Homework Set 6

Solutions are due Wednesday, October 31st.

Problem 1. Let X and Y be algebraic prevarieties, and let P and Q be points on X and Y, respectively.

i) Show that if $\phi\colon \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$ is a local ring homomorphism, then there are open subsets $U \subseteq X$ and $V \subseteq Y$, and a morphism $f\colon U \to V$ with $f(P) = Q$, such that f induces ϕ.
2) Deduce that if ϕ as above is an isomorphism, then then after possibly replacing U, V by $U' \subseteq U$ and $V' \subseteq V$, we have that f is an isomorphism.

Problem 2. Let $f\colon X \dashrightarrow Y$ be a rational map between the irreducible varieties X and Y. The *graph* Γ_f of f is defined as follows. If U is an open subset of X such that f is defined on U, then the graph of $f|_U$ is well-defined, and it is a closed subset of $U \times Y$. By definition, Γ_f is the closure of the graph of $f|_U$ in $X \times Y$.

i) Show that the definition is independent of the choice of U.
ii) Let $p\colon \Gamma_f \to X$ and $q\colon \Gamma_f \to Y$ be the morphisms induced by the two projections. Show that p is a birational morphism, and that q is birational if and only if f is.
iii) Show that if the fiber $p^{-1}(x)$ does not consist of only one point, then f is not defined at $x \in X$.

Problem 3. A *plane Cremona transformation* is a birational map of $\mathbf{P}^2$ into itself. Consider the following example of *quadratic* Cremona transformation: $\phi\colon \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$, given by $\phi(x\colon y\colon z) = (yz\colon xz\colon xy)$, when no two of x, y, or z are zero.

1) Show that ϕ is birational, and its own inverse.
2) Find open subsets $U, V \subset \mathbf{P}^2$ such that ϕ induces an isomorphism $U \simeq V$.
3) Describe the open sets on which ϕ and ϕ^{-1} are defined.

Problem 4. Generalize the construction of the blowing-up to dimension ≥ 2, as follows. Thinking of $\mathbf{P}^{n-1}$ as the set of lines in $\mathbf{A}^n$, define the *blowing-up of* $\mathbf{A}^n$ *at 0* as the set

$$\mathrm{Bl}_0(\mathbf{A}^n) := \{(P, \ell) \in \mathbf{A}^n \times \mathbf{P}^{n-1} \mid P \in \ell\}.$$

1) Show that $\mathrm{Bl}_0(\mathbf{A}^n)$ is a closed subset of $\mathbf{A}^n \times \mathbf{P}^{n-1}$.
2) Show that the restriction of the projection onto the first component gives a morphism $\pi\colon \mathrm{Bl}_0(\mathbf{A}^n) \to \mathbf{A}^n$ that is an isomorphism over $\mathbf{A}^n \smallsetminus \{0\}$.
3) Show that $\pi^{-1}(0) \simeq \mathbf{P}^{n-1}$.
4) Show that π is a closed map.
5) Show that $\mathrm{Bl}_0(\mathbf{A}^n)$ can be described as the graph of a suitable rational map.