

LECTURE 1. ZETA FUNCTIONS: AN OVERVIEW

Zeta functions encode the counting of certain objects of geometric, algebraic, or arithmetic behavior. What distinguishes them from other generating series are special analytic or algebraic properties.

Zeta functions come up in a lot of area of mathematics. The ones we will deal with come in two flavors: local and global. Here *local* means relative to a prime p in \mathbf{Z} , or in some ring of integers in a number field. In this case, one expects the zeta function to be a rational function, in a suitable variable. By a global zeta function we mean an object that takes into account all primes. In this case one expects to have a product formula in terms of local factors. The basic example is the well-known factorization of the Riemann zeta function:

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

A good understanding of the local factors of the zeta function can be used to show that the global zeta function is defined in some region $\{s \in \mathbf{C} \mid \operatorname{Re}(s) > \eta\}$, and then there are fundamental questions regarding analytic continuation and the existence of a functional equation. Again, the model is provided by the Riemann zeta function. However, very little is known in a more general setting. The general philosophy is that the analytic properties of the zeta function encode a lot of information about the geometric/arithmetic/algebraic of the object that is studied.

In what follows we give an overview of the types of zeta functions that we will discuss in the following lectures. In all this discussion, we restrict to the simplest possible setting.

1. THE HASSE-WEIL ZETA FUNCTION

This is one of the most famous zeta functions, and it played an important role in the development of algebraic geometry in the twentieth century. It is attached to a variety over a finite field, say $k = \mathbf{F}_q$. Suppose, for simplicity, that $X \subset \mathbf{A}_k^n$ is a closed subvariety defined by the equations f_1, \dots, f_d .

For every $m \geq 1$, let

$$N_m := |\{u \in X(\mathbf{F}_{q^m})\}| = |\{u \in \mathbf{F}_{q^m}^n \mid f_i(u) = 0 \text{ for all } i\}|.$$

The Hasse-Weil zeta function of X is

$$Z(X, t) := \exp\left(\sum_{m \geq 1} \frac{N_m}{m} t^m\right) \in \mathbf{Q}[[t]].$$

A fundamental result is that $Z(X, t)$ is a rational function. This was conjectured by Weil in [We2], who also proved it for curves and abelian varieties in [We1]. The general

case was proved by Dwork in [Dwo]. Another proof in the case of smooth projective varieties was later given by Grothendieck and its school using étale cohomology, see [Gro]. Both the methods of Grothendieck and of Dwork have been extremely influential for the development of arithmetic geometry.

When X is a smooth projective variety, $Z_X(t)$ satisfies

- The functional equation.
- A connection with the Betti numbers defined over \mathbf{C} .
- An analogue of the Riemann hypothesis.

These three properties, together with the rationality mentioned above, form the Weil conjectures [We2], now a theorem of Grothendieck [Gro] and Deligne [Del]. See Lecture 3 for the precise statements.

2. THE L -FUNCTION OF AN ALGEBRAIC VARIETY

Suppose now that $X \subset \mathbf{A}_{\mathbf{Q}}^n$ is an affine variety defined over \mathbf{Q} (or, more generally, over a number field). We may assume that the equations f_1, \dots, f_d defining X lie in $\mathbf{Z}[1/a][x_1, \dots, x_n]$ for some nonzero $a \in \mathbf{Z}$. If p is a prime not dividing a , then we may consider $f_1, \dots, f_d \in \mathbf{F}_p[x_1, \dots, x_n]$ defining $X_p \subseteq \mathbf{A}_{\mathbf{F}_p}^n$, and the corresponding $Z(X_p, t)$. After possibly changing finitely many factors, one puts

$$L_X(s) := \prod_{p \nmid a} Z(X_p, 1/p^s).$$

Let us consider the case $X = \text{Spec } \mathbf{Q}$, when we may take $X_p = \text{Spec } \mathbf{F}_p$ for every prime p . Note that

$$Z(X_p, t) = \exp\left(\sum_{e \geq 1} \frac{t^e}{e}\right) = \exp(-\log(1-t)) = (1-t)^{-1}.$$

Therefore $L_X(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s)$ is the Riemann zeta function.

In general, it is not hard to see that L_X is defined in some half-plane $\{s \in \mathbf{C} \mid \text{Re}(s) > \eta\}$ (we will discuss this in Lecture 5 below, with a precise value for η , as a consequence of the Lang-Weil estimates, which in turn follow from the Weil conjectures for curves).

