \subseteq \bigcup_{g \in G} gQ$. Indeed, if $u \in P$, then

$$\prod_{g \in G}(gu) \in P \cap A^G = Q \cap A^G,$$

hence there is $g \in G$ such that $gu \in Q$. The Prime Avoidance Lemma implies that $P \subseteq gQ$ for some $g \in G$. Similarly, we get $Q \subseteq hP$ for some $h \in G$. Since $P \subseteq ghP$, and gh is an automorphism, we must have $P = ghP$, hence $P = gQ$.

This proves ii), and the assertion in iii) is now clear since π is closed, being finite. It is easy to deduce iv) from the fact that if $f \in A^G$, then $(A_f)^G = (A_G)_f$. This completes the proof of the proposition. $\square$

Remark 1.2. Suppose that Y is a scheme with an action of the finite group G. If $\pi\colon Y \to W$ is a surjective morphism of schemes that satisfies ii)-iv) in Proposition 1.1, then π gives a quotient of Y by G. This is a consequence of the definition of morphisms of schemes. In particular, we see that the morphism $\pi\colon Y \to W$ in Proposition 1.1 is such a quotient.

Corollary 1.3. *If $\pi\colon Y \to W$ is as in the proposition, then for every open subset U of W, the induced morphism $\pi^{-1}(U) \to U$ is the quotient of $\pi^{-1}(U)$ by the action of G.*

Proof. It is clear that since π is a surjective morphism that satisfies ii)-iv) in the above proposition, the morphism $\pi^{-1}(U) \to U$ satisfies the same properties. $\square$

Suppose now that Y is a scheme over k, with an action of G. We assume that every $y \in Y$ has an affine open neighborhood that is preserved by the G-action. This happens, for example, if Y is quasiprojective. Indeed, in this case for every $y \in Y$, the finite set $\{\sigma_g(y) \mid g \in G\}$ is contained in some affine open subset U of Y[1]. After replacing U by $\cap_{g\in G}\sigma_g(U)$ (this is again affine, since Y is separated), we may assume that U is affine, and preserved by the action of G.

By assumption, we can thus cover Y by $U_1, \ldots, U_r$, where each U_i is affine, and preserved by the G-action. By what we have discussed so far, we may construct the quotient morphisms $\pi_i\colon U_i \to W_i = U_i/G$. Furthermore, it follows from Corollary 1.3 that for every i and j we have canonical isomorphisms $\pi_i(U_i \cap U_j) \simeq \pi_j(U_i \cap U_j)$. We can thus glue these morphisms to get a quotient $\pi\colon Y \to Y/G$ of Y with respect to the G-action. Note that this is a finite surjective morphism that satisfies conditions ii)-iv) in Proposition 1.1, hence gives a quotient of Y by the action of G.

Remark 1.4. It follows from the above construction that if Y is reduced, then Y/G is reduced too.

Remark 1.5. The above construction is compatible with field extensions in the following sense. Suppose that Y is a scheme over k with an action of the finite group G, such that every point on Y has an affine open neighborhood preserved by the G-action. Suppose that K/k is a field extension, and $Y_K = Y \times_{\operatorname{Spec} k} \operatorname{Spec} K$. Note that Y_K has an induced G-action, and every point on Y_K has an affine open neighborhood preserved by the G-action. We have an isomorphism of K-varieties $Y_K/G \simeq (Y/G) \times_{\operatorname{Spec} k} \operatorname{Spec} K$. Indeed, it

[1]If Y is a locally closed subset of $\mathbf{P}^n_k$, and $x_1, \ldots, x_n \in Y$, then there is a hypersurface H in $\mathbf{P}^n_k$ that contains $\overline{Y} \smallsetminus Y$, but does not contain $x_1, \ldots, x_n$. Indeed, by the graded version of Prime Avoidance Lemma, there is a homogeneous element of positive degree in the ideal of $\overline{Y} \smallsetminus Y$ (if this set is empty, we take this ideal to be the "irrelevant" maximal ideal), but that does not lie in the ideal of any $\overline{\{x_i\}}$. The complement of H in Y is an affine open subset of Y that contains all the x_i.

is enough to consider the case when $Y = \operatorname{Spec} A$, and in this case the assertion follows from the lemma below.

