Homework Set 2

Solutions are due Friday, September 28th.

Problem 1. Let X be an algebraic prevariety, and consider a finite open cover

$$X = U_1 \cup \ldots \cup U_n,$$

where each U_i is nonempty. Show that X is irreducible if and only if the following hold:

i) Each U_i is irreducible.
ii) For every i and j, we have $U_i \cap U_j \neq \emptyset$.

Problem 2. If X is an affine algebraic variety, and if $u \in \mathcal{O}(X)$, then we denote by $D(u)$ the open subset of X

$$D(u) = \{x \in X \mid u(x) \neq 0\}$$

(we have seen in class that this is again an affine variety). Suppose that $f \colon X \to Y$ is a morphism of affine algebraic varieties, and denote by $f^\sharp \colon \mathcal{O}(Y) \to \mathcal{O}(X)$ the induced ring homomorphism, that takes $\phi \in \mathcal{O}(Y)$ to $\phi \circ f$. Show that if $u \in \mathcal{O}(Y)$, then

i) We have $f^{-1}(D(u)) = D(w)$, where $w = f^\sharp(u)$.
ii) The induced ring homomorphism

$$\mathcal{O}(D(u)) \to \mathcal{O}(D(w))$$

can be identified with the homomorphism

$$\mathcal{O}(Y)_u \to \mathcal{O}(X)_w$$

induced by $f^\sharp$ by localization.

Problem 3. Let $f \colon X \to Y$ be a morphism of affine algebraic varieties. Show that the closure of $\mathrm{Im}(f)$ is the closed subset of Y defined by $\ker\big(f^\sharp \colon \mathcal{O}(Y) \to \mathcal{O}(X)\big)$.

Problem 4. Show that the image of a morphism of algebraic prevarieties $f \colon X \to Y$ might not be locally closed in Y (you can use, for example, the morphism $f \colon \mathbb{A}^2 \to \mathbb{A}^2$ given by $f(x, y) = (x, xy)$).