Homework Set 8

Solutions are due Friday, November 16th.

Problem 1. Prove that if X and Y are algebraic varieties, then
$$\dim(X \times Y) = \dim(X) + \dim(Y).$$

We have seen in class that if X and Y are closed subsets in $\mathbf{P}^n$, with $\dim(X) + \dim(Y) \geq n$, then $X \cap Y \neq \emptyset$. The following problem shows that this is sharp in a strong sense.

Problem 2. Let $X \subseteq \mathbf{P}^n$ be a closed subset of dimension r. Show that there is a linear space $L \subseteq \mathbf{P}^n$ of dimension $(n - r - 1)$ such that $L \cap X = \emptyset$. (Hint: use Pb. 1 from Problem session 10).

Problem 3. Let $r \leq \min\{m, n\}$, and consider the *generic determinantal variety*
$$D_r(m, n) = \{A \in M_{m,n}(k) \mid \mathrm{rk}(A) \leq r\}.$$
Show that $D_r(m, n)$ is irreducible, and compute its dimension, as follows.

i) We identify in the obvious way $M_{m,n}(k)$ with $Hom_k(k^n, k^m)$. Show that the following set
$$Z := \{(A, L) \in M_{m,n}(k) \times G(n - r, n) \mid L \subseteq \ker(A)\}$$
is closed in $M_{m,n}(k) \times G(n - r, n)$.

ii) Let $p\colon Z \to M_{m,n}(k)$ and $q\colon Z \to G(n - r, n)$ be the two projections. Show that $G(n-r, n)$ can be covered by affine open subsets U_i, such that $q^{-1}(U_i) \simeq U_i \times \mathbf{A}^{rm}$.

iii) Deduce that Z is irreducible, and $\dim(Z) = mr + nr - r^2$.

iv) Deduce that $D_r(m, n)$ is irreducible, and $\mathrm{codim}(D_r(m, n), M_{m,n}(k)) = (m-r)(n-r)$. (Hint: you can use the result we'll prove in class that if $f\colon X \to Y$ is dominant, then there is an open subset U of Y such that $\dim(f^{-1}(y)) = \dim(X) - \dim(Y)$ for every $y \in Y$).