

## LECTURE 3. THE STATEMENTS OF THE WEIL CONJECTURES

In this lecture we give the statements of the Weil Conjectures, make some general comments, and give some examples.

### 1. THE STATEMENTS

Suppose that  $X$  is a smooth, geometrically connected, projective variety, of dimension  $n$ , defined over a finite field  $k = \mathbf{F}_q$ . We put  $Z(t) = Z(X, t)$ .

**Conjecture 1.1** (Rationality).  $Z(t)$  is a rational function, i.e. it lies in  $\mathbf{Q}(t)$ .

**Conjecture 1.2** (Functional equation). If  $E = (\Delta^2)$  is the self-intersection of the diagonal  $\Delta \hookrightarrow X \times X$ , then

$$Z\left(\frac{1}{q^nt}\right) = \pm q^{nE/2} t^E Z(t).$$

**Conjecture 1.3** (Analogue of Riemann hypothesis). One can write

$$Z(t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)},$$

with  $P_0(t) = 1 - t$ ,  $P_{2n}(t) = 1 - q^nt$ , and for  $1 \leq 2n - 1$ , we have  $P_i(t) \in \mathbf{Z}[t]$ ,

$$P_i(t) = \prod_j (1 - \alpha_{i,j}t),$$

with  $\alpha_{i,j}$  algebraic integers with  $|\alpha_{i,j}| = q^{i/2}$ .

Note that the conditions in the above conjecture uniquely determine the  $P_i$ .

**Conjecture 1.4.** Assuming Conjecture 1.3, define the “ $i^{\text{th}}$  Betti number of  $X$ ” as  $b_i(X) := \deg(P_i(t))$ . In this case, the following hold:

- i)  $E = \sum_{i=0}^{2n} (-1)^i b_i(X)$ .
- ii) Suppose that  $R$  is a finitely generated  $\mathbf{Z}$ -subalgebra of the field  $\mathbf{C}$  of complex numbers,  $\tilde{X}$  is a smooth projective scheme over  $\text{Spec } R$ , and  $P \in \text{Spec } R$  is a prime ideal such that  $R/P = \mathbf{F}_q$  and  $\tilde{X} \times_{\text{Spec } R} \text{Spec } R/P = X$ . Then

$$b_i(X) = \dim_{\mathbf{Q}} H^i\left(\left(\tilde{X} \times_{\text{Spec } R} \text{Spec } \mathbf{C}\right)^{\text{an}}, \mathbf{Q}\right).$$

As we will see in Lecture 5, one can in fact formulate Conjecture 1.4 without assuming Conjecture 1.3. We will give in the next lecture the proofs of the above conjectures in the case of curves. In Lecture 5 we will give a brief introduction to  $\ell$ -adic cohomology, and explain how this formalism allows one to prove Conjectures 1.1, 1.2, and 1.4 (where in Conjecture 1.1 one just has to assume that  $X$  is of finite type over  $\mathbf{F}_q$ ). The harder

Conjecture 1.3 was proved by Deligne [Del], and a later proof was given by Laumon [Lau], but both these proofs go far beyond the scope of our lectures. We should also mention that the first proof of Conjecture 1.1, for arbitrary schemes of finite type over  $\mathbf{F}_q$ , was obtained by Dwork [Dwo] using  $p$ -adic analysis.

## 2. COMMENTS ON THE CONJECTURES

**Remark 2.1.** Let  $X$  be an arbitrary variety over  $k = \mathbf{F}_q$ , and let  $Y \hookrightarrow X$  be a closed subvariety, and  $U = X \setminus Y$ . It follows from Proposition 3.5 in Lecture 2 that  $Z(X, t) = Z(Y, t) \cdot Z(U, t)$ . Therefore if two of  $Z(X, t)$ ,  $Z(Y, t)$  and  $Z(U, t)$  are known to be rational, then the third one is rational, too.

