Homework Set 10

Solutions are due Friday, December 7th.

Problem 1. Let $f\colon X \to Y$ be a dominat morphism between irreducible algebraic varieties. One says that f is *generically finite* if there are nonempty open subsets $U \subseteq X$ and $V \subseteq Y$ such that f induces a finite morphism $U \to V$.

1) Show that f is generically finite if and only if $\dim(X) = \dim(Y)$.
2) Show that if f is generically finite, then in fact there is a nonempty open subset $V \subseteq Y$ such that the induced morphism $f^{-1}(V) \to V$ is finite.

Problem 2. Let X and Y be algebraic varieties, and x and y be points on X and Y, respectively.

1) Show that there is a canonical isomorphism $T_{x,y}X \times Y \simeq T_xX \times T_yY$.
2) Deduce that $(x, y) \in X \times Y$ is a nonsingular point if and only if $x \in X$ and $y \in Y$ are both nonsingular points.

Problem 3. Let G be a linear algebraic group acting on the variety X. Show that every orbit of G in X is nonsingular.

The following is a very useful interpretation of the tangent space at a point.

Problem 4. Let X be an affine algebraic variety, and $x \in X$ a point. Show that the tangent space T_xX is in natural bijection with the set of k-algebra homomorphisms $f\colon \mathcal{O}(X) \to k[t]/(t^2)$ with the property that if $p\colon k[t]/(t^2) \to k$ is the canonical surjection, then $p \circ f$ is the map to k corresponding to $x \in X$.

Problem 5. Recall that $D_r(m, n) \subseteq M_{m,n}(k)$ denotes the set of matrices A such that $\mathrm{rk}(A) \leq r$.

1) Show that the group $Gl_m(k) \times Gl_n(k)$ has a natural action on $M_{m,n}(k)$ such that the orbits are the sets $D_r(m, n) \smallsetminus D_{r-1}(m, n)$. Deduce that every point in $D_r(m, n) \smallsetminus D_{r-1}(m, n)$ is a nonsingular point of $D_r(m, n)$.
2) Let $A = (a_{ij}) \in D_r(m, n)$. Show that $T_AD_r(m, n)$ is isomorphic to the vector space of matrices $A + tB \in M_{m,n}(k[t]/(t^2))$, having all $(r + 1)$-minors equal to zero.
3) Deduce that if $A \in D_{r-1}(m, n)$, then $\dim_k T_AD_r(m, n) = mn$, hence A is a singular point of $D_r(m, n)$.