Lemma 1.6. *Let V and W be k-vector spaces, and suppose that a group G acts on V on the left by k-linear automorphisms. If we consider on $V \otimes_k W$ the induced G-action, then we have a canonical isomorphism $(V \otimes_k W)^G \simeq V^G \otimes_k W$.*

Proof. We clearly have an inclusion $V^G \otimes_k W \hookrightarrow (V \otimes_k W)^G$. Consider $u \in V \otimes_k W$. If $(b_i)_{i \in I}$ is a k-basis of W, we can write $u = \sum_i a_i \otimes b_i$ for a unique tuple $(a_i)_{i \in I}$. Since $gu = \sum_i (ga_i) \otimes b_i$, it follows that $gu = u$ if and only if $ga_i = a_i$ for every i. Therefore $u \in (V \otimes_k W)^G$ if and only if all a_i lie in V^G. $\square$

Proposition 1.7. *Let G and H be finite groups, acting by algebraic automorphisms over k on the schemes X and Y, respectively, where X and Y are of finite type over k. If both X and Y can be covered by affine open subsets preserved by the action of the corresponding group, then $X \times Y$ satisfies the same property with respect to the product action of $G \times H$, and $X \times Y/G \times H \simeq X/G \times Y/H$.*

Proof. Let $X = \bigcup_i U_i$ and $Y = \bigcup_j V_j$ be covers by affine open subsets, preserved by the respective group actions. It is clear that $X \times Y = \bigcup_{i,j} U_i \times V_j$ is a cover by affine open subsets preserved by the $G \times H$-action. Furthermore, using Lemma 1.6 twice, we obtain

$$(\mathcal{O}(U_i) \otimes_k \mathcal{O}(V_j))^{G \times H} \simeq \mathcal{O}(U_i)^G \otimes_k \mathcal{O}(V_j)^H,$$

and these isomorphisms glue together to give the isomorphism in the proposition. $\square$

Proposition 1.8. *Let G be a finite group acting by algebraic automorphisms on a scheme X of finite type over k, such that X has an affine open cover by subsets preserved by the G-action. Suppose that H is a subgroup of G, and Y is an open subset of X such that*

i) *Y is preserved by the action of H on X.*
ii) *If $Hg_1, \ldots, Hg_r$ are the right equivalence classes of G mod H, then $X = \bigcup_{i=1}^r Yg_i$ is a disjoint cover.*

In this case the natural morphism $Y/H \to X/G$ is an isomorphism.

Proof. Note that by ii), Y is also closed in X. Consider a cover $X = \bigcup_j U_j$ by affine open subsets preserved by the G-action. Each $V_j = Y \cap U_j$ is an affine open subset of Y preserved by the H-action (note that $U_j \cap Y$ is nonempty since U_j must intersect some Yg_i). Therefore we have the quotient Y/H, and since the natural morphism $Y \to X/G$ is H-invariant, we obtain a morphism $\varphi \colon Y/H \to X/G$.

We claim that each $Y \cap U_j \hookrightarrow U_j$ still satisfies i) and ii). Indeed, it is clear that $Y \cap U_j$ is preserved by the H-action, and we have $U_j = \bigsqcup_{i=1}^r (Y \cap U_j) g_i$. Therefore we may assume that X and Y are affine.

It follows from ii) that $\mathcal{O}(X) = \prod_{i=1}^r \mathcal{O}(Yg_i)$, and it is clear that if $\varphi \in \mathcal{O}(X)^G$, then $\varphi = (\psi g_1^{-1}, \ldots, \psi g_r^{-1})$ for some $\psi \in \mathcal{O}(Y)$, and in fact we must have $\psi \in \mathcal{O}(Y)^H$. This shows that the natural homomorphism $\mathcal{O}(X)^G \to \mathcal{O}(Y)^H$ is an isomorphism. $\square$

Remark 1.9. Given X as in the above proposition, suppose that Y is an open subset of X such that for every $g, h \in G$, the sets Yg and Yh are either equal, or disjoint. In this case i) and ii) are satisfied if we take $H = \{g \in G \mid Yg = Y\}$ and if we replace X by $\bigcup_{g\in G} Yg$.

