Homework Set 3

Solutions are due Monday, October 8th.

Problem 1. Recall that $GL_n(k)$ denotes the set of invertible $n \times n$ matrices with entries in k. Let $PGL_n(k)$ denote the quotient $GL_n(k)/k^*$, where k^* acts by

$$\lambda \cdot (a_{i,j}) = (\lambda a_{i,j}).$$

i) Show that $PGL_n(k)$ has a natural structure of affine variety (you may use Pb. 4 on Problem set 5). Show that $PGL_n(k)$ is irreducible.
ii) Prove that $PGL_n(k)$ acts naturally on $\mathbb{P}^n$, such that for every $A \in PGL_n(k)$, multiplication by A gives an isomorphism of prevarieties $\mathbb{P}^n \xrightarrow{A\cdot} \mathbb{P}^n$.
iii) Show that given two sets of points in $\mathbb{P}^n$

$$\Gamma = \{P_0, \ldots, P_n\} \text{ and } \Gamma' = \{Q_0, \ldots, Q_n\},$$

neither of them being contained in a hyperplane, there is a unique $A \in PGL_n(k)$ such that $A \cdot P_i = Q_i$ for every i (recall that a *hyperplane* in $\mathbb{P}^n$ is a closed subset defined by a nonzero linear polynomial).

Two subsets of $\mathbb{P}^n$ are *projectively equivalent* if they differ by an automorphism as above (one can show that these are, indeed, all automorphisms of $\mathbb{P}^n$).

Problem 2.

i) Show that every hyperplane in $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^{n-1}$.
ii) Show that the hyperplanes in $\mathbb{P}^n$ are parametrized by $\mathbb{P}^n$ (one sometimes writes $\mathbb{P}^{n*}$ for the projective space of hyperplanes in $\mathbb{P}^n$).

Problem 3. Show that there is a morphism

$$\nu_d \colon \mathbb{P}^1 \to \mathbb{P}^d,$$

defined by $\nu_d(u\colon v) = (u^d \colon u^{d-1}v \colon \ \ldots \colon v^d)$. Moreover,

i) Show that if we consider the ring homomorphism

$$f_d \colon k[x_0, \ldots, x_d] \to k[u, v],$$

given by $f(x_i) = u^{d-i}v^i$, then the kernel I of f_d is homogeneous, and the closed subset it defines is the image of ν_d. Any subset projectively equivalent to $\mathrm{Im}(\nu_d)$ is a *rational normal curve* in $\mathbb{P}^d$.
ii) Show that ν_d is a closed immersion.