It is conjectured that if X is a smooth projective variety over \mathbf{Q} , then L_X admits analytic continuation as an entire function (after possibly changing the local factors where X does not have good reduction). One also expects that after a suitable normalization (necessary for taking into account the infinite prime) L_X satisfies a functional equation. Very little is known in this direction. Both properties are known for \mathbf{P}^n and related varieties (such as toric varieties or flag varieties). The case of elliptic curves is known as a consequence of the Taniyama-Shimura conjecture (proved by Wiles [Wil], Taylor-Wiles [TW] and Breuil-Conrad-Diamond-Taylor [BCDT]), which implies that in this case L_X can be described as the L -function attached to a modular form.

3. THE IGUSA ZETA FUNCTION

Suppose now, for simplicity, that p is a prime in \mathbf{Z} , and $X \hookrightarrow \mathbf{A}_{\mathbf{Z}_p}^n$ is defined by $f \in \mathbf{Z}_p[x_1, \dots, x_n]$. The Igusa zeta function of f is defined by

$$Z_f(s) := \int_{\mathbf{Z}_p^n} |f(x)|_p^s dx.$$

This is defined using the p -adic absolute value $|\cdot|_p$ and the Haar measure on \mathbf{Z}_p . It is easy to see using the definition that Z_f is analytic in the half-plane $\{s \mid \operatorname{Re}(s) > 0\}$. Let us give some motivation for this definition.

3.1. The Archimedean analogue of Z_f . The following analogue in the Archimedean setting (over \mathbf{R} or \mathbf{C}) appeared before Igusa's zeta function, in the setting of complex powers. Suppose, for example, that $f \in \mathbf{R}[x_1, \dots, x_n]$, and we want to define $|f(x)|^s$ for $s \in \mathbf{C}$ as a distribution.

Given a test function Φ , consider the map

$$s \rightarrow \int_{\mathbf{R}^n} |f(x)|^s \Phi(x) dx.$$

It is not hard to see that this is well-defined and analytic in the half-space $\{s \in \mathbf{C} \mid \operatorname{Re}(s) > 0\}$. Gelfand conjectured that it has a meromorphic continuation to \mathbf{C} .

This conjecture was proved by two methods. The first solution, given independently by Atiyah [Ati] and by Bernstein-Gelfand [BG], used Hironaka's theorem on resolution of singularities. This essentially allows replacing f by a monomial, in which case the assertion can be easily proved via integration by parts. A second proof due to Bernstein [Ber] directly used integration by parts, relying on the existence of what is nowadays called the Bernstein-Sato polynomial of f (in the process of proving the existence of this polynomial, Bernstein established the basics of the algebraic D -module theory).

3.2. The Poincaré power series of f . For every $m \geq 0$, let

$$c_m := |\{u \in (\mathbf{Z}/p^m\mathbf{Z})^n \mid f(u) = 0\}|$$

(with the convention $c_0 = 1$). The Poincaré series of f is $P_f := \sum_{m \geq 0} \frac{c_m}{p^{mn}} t^m \in \mathbf{Q}[[t]]$. It was a conjecture of Borevich that P_f is a rational function.

It is not hard to see, using the definition of the Haar measure on \mathbf{Z}_p^n that

$$P_f(t) = \frac{1 - tZ_f(s)}{1 - t},$$

where $t = (1/p)^s$. The usefulness of the integral expression for P_f via Z_f is that allows the use of the same methods employed in the Archimedean case. Using embedded resolution of singularities and the change of variable formula for p -adic integrals, Igusa showed that $Z_f(s)$ is a rational function of $(1/p)^s$, see [Igu]. In particular, this proved Borevich's conjecture about the rationality of P_f .

Note that if $X = V(f)$ is smooth over \mathbf{Z}_p , then the information contained in P_f is equivalent with that of $X(\mathbf{F}_p)$. It is remarkable, in fact, that in general the behavior of P_f can be linked to invariants of singularities of f . Since an embedded resolution of singularities comes up in the proof of rationality, it is maybe not too surprising that invariants that come up via resolutions are related to the poles of Z_f . On the other hand, a very interesting open problem in this field, due to Igusa, concerns a relation between these poles and the roots of the Bernstein-Sato polynomial of f (compare with the Archimedean case; note, however, that there is no analogue of integration by parts in the p -adic setting).