Proposition 1.10. *Let G be a finite group acting by algebraic automorphisms on a scheme X of finite type over k, such that X has an affine open cover by subsets preserved by the $G \times H$-action. If H is a normal subgroup of G, then X/H has an induced G/H-action, and the quotient by this action is isomorphic to X/G.*

Proof. Let $X = \bigcup_i U_i$ be an affine open cover of X, with each U_i preserved by the G-action. In particular, each U_i is preserved by the G-action, hence the quotient X/G exists. The action of G on X induces an action of G/H on X/H by the universal property of the quotient. Note that the U_i/H give an affine open cover of X/H by subsets preserved by the G/H-action. Since we clearly have $\mathcal{O}(U_i)^G = (\mathcal{O}(U_i)^H)^{G/H}$, we get isomorphisms of the quotient of U_i/H by the G/H-action with U_i/G. These isomorphisms glue to give the required isomorphism. □

2. Ground field extension for algebraic varieties

Let X be a variety over a field k (recall that this means that X is a reduced scheme of finite type over k). Let K/k be a finite Galois extension, with group G, and put $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. Note that this is a variety over K, since the extension K/k is separable. Since K is flat over k, we see that the canonical projection $|pi \colon X_K \to X$ is flat.

The left action of G on K induces a right action of G on $\operatorname{Spec} K$, hence on X_K (note that the corresponding automorphisms of X_K are k-linear, but not K-linear). If $x \in X_K$ and V is an affine open neighborhood of $\pi(x)$, then $\pi^{-1}(U)$ is an affine open neighborhood of x, preserved by the G-action. Therefore we may apply to the G-action on X_K the considerations in the previous section. In fact, π is the quotient of X_K by the action of G. Indeed, it is enough to note that if $U \simeq \operatorname{Spec}(A)$ is an affine open subset of X, then Lemma 1.6 gives

$$(A \otimes_k K)^G = A \otimes_k K^G = A.$$

By the discussion in the previous section, it follows that π identifies X with the set of G-orbits of X_K, with the quotient topology.

If $Y \hookrightarrow X$ is a closed subvariety, then $Y_K \hookrightarrow X_K$ is a closed subvariety preserved by the G-action. The following proposition gives a converse.

Proposition 2.1. *With the above notation, suppose that W is a closed subvariety of X_K preserved by the G-action. If $Y = \pi(W)$, then W is a closed subvariety of X, and $W = Y_K$.*

Proof. Since π is finite, it follows that Y is closed in X. We clearly have an inclusion $W \subseteq Y_K$. This is an equality of sets since W is preserved by the G-action, and π identifies

X with the set of G-orbits in X_K. Since both W and Y_K are reduced, it follows that $W = Y_K$. $\square$

The above considerations can be easily extended to the case of infinite Galois extensions. In what follows, we assume that k is perfect, and consider an algebraic closure $\overline{k}$ of k. Note that $\overline{k}$ is the union of the finite Galois subextensions K of k, and we have $G(\overline{k}/k) \simeq \varprojlim_K G(K/k)$. As above, if X is a variety over k, we put $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and let $\pi \colon X_{\overline{k}} \to X$ be the canonical projection. Note that since k is perfect, all fibers of π are reduced. We have a right action of $G(\overline{k}/k)$ on $X_{\overline{k}}$, induced by its left action on $\overline{k}$.

Proposition 2.2. *If W is a closed subvariety of $X_{\overline{k}}$ that is preserved by the G-action, and if $Y = \pi(W)$, then Y is a closed subvariety of X, and $W = Y_{\overline{k}}$ (in this case we say that W is defined over k).*

Proof. The fact that Y is closed in X follows from the fact that π is an integral morphism. There is a finite Galois extension K of k such that for some closed subscheme V of X_K, we have $V_{\overline{k}} = W$. After replacing V by V_{red}, we may assume that V is reduced, in which case we see that it is the image of W via the canonical projection $X_{\overline{k}} \to X_K$. Since W is preserved by the $G(\overline{k}/k)$-action, it follows that V is preserved by the $G(K/k)$-action (recall that $G(K/k)$ is the quotient of $G(\overline{k}/k)$ by $G(\overline{k}/K)$). We may thus apply Proposition 2.1 to conclude that $V = Y_K$, and therefore $W = Y_{\overline{k}}$. $\square$

Proposition 2.3. *The fibers of the projection $\pi \colon X_{\overline{k}} \to X$ are the orbits of the $G(\overline{k}/k)$-action on $X_{\overline{k}}$.*

