Homework Set 7

Solutions are due Friday, November 9th.

Problem 1. Let X be a (not necessarily irreducible) algebraic prevariety. Show that every irreducible component of X is connected. Deduce that the connected components of X are both open and closed in X.

Problem 2. Let X and Y be algebraic varieties, with X projective and Y affine. Show that every morphism $f\colon X \to Y$ is constant on each connected component of X.

We have seen that the hyperplanes in $\mathbf{P}^n$ are parametrized by a projective space $\mathbf{P}^{n*}$. In the next exercise we generalize this construction to the case of hypersurfaces of higher degree.

Problem 3. A *hypersurface* of degree d in $\mathbf{P}^n$ is a closed subset of $\mathbf{P}^n$ such that its ideal is generated by a (nonzero) homogeneous polynomial of degree d. Denote by $\mathcal{H}_d$ the set of hypersurfaces of degree d.

Let $V = k[x_0, \ldots, x_n]_d$ be the k-vector space of hmogeneous polynomials of degree d in $x_0, \ldots, x_n$. A point in the projective space $\mathbf{P}(V)$ is given by the class $[Q]$ of a nonzero polynomial $Q \in k[x_0, \ldots, x_n]_d$, uniquely defined up to multiplication by a nonzero scalar. Note that $V \simeq k^{N+1}$, where $N = \binom{n+d}{d} - 1$, hence the corresponding projective space $\mathbf{P}(V)$ is isomorphic to $\mathbf{P}^N$.

i) Show that the set
$$\mathcal{U}_d := \{[Q] \in \mathbf{P}(V) \mid Q \text{ is reduced}\}$$
is open in $\mathbf{P}(V)$. (Hint: show that this set is the complement of the union of the images of various products of projective spaces).

ii) Show that we have a bijection $\tau\colon \mathcal{U}_d \to \mathcal{H}_d$. This makes $\mathcal{H}_d$ an (irreducible) algebraic variety.

iii) Show that there is a *universal hypersurface* over $\mathcal{H}_d$, that is, a closed subset in $\mathcal{H}_d \times \mathbf{P}^n$, whose fiber over every $\{H\} \in \mathcal{H}_d$ is the hypersurface $H \subset \mathbf{P}^n$.

iv) Show that for every point $p \in \mathbf{P}^n$, the set of points $[Q] \in \mathbf{P}(V)$ such that $Q(p) = 0$ is a hyperplane in $\mathbf{P}(V)$.

v) Deduce that if $p_1, \ldots, p_m \in \mathbf{P}^n$, and if $m \leq N$, then there is $[Q] \in \mathbf{P}(V)$ such that $Q(p_i)$ for every i.

vi) Show that if, in addition, $m = N$, and the points $p_1, \ldots, p_N \in \mathbf{P}^n$ are general (that is, $(p_1, \ldots, p_N)$ belongs to a suitable nonempty open subset of $(\mathbf{P}^n)^N$), then there is a unique $[Q] \in \mathbf{P}(V)$ such that $Q(p_i) = 0$ for every i (and moreover, $[Q]$ gives a point in $\mathcal{H}_d$).