Homework Set 9

Solutions are due Monday, November 26th.

Problem 1. Let G be an irreducible algebraic group acting algebraically on the variety X. Show that every irreducible component of X is preserved by the action of G.

Problem 2. Show that $PGL_{n+1}(k)$ is a linear algebraic group that acts algebraically on $\mathbf{P}^n$.

A *homogeneous variety* is a projective variety with a transitive action of a linear algebraic group. One such example is $\mathbf{P}^n$. The following problem shows that the Grassmannian and the flag variety are homogeneous varieties.

Problem 3. Consider the linear algebraic group $G = GL_n(k)$.

i) Show that G has a transitive algebraic action on the Grassmannian $G(r, k^n)$.
ii) Show that G has a transitive algebraic action on the variety of complete flags $\mathrm{Fl}(k^n)$. Show that the stabilizer of any element is conjugate to the subgroup $B \subseteq G$ of all upper-triangular matrices. Use this to give another proof of the fact that $\dim \mathrm{Fl}(V) = n(n-1)/2$.

Problem 4. We think of the Grassmannian $G(r+1, V)$ as parametrizing the r-dimensional linear subspaces in $\mathbf{P}(V)$.

i) Let T be a closed subset of $G(r+1, V)$. Show that the union $\widetilde{T} := \bigcup_{[W]\in T} W \subseteq \mathbf{P}(V)$ is a closed subset of $\mathbf{P}(V)$, and $\dim(\widetilde{T}) \leq r + \dim(T)$.
ii) Deduce that if X and Y are disjoint closed subsets in $\mathbf{P}(V)$, and if $J(X, Y)$ is their join, then
$$\dim J(X, Y) \leq \dim(X) + \dim(Y) + 1.$$
(see Pb. set 10 for the definition of the join).

Problem 5. Let G be an algebraic group acting algebraically on the variety X.

i) Show that X contains at least one closed orbit.
ii) Show that if X is irreducible, and the action has only finitely many orbits, then X contains precisely one open orbit.