**Remark 2.2.** The above remark implies that if we assume resolution of singularities over finite fields, then a positive answer to Conjecture 1.1 for smooth projective varieties implies the rationality of  $Z(X, t)$  for every variety  $X$  over a finite field. Indeed, suppose by induction on dimension that the assertion is known for varieties of dimension  $< n$ . Remark 2.1 and the induction hypothesis imply that if  $X$  and  $Y$  are varieties of dimension  $n$  that have dense open subsets  $U$ , respectively  $V$ , that are isomorphic, then  $Z(X, t)$  is rational if and only if  $Z(Y, t)$  is rational. Given any  $n$ -dimensional variety  $X$ , there is a projective variety  $Y$  such that there are  $U$  and  $V$  as above. Furthermore, if we have resolution of singularities over our ground field, then we may assume that  $Y$  is also smooth. Therefore  $Z(Y, t)$  is rational by Conjecture 1.1 (note that the irreducible components  $Y_i$  of  $Y$  are disjoint, hence  $Z(Y, t) = \prod_i Z(Y_i, t)$ , any we apply Conjecture 1.1 to each  $Y_i$ ).

**Remark 2.3.** In a similar vain, in order to prove that  $Z(X, t)$  is rational for every variety  $X$ , it is enough to prove it in the case when  $X$  is an irreducible hypersurface in  $\mathbf{A}_{\mathbf{F}_q}^n$ . Indeed, arguing as in the previous remark we see that we may assume that  $X$  is affine and irreducible, in which case it is birational (over  $\mathbf{F}_q$ ) with a hypersurface in an affine space.

**Remark 2.4.** There is the following general formula in intersection theory: if  $i: Y \hookrightarrow X$  is a closed embedding of nonsingular varieties of pure codimension  $r$ , then  $i^*(i_*(\alpha)) = c_r(N_{Y/X}) \cap \alpha$  for every  $\alpha \in A^*(X)$ , where  $N_{Y/X}$  is the normal bundle of  $Y$  in  $X$ .

In particular, if  $X$  is smooth, projective, of pure dimension  $n$ , and  $\Delta: X \hookrightarrow X \times X$  is the diagonal embedding, then  $N_{X/X \times X} = T_X$ , and therefore

$$(\Delta^2) = \deg(c_n(T_X)).$$

**Example 2.5.** Let us check the Weil conjectures when  $X = \mathbf{P}_{\mathbf{F}_q}^n$ . As we have seen in Corollary 3.6 in Lecture 3, we have

$$(1) \quad Z(\mathbf{P}^n, t) = \frac{1}{(1-t)(1-qt) \cdots (1-q^n t)}.$$

In particular, it is clear that Conjectures 1.1 and 1.3 hold in this case. It follows from (1) that

$$Z(X, 1/q^n t) = \frac{1}{\left(1 - \frac{1}{q^n t}\right) \left(1 - \frac{1}{q^{n-1} t}\right) \cdots \left(1 - \frac{1}{t}\right)} = (-1)^{n+1} t^{n+1} q^{n(n+1)/2} Z(X, t).$$

Hence in order to check Conjecture 1.2, it is enough to show that  $E = n + 1$ . The Euler exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^n} \rightarrow \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus(n+1)} \rightarrow T_{\mathbf{P}^n} \rightarrow 0$$

implies  $c(T_{\mathbf{P}^n}) = c(\mathcal{O}_{\mathbf{P}^n}(1)^{\oplus(n+1)}) = (1 + h)^{n+1}$ , where  $h = c_1(\mathcal{O}_{\mathbf{P}^n}(1))$ . This implies that  $\deg(c_n(T_{\mathbf{P}^n})) = n + 1$ .

Since  $H^*(\mathbf{P}_{\mathbf{C}}^n, \mathbf{Q}) \simeq \mathbf{Q}[t]/(t^{n+1})$ , with  $\deg(t) = 2$ , the assertions in Conjecture 1.4 also follow.

**Remark 2.6.** Let  $X$  be a variety over  $\mathbf{F}_q$ , and suppose we know that  $Z(X, t)$  is rational. Let us write  $Z(X, t) = \frac{f(t)}{g(t)}$ , with  $f, g \in \mathbf{Q}[t]$ . After dividing by the possible powers of  $t$ , we may assume that  $f(0), g(0) \neq 0$ , and after normalizing, that  $f(0) = 1 = g(0)$ .

