Homework Set 4

Solutions are due Monday, October 15th.

Problem 1. Recall that if J is an ideal in $k[x_1, \ldots, x_n]$, then

$$J^{\text{hom}} := (f^{\text{hom}} \mid f \in J) \subseteq k[x_0, \ldots, x_n],$$

where $f^{\text{hom}} = x_0^{\deg(\text{f})} \cdot f(x_1/x_0, \ldots, x_n/x_0)$.

Consider the closed subset in $\mathbb{A}^3$ given by the image of the morphism $f\colon \mathbb{A}^1 \to \mathbb{A}^3$, $f(t) = (t, t^2, t^3)$. Show on this example that if an ideal J is generated by $f_1, \ldots, f_m$, then it is not necessarily true that J^{hom} is generated by $(f_1)^{\text{hom}}, \ldots, (f_m)^{\text{hom}}$. However, show that this is true if $m = 1$ (that is, when J is a principal ideal).

Problem 2. A *linear subspace* in $\mathbb{P}^n$ is a closed subset whose ideal is generated by linear forms (that is, by homogeneous polynomials of degree one). Show that a closed subset W of $\mathbb{P}^n$ is a linear subspace if and only if the affine cone $C(W)$ over W is a vector subspace of $\mathbb{A}^{n+1}$. Moreover, the map $W \to C(W)$ gives a bijection between the linear subspaces of $\mathbb{P}^n$ and the vector subspaces of $\mathbb{A}^{n+1}$.

The *dimension* of a linear subspace W in $\mathbb{P}^n$ is defined as $\dim(C(W)) - 1$, where $C(W)$ is the affine cone over W. Recall that a *hyperplane* in $\mathbb{P}^n$ is a linear subspace of dimension $n - 1$. A *line* in $\mathbb{P}^n$ is a one-dimensional linear subspace.

Problem 3. Show that if W is a linear subspace of $\mathbb{P}^n$ of dimension r, then W is isomorphic to $\mathbb{P}^r$.

Problem 4. Suppose that $\text{char}(k) \neq 2$. A *quadric* in $\mathbb{P}^n$ is a closed subset of $\mathbb{P}^n$ defined by a reduced homogeneous polynomial of degree two.

i) Show that every quadric is projectively equivalent to some $V(f_r)$, where $f_r = x_1^2 + \ldots + x_r^2$, with $2 \leq r \leq n$.
ii) Show that such a quadric is irreducible if and only if $r \geq 3$.

Problem 5. Let P be a point and H a hyperplane in $\mathbb{P}^n$, such that $P \notin H$. The *projection* from P to H is the map $\phi\colon \mathbb{P}^n \smallsetminus \{P\} \to H$, such that $\phi(Q)$ is the intersection of the line containing P and Q with H.

i) Show that ϕ is a morphism.
ii) Suppose that $P = (0\colon 1\colon 0\colon 0) \in \mathbb{P}^3$, and $H = (x_1 = 0) \subseteq \mathbb{P}^3$. Let $X \subseteq \mathbb{P}^3$ be the rational normal curve that is the image of $f\colon \mathbb{P}^1 \to \mathbb{P}^3$, given by $f(u\colon v) = (u^3\colon u^2v\colon uv^2\colon v^3)$. Show that if ϕ is the projection from P to H, then $\phi(X)$ is isomorphic to the *cuspidal curve* defined by the equation $y^3 = xz^2$ in $\mathbb{P}^2$.