Proof. It is clear from definition that $G(\overline{k}/k)$ acts on $X_{\overline{k}}$ by automorphisms over X. Suppose now that $x, y \in X_{\overline{k}}$ are such that $\pi(x) = \pi(y)$. There is a finite Galois extension K of k such that both $\overline{\{x\}}$ and $\overline{\{y\}}$ are defined over K, and let x_K and y_K denote the images of x and y, respectively, in X_K. Since x_K and y_K lie in the same fiber of $X_K \to X$, we can find $\sigma \in G(K/k)$ such that $x_K\sigma = y_K$. In this case, for every $\widetilde{\sigma} \in G(\overline{k}/k)$ that extends σ, we have $x\sigma = y$. $\square$

Proposition 2.4. *If X is an irreducible variety over k, then $G = G(\overline{k}/k)$ acts transitively on the set of irreducible components of $X_{\overline{k}}$.*

Proof. Note first that every automorphism of $X_{\overline{k}}$ maps an irreducible component to an irreducible component, hence G indeed has an induced action on the set of irreducible components of $X_{\overline{k}}$. Let V and W be irreducible components of $X_{\overline{k}}$. Since $X_{\overline{k}}$ is flat over X, and X is irreducible, it follows that both V and W dominate X. Therefore the generic points of V and W lie in the same fiber of π, and we conclude by applying the previous proposition. $\square$

Proposition 2.5. *If X is a variety over k and $\pi \colon X_{\overline{k}} \to X$ is the canonical projection, then taking $x \in X$ to the sum of the elements in $\pi^{-1}(x)$ induces a bijection between the set of effective 0-cycles on X of degree n and the set of effective 0-cycles on $X_{\overline{k}}$ that have degree n and that are fixed by $G(\overline{k}/k)$.*

Proof. By Proposition 2.3, an effective cycle α on $X_{\overline{k}}$ is invariant by $G(\overline{k}/k)$ if and only if for every closed point $x \in X_{\overline{k}}$ that appears in α, all $y \in \pi^{-1}(\pi(x))$ appear in α with the same coefficient. In other words, α can be written as $\sum_{i=1}^{r}\sum_{y\in\pi^{-1}(u_i)} y$ for some $u_1, \ldots, u_r \in X$. In order to complete the proof, it is enough to note that for every $u \in X$, we have $\deg(k(u)/k) = |\pi^{-1}(u)|$ (recall that $\pi^{-1}(u)$ is reduced). □

Suppose now that $k = \mathbf{F}_q$ is a finite field. Recall that $G(\overline{k}/k) \simeq \widehat{\mathbf{Z}}$, and we may take as a topological generator either the arithmetic Frobenius element $x \to x^q$, or its inverse, the geometric Frobenius element. Let σ denote the automorphism of $X_{\overline{k}}$ corresponding to the action of the arithmetic Frobenius element. Recall that the endomorphism $\mathrm{Frob}_{X,q}$ on X induces by base extension the $\overline{k}$-linear endomorphism $F = \mathrm{Frob}_{X_{\overline{k}},q}$ of $X_{\overline{k}}$.

Proposition 2.6. *Let X be a variety over $k = \mathbf{F}_q$, and W a closed subvariety of $X_{\overline{k}}$. There is a closed subvariety Y of $X_{\mathbf{F}_{q^r}}$ such that $W = Y_{\overline{k}}$ (in which case Y is the image of W in $X_{\mathbf{F}_{q^r}}$) if and only if $F^r(W) \subseteq W$.*

Proof. After replacing X by $X_{\mathbf{F}_{q^r}}$, we may assume that $r = 1$. We have seen in Exercise 2.5 in Lecture 2 that $\sigma \circ F = F \circ \sigma$, and this is the absolute q-Frobenius morphism of $X_{\overline{k}}$ (let's denote it by T). Since $T(W) = W$ for every closed subvariety W of $X_{\overline{k}}$, it is easy to see that $\sigma^{-1}(W) \subseteq W$ if and only if $F(W) \subseteq W$ (in which case $F(W) = W$).

Applying Proposition 2.2, we are done if we show that if $\sigma^{-1}(W) \subseteq W$, then W is preserved by $G(\overline{k}/k)$. Since the geometric Frobenius element is a topological generator of $G(\overline{k}/k)$, this follows from the fact that the action of $G(\overline{k}/k)$ on $X_{\overline{k}}$ is continuous, where on $X_{\overline{k}}$ we consider the discrete topology. Continuity simply means that the stabilizer of every point in $X_{\overline{k}}$ contains a subgroup of the form $G(\overline{k}/K)$, for some finite Galois extension K of k. This is clear for $X_{\overline{k}}$, since it is clear for $\mathbf{A}^n_{\overline{k}}$: for the point $(u_1, \ldots, u_n) \in \overline{k}^n$, we may simply take K to be the Galois closure of $k(u_1, \ldots, u_n)$. □

3. Radicial morphisms

We will need the notion of radicial morphism in the next section, in order to discuss quotients of closed subschemes. In this section we recall the definition of this class of morphisms and prove some basic properties.

