Homework Set 5

Solutions are due Wednesday, October 24th.

Problem 1. *The Veronese embedding.* Let n and d be positive integers, and let $M_0, \ldots, M_N$ be all monomials in $k[x_0, \ldots, x_n]$ of degree d (hence $N = \binom{n+d}{d} - 1$).

1) Show that there is a morphism $\rho_d \colon \mathbb{P}^n \to \mathbb{P}^N$ that takes the point $(a_0 \colon \ldots \colon a_n)$ to the point $(M_0(a) \colon \ldots \colon M_N(a))$.
2) Consider the ring homomorphism $f_d \colon k[z_0, \ldots, z_N] \to k[x_0, \ldots, x_n]$ defined by $f_d(z_i) = M_i$. Show that $\ker(f_d)$ is a homogeneous prime ideal that defines in $\mathbb{P}^N$ the image of ρ_d (in particular, this image is closed).
3) Show that ρ_d is a closed immersion.
4) Show that if Z is a hypersurface of degree d in $\mathbb{P}^n$ (this means that Z is defined by a homogeneous polynomial of degree d), then there is a hyperplane H in $\mathbb{P}^N$ such that for every projective variety $X \subseteq \mathbb{P}^d$, the morphism ρ_d induces an isomorphism between $X \cap Z$ and $\rho_d(X) \cap H$. (In other words, the Veronese embedding allows to reduce the intersection with a hypersurface to the intersection with a hyperplane).

Remark. We have seen before the case $n = 1$ of the above embedding, in which case the image is a rational normal curve.

Problem 2. *The incidence correspondence.* Consider the dual projective space $\mathbb{P}^{n*}$ parametrizing the hyperplanes in $\mathbb{P}^n$. Let

$$\Gamma = \{(x, H) \in \mathbb{P}^n \times \mathbb{P}^{n*} \mid x \in H\}.$$

1) Show that Γ is closed in $\mathbb{P}^n \times \mathbb{P}^{n*}$.
2) Let $\alpha \colon \Gamma \to \mathbb{P}^n$ and $\beta \colon \Gamma \to \mathbb{P}^{n*}$ be the morphisms induced by the two projections. Show that for $x \in \mathbb{P}^n$ and for $H \in \mathbb{P}^{n*}$, we have
$$\alpha^{-1}(x) \simeq \{H' \mid x \in H'\} \simeq \mathbb{P}^{n-1}, \ \beta^{-1}(H) \simeq H \simeq \mathbb{P}^{n-1}.$$
3) Show that if X is a closed subset of $\mathbb{P}^n$, then by taking
$$W = \alpha^{-1}(X) \to \mathbb{P}^{n*},$$
we get a morphism such that the fiber over every hyperplane $H \in \mathbb{P}^{n*}$ is the *hyperplane section* $H \cap X$.