We write  $f(t) = \prod_{i=1}^r (1 - \alpha_i t)$  and  $g(t) = \prod_{j=1}^s (1 - \beta_j t)$ . If  $N_m = |X(\mathbf{F}_{q^m})|$ , then

$$\sum_{m \geq 1} \frac{N_m}{m} t^m = \sum_{i=1}^r \log(1 - \alpha_i t) - \sum_{j=1}^s \log(1 - \beta_j t),$$

hence  $N_m = \sum_{j=1}^s \beta_j^m - \sum_{i=1}^r \alpha_i^m$  for every  $m \geq 1$ .

### 3. TWO EXAMPLES: COMPUTING THE BETTI NUMBERS FOR GRASSMANNIANS AND FULL FLAG VARIETIES

One can use the above Conjecture 1.4 to compute the Betti numbers of smooth complex projective varieties. We illustrate this by computing the Poincaré polynomials for Grassmannians and full flag varieties. For a famous example, in which the Weil conjectures are used to compute the Betti numbers of the Hilbert schemes of points on smooth projective surfaces, see [Göt].

Recall that both the Grassmannian and the flag variety can be defined over  $\mathbf{Z}$ . More precisely, if  $1 \leq r \leq n - 1$ , there is a scheme  $\mathrm{Gr}(r, n)$  defined over  $\mathrm{Spec} \mathbf{Z}$ , such that for every field  $K$ , the  $K$ -valued points of  $\mathrm{Gr}(r, n)$  are in bijection with the  $r$ -dimensional subspaces of  $K^n$ . Similarly, we have a scheme  $\mathrm{Fl}(n)$  defined over  $\mathrm{Spec} \mathbf{Z}$  such that for every field  $K$ , the  $K$ -valued points of  $\mathrm{Fl}(n)$  are in bijection with the full flags on  $K^n$ , that is, with the  $n$ -tuples of linear subspaces  $V_1 \subset V_2 \subset \dots \subset V_n = K^n$ , with  $\dim(V_i) = i$  for every  $i$ . It is well-known that both  $\mathrm{Gr}(r, n)$  and  $\mathrm{Fl}(n)$  are smooth, geometrically connected, and projective over  $\mathrm{Spec} \mathbf{Z}$ . For every field  $K$ , we put  $\mathrm{Gr}(r, n)_K = \mathrm{Gr}(r, n) \times \mathrm{Spec} K$  and  $\mathrm{Fl}(n)_K = \mathrm{Fl}(n) \times \mathrm{Spec} K$ .

Recall that the Poincaré polynomial of a complex algebraic variety  $X$  is given by  $P_X(y) = \sum_{i=0}^{2 \dim(X)} (-1)^i \dim_{\mathbf{Q}} H^i(X(\mathbf{C})^{\mathrm{an}}, \mathbf{Q}) y^i$ . In order to compute the Poincaré polynomials of  $\mathrm{Gr}(r, n)_{\mathbf{C}}$  and  $\mathrm{Fl}(n)_{\mathbf{C}}$ , we need to compute the zeta functions  $Z(\mathrm{Gr}(r, n)_{\mathbf{F}_q}, t)$  and  $Z(\mathrm{Fl}(n)_{\mathbf{F}_q}, t)$ . Therefore we need to determine the numbers  $a_q(r, n)$  and  $b_q(n)$  of  $r$ -dimensional linear subspaces of  $\mathbf{F}_q^n$ , respectively, of full flags on  $\mathbf{F}_q^n$ .

In fact, we first compute  $b_q(n)$ , and then use this to compute  $a_q(r, n)$ . In order to give a full flag in  $\mathbf{F}_q^n$ , we first need to give a line  $L_1$  in  $\mathbf{F}_q^n$ , then a line  $L_2$  in  $\mathbf{F}_q^n/L_1$ , and

so on. This shows that

$$b_q(n) = |\mathbf{P}^{n-1}(\mathbf{F}_q)| \cdot |\mathbf{P}^{n-2}(\mathbf{F}_q)| \cdots |\mathbf{P}^1(\mathbf{F}_q)|,$$

and therefore

$$(2) \quad b_q(n) = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{(q - 1)^n} = (1 + q)(1 + q + q^2) \cdots (1 + q + \dots + q^{n-1}).$$