One can define a global analogue of Igusa's zeta function, though this has been a lot less studied. Suppose that f is a polynomial with coefficients in \mathbf{Z} (or, more generally, in a ring of integers in some number field). For every prime p , we may consider the image f_p of f in $\mathbf{Z}_p[x_1, \dots, x_n]$, and the corresponding zeta function $Z_{f_p}(s)$. If a_p is the constant coefficient of the power series in $(1/p)^s$ representing $Z_{f_p}(s)$, then one can define

$$Z(s) := \prod_{p \text{ prime}} (a_p^{-1} Z_{f_p}(s)).$$

All non-trivial results concerning Z are due to du Sautoy and Grunewald [dSG]. They showed that this function has a rational abscissa of convergence, and that it can be meromorphically continued to the left of this abscissa. However, it is known that even in simple examples, Z does not have a meromorphic continuation to \mathbf{C} . It is also not clear how properties of the singularities of f can be recast into analytic properties of Z .

4. MOTIVIC VERSIONS OF THE ABOVE (LOCAL) ZETA FUNCTIONS

Both the Hasse-Weil zeta functions and the Igusa zeta functions have motivic versions. In this setting, *motivic* means working with coefficients in the Grothendieck ring of varieties over a field k . Recall that this is the quotient $K_0(\text{Var}/k)$ of the free abelian group on the set of isomorphism classes of varieties over k , by the relations

$$[X] = [Y] + [X \setminus Y],$$

where Y is a closed subvariety of X .

The motivic analogue of the Hasse-Weil zeta function was introduced by Kapranov [Kap]. If k is any field, and X is a variety over k , let $\text{Sym}^n(X)$ denote the n^{th} symmetric product of X . Kapranov's zeta function is

$$Z_{\text{mot}}(X, t) := \sum_{n \geq 0} [\text{Sym}^n(X)] t^n \in K_0(\text{Var}/k)[[t]].$$

If k is a finite field, then there is a ring homomorphism $K_0(\text{Var}/k) \rightarrow \mathbf{Z}$, that takes $[V]$ to $|V(k)|$. One can show that the induced map $K_0(\text{Var}/k)[[t]] \rightarrow \mathbf{Z}[[t]]$ takes $Z_{\text{mot}}(X, t)$ to $Z(X, t)$. Kapranov proved in [Kap] that if X is any curve, then $Z_{\text{mot}}(X, t)$ is a rational function. On the other hand, Larsen and Lunts [LL] showed that if X is a smooth complex surface, then $Z_{\text{mot}}(X, t)$ is rational if and only if X has negative Kodaira dimension. However, it is still open whether $Z_{\text{mot}}(X, t)$ is always rational when inverting the class \mathbf{L} of \mathbf{A}^1 in $K_0(\text{Var}/k)$.

Igusa's zeta function also has a motivic version, due to Denef and Loeser, see [DL]. The idea is to replace \mathbf{Z}_p by $\mathbf{C}[[t]]$ (in this case f is a polynomial with complex coefficients). The space of integration \mathbf{Z}_p^n is replaced by $(\mathbf{C}[[t]])^n$, and p -adic integrals by the so-called motivic integrals. Once the framework of motivic integration is in place, the results about Igusa's zeta function extend to this framework without much effort.

5. ZETA FUNCTIONS IN GROUP THEORY

5.1. Subgroup growth zeta functions. Let G be a finitely generated group. For every $n \geq 1$, put $a_n(G) := |\{H \leq G \mid [G : H] = n\}|$, and let

$$\zeta_G(s) = \sum_{n \geq 1} \frac{a_n(G)}{n^s}.$$

This is a global type of zeta function.

The following facts are known:

- If G is solvable, then ζ_G is analytic in a half-plane of the form $\{s \mid \operatorname{Re}(s) > \alpha(G)\}$.
- If G is nilpotent, then there is a product formula

$$\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s),$$

where $\zeta_{G,p}(s) = \sum_{n \geq 0} \frac{a_{p^n}(G)}{p^{ns}}$. Furthermore, each $\zeta_{G,p}$ is a rational function of $(1/p)^s$.

A key point in the study of $\zeta_{G,p}(s)$ is the fact that it can be computed by a p -adic integral, very similar to the ones that come up in the definition of Igusa zeta functions. A fundamental problem concerns the behavior of $\zeta_{G,p}$ when p varies. In general, it turns out that this can be rather wild. Some of the key results in the understanding of this variation of $\zeta_{G,p}$ are due to du Sautoy and Grunewald [dSG]. For some recent developments concerning functional equations in this context, see [Voll].