Proposition 3.1. *If $f\colon X \to Y$ is a morphism of schemes, then the following are equivalent:*

i) *For every field K (which may be assumed algebraically closed), the induced map*
$$\mathrm{Hom}(\mathrm{Spec}\, K, X) \to \mathrm{Hom}(\mathrm{Spec}\, K, Y)$$
is injective.

ii) *For every scheme morphism $Y' \to Y$, the morphism induced by base-change $X \times_Y Y' \to Y'$ is injective.*

iii) *f is injective, and for every $x \in X$, the extension of residue fields $k(f(x)) \hookrightarrow k(x)$ is purely inseparable.*

If f satisfies the above equivalent conditions, one says that f is *radicial*.

Proof. We first prove i)$\Rightarrow$ii). Let $Y' \to Y$ be a morphism of schemes, and suppose that $x_1, x_2 \in X \times_Y Y'$ are two distinct points that map to the same point $y \in Y'$. Let K be a field extension of $k(y)$ containing both $k(x_1)$ and $k(x_2)$ (note that we may take K to be algebraically closed). The inclusions $k(x_1), k(x_2) \hookrightarrow K$ give two distinct morphisms $\operatorname{Spec} K \to X \times_Y Y'$ such that the induced morphisms to Y' are equal. In particular, the induced morphisms to Y are equal, hence by i) the induced morphisms to X are equal. The universal property of the fiber product shows that we have a contradiction.

We now prove ii)$\Rightarrow$i). Suppose that $\varphi, \psi \colon \operatorname{Spec} K \to X$ induce the same morphism $\operatorname{Spec} K \to Y$, and let $X_K = X \times_Y \operatorname{Spec} K$. By the universal property of the fiber product, φ and ψ induce morphisms $\widetilde{\varphi}, \widetilde{\psi} \colon \operatorname{Spec} K \to X_K$ over $\operatorname{Spec} K$. These correspond to two points $x_1, x_2 \in X_K$ and to isomorphisms $K \simeq k(x_i)$. By ii) we have $x_1 = x_2$, hence $\widetilde{\varphi} = \widetilde{\psi}$ and $\varphi = \psi$.

Suppose now that i) holds, and let us deduce iii). The fact that f is injective follows since we know i)$\Rightarrow$ii), so let us suppose that $x \in X$ and $y = f(x)$ are such that $k(y) \hookrightarrow k(x)$ is not purely inseparable. In this case there is a field K and two homomorphisms $\alpha, \beta \colon k(x) \to K$ such that α and β agree on $k(y)$. We thus get two scheme morphisms $\operatorname{Spec} K \to X$ taking the unique point to x, such that they induce the same morphism $\operatorname{Spec} K \to Y$. This contradicts i).

In order to complete the proof of the proposition, it is enough to show that iii)$\Rightarrow$i). Suppose that $u, v \colon \operatorname{Spec} K \to X$ are such that $f \circ u = f \circ v$. Since f is injective, it follows that both u and v take the unique point to the same $x \in X$. We thus have two homomorphisms $k(x) \to K$ whose restrictions to $k(f(x))$ are equal. This shows that $k(x)$ is not purely inseparable over $k(f(x))$, a contradiction. $\square$

Example 3.2. It is clear that every closed immersion is radicial. For a more interesting example, consider a scheme X over $\mathbf{F}_p$, and let $f \colon X \to X$ be the absolute Frobenius morphism. It is clear that f is a surjective, radicial morphism (use description iii) in the above proposition).

Remark 3.3. It follows from either of the descriptions in Proposition 3.1 that the class of radicial morphisms is closed under composition and base-change. Of course, the same holds for radicial surjective morphisms.

Remark 3.4. If $f \colon X \to Y$ is a morphism of schemes, it is a consequence of the description iii) in Proposition 3.1 that f is radicial of and only if $f_{\rm red} \colon X_{\rm red} \to Y_{\rm red}$ has this property.