Similar zeta functions can be defined to measure the rate of growth of other algebraic subobjects. For example, this can be done for Lie subalgebras of a Lie algebra that is finitely generated as an abelian group over \mathbf{Z} , or for ideals in a ring that is finitely generated as an abelian group over \mathbf{Z} . The corresponding zeta functions have similar properties with the ones measuring the rate of growth of subgroups, see [dSG].

5.2. Representation zeta functions. Given a group G , let $r_n(G)$ denote the number of equivalence classes of n -dimensional representations of G (with suitable restrictions: for example, the representations are assumed to be rational if G is an algebraic group). The representation zeta function of G is

$$\zeta_G^{\text{rep}}(s) = \sum_{n \geq 1} \frac{r_n(G)}{n^s}.$$

An interesting example is given by $G = SL_n(\mathbf{Z})$. One can show that if $n \geq 3$, then

$$\zeta_{SL_n(\mathbf{Z})}^{\text{rep}}(s) = \zeta_{SL_n(\mathbf{C})}^{\text{rep}}(s) \cdot \prod_{p \text{ prime}} \zeta_{SL_n(\mathbf{Z}_p)}^{\text{rep}}(s).$$

It is somewhat surprising that in the few known examples, the dependence on p of the p -factors of the representation zeta function is better behaved than in the case of the subgroup growth zeta functions. Again, a key ingredient in the study of the the p -factors is given by p -adic integration. We refer to [AKOV] for some interesting new results on representation zeta functions.

REFERENCES

- [AKOV] N. Avni, B. Klopsch, U. Onn, and C. Voll, Representation zeta functions of some compact p -adic analytic groups, arXiv:1011.6533. [6](#)
- [Ati] M. F. Atiyah, Resolution of singularities and division of distributions, *Comm. Pure Appl. Math.* **23** (1970), 145–150. [3](#)
- [Ber] I. N. Bernstein, The analytic continuation of generalized functions with respect to a parameter, *Functional Anal. Appl.* **6** (1972), 273–285. [3](#)
- [BG] I. N. Bernstein and S. I. Gelfand, Meromorphic property of the functions P^λ , *Functional Anal. Appl.* **3** (1969), 68–69. [3](#)
- [BCDT] C. Breuil, B. Conrad, F. Diamond, and R. Taylor, On the modularity of elliptic curves over \mathbf{Q} : wild 3-adic exercises, *J. Amer. Math. Soc.* **14** (2001), 843–939. [2](#)
- [Del] P. Deligne, La conjecture de Weil, I, *Publ. Math. IHES* **43** (1974), 273–307. [2](#)
- [DL] J. Denef and F. Loeser, Motivic Igusa zeta functions, *J. Algebraic Geom.* **7** (1998), 505–537. [5](#)
- [Dwo] B. Dwork, On the rationality of the zeta function of an algebraic variety, *Amer. J. Math.* **82** (1960), 631–648. [2](#)
- [dSG] M. du Sautoy and F. Grunewald, Analytic properties of zeta functions and subgroup growth, *Ann. of Math. (2)* **152** (2000), 793–833. [4](#), [5](#)
- [Gro] A. Grothendieck, Formule de Lefschetz et rationalité de fonctions L, *Séminaire Bourbaki* 279 (1965). [2](#)
- [Igu] J.-i. Igusa, *An introduction to the theory of local zeta functions*, AMS/IP Studies in Advanced Mathematics 14, American Mathematical Society, Providence, RI, International Press, Cambridge, MA, 2000. [3](#)
- [Kap] M. Kapranov, The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups, arXiv: math.AG/0001005. [4](#)
- [LL] M. Larsen and V. A. Lunts, Rationality criteria for motivic zeta functions, *Compos. Math.* **140** (2004), 1537–1560. [4](#)
- [TW] R. Taylor and A. Wiles, Ring-theoretic properties of certain Hecke algebras, *Ann. of Math. (2)* **141** (1995), 553–572. [2](#)
- [Voll] C. Voll, Functional equations for zeta functions of groups and rings, *Ann. of Math. (2)* **172** (2010), 1181–1218. [5](#)
- [We1] A. Weil, *Variétés Abéliennes et Courbes Algébriques*, Hermann, Paris (1948). [1](#)
- [We2] A. Weil, Number of solutions of equations over finite fields, *Bull. Amer. Math. Soc.* **55** (1949), 497–508. [1](#), [2](#)
- [Wil] A. Wiles, Modular elliptic curves and Fermat’s last theorem, *Ann. of Math. (2)* **141** (1995), 443–551. [2](#)