Remark 3.5. The notion of radicial morphism is local on the target: $f \colon X \to Y$ is radicial if and only if there is an open cover $Y = \bigcup_i V_i$ such that each $f^{-1}(V_i) \to V_i$ is radicial (one can use for this any of the descriptions in Proposition 3.1).

Remark 3.6. A morphism $f \colon X \to Y$ of schemes over a field k is radicial and surjective if and only if for every algebraically closed field K containing k, the induced map $f_K \colon X(K) \to Y(K)$ is bijective. Indeed, Proposition 3.1 shows that f is radicial if and only if all f_K are injective. Assuming that this is true, it is easy to see that if all f_K are surjective, then f is surjective, and the converse follows from the fact that for every $x \in X$, the extension of residue fields $k(f(x)) \hookrightarrow k(x)$ is algebraic.

Example 3.7. If $\varphi\colon R \to S$ is a morphism of rings of characteristic p such that

i) The kernel of φ is contained in the nilradical of R.
ii) For every $b \in S$, there is m such that $b^{p^m} \in \mathrm{Im}(\varphi)$,

then the induced morphism $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is radicial and surjective. Indeed, if $\mathfrak{p}$ is a prime ideal of R, then there is a unique prime ideal $\mathfrak{q}$ of S such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, namely

$$\mathfrak{q} = \{b \in S \mid b^{p^m} = \varphi(a) \,\text{for some}\, a \in \mathfrak{p} \,\text{and}\, m \geq 1\}.$$

Furthermore, for every $u \in S/\mathfrak{q}$, there is $m \geq 1$ such that u^{p^m} lies in the image of $R/\mathfrak{p}$, hence $R/\mathfrak{p} \hookrightarrow S/\mathfrak{q}$ is purely inseparable.

Proposition 3.8. *If $f\colon X \to Y$ is a morphism of schemes of finite type over a field k of characteristic zero, then the following are equivalent:*

i) *f is radicial and surjective.*
ii) *$X(\overline{k}) \to Y(\overline{k})$ is bijective, where $\overline{k}$ is an algebraic closure of k.*
iii) *f is a piecewise isomorphism, that is, there is a disjoint cover $Y = Y_1 \sqcup \ldots \sqcup Y_m$ by locally closed subsets, such that all induced morphisms $f^{-1}(Y_i)_{\mathrm{red}} \to (Y_i)_{\mathrm{red}}$ are isomorphisms.*

Proof. The implication i)$\Rightarrow$ii) follows from Remark 3.6. Suppose now that f is a piecewise isomorphism and $Y = \bigsqcup_i Y_i$ is a disjoint cover as in iii). Given a morphism $\varphi\colon Y' \to Y$, let $g\colon X \times_Y Y' \to Y'$ be the morphism obtained by base-change from f. We get a locally closed disjoint cover $Y' = \bigsqcup_i Y'_i$, where $Y'_i = \varphi^{-1}(Y_i)$, such that each $g^{-1}(Y'_i)_{\mathrm{red}} \to (Y'_i)_{\mathrm{red}}$ is an isomorphism. Therefore f is radicial, and it is clear that f is surjective. Therefore in order to finish the proof of the proposition it is enough to show that if f satisfies ii), then f is a piecewise isomorphism.

Arguing by Noetherian induction, we may assume that the property holds for $f^{-1}(Z) \to Z$, for every proper closed subset Z of Y. Therefore whenever it is convenient, we may replace f by $f^{-1}(U) \to U$, where U is a nonempty open subset of Y. We may put on both X and Y their reduced scheme structures, and therefore assume that they are reduced. If $Y_1, \ldots, Y_r$ are the irreducible components of Y, we may replace Y by $Y_1 \smallsetminus \cup_{i \neq 1} Y_i$, and therefore assume that Y is irreducible.

Since $X(\overline{k}) \to Y(\overline{k})$ is injective, we deduce that there is a unique irreducible component of X that dominates Y. Therefore there is an open subset U in Y such that $f^{-1}(U)$ does not meet the other irreducible components of X. After replacing Y by U, we may assume that both X and Y are irreducible. Let $d = \deg(K(X)/K(Y))$. It is enough to show that $d = 1$, since in this case f is birational, hence there is an open subset U of X such that $f^{-1}(U) \to U$ is an isomorphism.

Since we are in characteristic zero, f is generically smooth, that is, there are open subsets $V \subseteq X$ and $W \subseteq Y$ such that f induces a smooth morphism $g\colon V \to W$. It follows from [Har, Exercise II.3.7] that there is an open subset W' of W such that $g^{-1}(W') \to W'$ is finite. After restricting further to an open subset of W', we may assume that W' is affine, and $\mathcal{O}(g^{-1}(W'))$ is free of rank d over $\mathcal{O}(W')$. Since all fibers of $g^{-1}(W') \times_k \overline{k} \to W' \times_k \overline{k}$ are reduced, it follows that each such fiber has d elements, so by assumption $d = 1$. This completes the proof of the proposition.

□

4. Quotients of locally closed subschemes

Proposition 4.1. *Let X be a scheme of finite type over k, and G a finite group acting on X by algebraic automorphisms over k. We assume that X is covered by affine open subsets preserved by the G-action, and let $\pi\colon X \to X/G$ be the quotient morphism. If W is a locally closed subscheme of X such that G induces an action on W, then the canonical morphism $W/G \to \pi(W)$ is radicial and surjective.*

Proof. We first need to show that W/G exists, and that we have an induced morphism $W/G \to X/G$. If $\overline{W}$ is the closure of W (with the image scheme structure), then W is an open subscheme of $\overline{W}$, which is a closed subscheme of X. Furthermore, G has an induced action on $\overline{W}$. It follows that it is enough to consider separately the cases when W is an open or a closed subscheme of X. If W is an open subscheme, then the assertion is clear: $\pi(W)$ is open in X/G, and we have seen that $W = \pi^{-1}(\pi(W)) \to \pi(W)$ is the quotient of W by the G-action.

Suppose now that W is a closed subscheme of X, and consider $\pi(W)$ (with the image scheme structure). Note first that since $\pi(W)$ can be covered by affine open subsets, and π is finite, it follows that W is covered by affine open subsets that are preserved by the G-action. In particular, W/G exists, and the G-invariant morphism $W \to X \to X/G$ induces a morphism $\varphi\colon W/G \to X/G$. It is clear that the image of this morphism is $\pi(W)$. In order to show that φ is radicial, we may assume that $X = \operatorname{Spec} A$ is affine (simply consider an affine cover of X by affine open subsets preserved by the G-action). Let I denote the ideal defining W. If B is the image of $A^G \to (A/I)^G$, then it is enough to prove that $\operatorname{Spec}(A/I)^G \to \operatorname{Spec} B$ is radicial. In light of Example 3.7, this is a consequence of the more precise statement in the lemma below. □

Lemma 4.2. *Let A be a finitely generated k-algebra, and let G be a finite group acting on A by k-algebra automorphisms. Suppose that $I \subseteq A$ is an ideal preserved by the G-action. If p^n is the largest power of $p = \operatorname{char}(k)$ that divides $|G|$ (we make the convention that $p^n = 1$ if $\operatorname{char}(k) = 0$), then for every $b \in (A/I)^G$, we have $b^{p^n} \in \operatorname{Im}(A^G \to (A/I)^G)$.*

Proof. The argument that follows is inspired from [KM, p.221]. We write it assuming $p > 0$, and leave for the reader to do the translation when $\operatorname{char}(k) = 0$.

Let $u \in A$ be such that $b = \overline{u} \in A/I$ is G-invariant. Since $gu - u \in I$ for every $g \in G$, we have the following congruence in the polynomial ring $A[x]$:

$$\prod_{g \in G}(1 + (gu)x) \equiv (1 + ux)^{|G|} \bmod IA[x].$$

The polynomial on the left-hand side has coefficients in A^G, hence by considering the coefficient of x^{p^n} on the right-hand side, we conclude that $\binom{|G|}{p^n}u^{p^n}$ is congruent mod I to an element in A^G. Since $\binom{|G|}{p^n}$ is invertible in k[2], it follows that $\overline{u}^{p^n}$ lies in the image of R^G. $\square$

Remark 4.3. It follows from the proof of Proposition 4.1 and Lemma 4.2 that if $\mathrm{char}(k)$ does not divide $|G|$, then under the assumptions in Proposition 4.1, the morphism $W/G \to \pi(W)$ is an isomorphism. In particular, this is the case for every G if $\mathrm{char}(k) = 0$.

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[2]It is easy to show this by computing the exponent of p in this binomial coefficient. On the other hand, this is also a consequence of Lucas' theorem, see [Gra]: if $|G| = p^n m$, with m and p relatively prime, then $\binom{|G|}{p^n} \equiv m \pmod{p}$.