We now compute  $a_q(r, n)$ . Note that we have an action of  $GL_n(\mathbf{F}_q)$  on  $\text{Gr}(r, n)(\mathbf{F}_q)$  induced by the natural action on  $\mathbf{F}_q^n$ . This action is transitive, and the stabilizer of the subspace  $W$  generated by  $e_1, \dots, e_r$  (where  $e_1, \dots, e_n$  is the standard basis of  $\mathbf{F}_q^n$ ) is the set of matrices

$$\{A = (a_{i,j}) \in GL_n(\mathbf{F}_q) \mid a_{i,j} = 0 \text{ for } r+1 \leq i \leq n, 1 \leq j \leq r\}.$$

If  $A = (a_{i,j}) \in M_n(\mathbf{F}_q)$  is such that  $a_{i,j} = 0$  for  $r+1 \leq i \leq n$  and  $1 \leq j \leq r$ , then  $A$  is invertible if and only if the two matrices  $(a_{i,j})_{i,j \leq r}$  and  $(a_{i,j})_{i,j \geq r+1}$  are invertible. We conclude that the number of elements in the stabilizer of  $W$  is

$$(3) \quad |GL_r(\mathbf{F}_q)| \cdot |GL_{n-r}(\mathbf{F}_q)| \cdot |M_{r,n-r}(\mathbf{F}_q)|.$$

In order to compute  $|GL_r(\mathbf{F}_q)|$ , we use the transitive action of  $GL_r(\mathbf{F}_q)$  on  $\text{Fl}(r)(\mathbf{F}_q)$  induced by the natural action on  $\mathbf{F}_q^r$ . The stabilizer of the flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \mathbf{F}_q^r$$

is the set of upper triangular matrices in  $GL_r(\mathbf{F}_q)$ , and there are  $(q-1)^r q^{r(r-1)/2}$  such matrices. We conclude that

$$|GL_r(\mathbf{F}_q)| = b_q(r) q^{r(r-1)/2} (q-1)^r = q^{r(r-1)/2} (q^r - 1)(q^{r-1} - 1) \cdots (q - 1).$$

We now deduce from (3) that

$$(4) \quad a_q(r, n) = \frac{|GL_n(\mathbf{F}_q)|}{|GL_r(\mathbf{F}_q)| \cdot |GL_{n-r}(\mathbf{F}_q)| \cdot q^{r(n-r)}} = \frac{(q^n - 1) \cdots (q - 1)}{(q^r - 1) \cdots (q - 1)(q^{n-r} - 1) \cdots (q - 1)} \\ = \frac{(q^n - 1) \cdots (q^{n-r+1} - 1)}{(q^r - 1) \cdots (q - 1)}.$$

The expression in (4) is called the *Gaussian binomial coefficient*, and it is denoted by  $\binom{n}{r}_q$  (there are many analogies with the usual binomial coefficients; in any case, note that  $\lim_{q \rightarrow 1} \binom{n}{r}_q = \binom{n}{r}$ ).

**Exercise 3.1.** *Prove the following properties of Gaussian binomial coefficients:*

- i)  $\binom{n}{r}_q = q^r \binom{n-1}{r}_q + \binom{n-1}{r-1}_q$  (*generalized Pascal identity*).
- ii) *Using i) and induction on  $n$ , show that if  $\lambda_{n,r}(j)$  denotes the number of partitions of  $j$  into  $\leq n-r$  parts, each of size  $\leq r$ , then*

$$(5) \quad \binom{n}{r}_q = \sum_{j=0}^{r(n-r)} \lambda_{n,r}(j) q^j.$$

A variety  $X$  over  $\mathbf{F}_q$  is called of *polynomial count* if there is a polynomial  $P \in \mathbf{Z}[y]$  such that the number of  $\mathbf{F}_{q^m}$ -valued points of  $X$  is  $P(q^m)$  for every  $m \geq 1$ . It follows from (2) that  $\mathrm{Fl}(n)_{\mathbf{F}_q}$  is of polynomial count. Similarly,  $\mathrm{Gr}(r, n)_{\mathbf{F}_q}$  is of polynomial count by (4) and part ii) in the above exercise.

**Lemma 3.2.** *Suppose that  $X$  is a variety over  $\mathbf{F}_q$ , and  $P(y) = a_d y^d + a_{d-1} y^{d-1} + \dots + a_0$ , with all  $a_i \in \mathbf{Z}$ , is such that  $|X(\mathbf{F}_q^m)| = P(q^m)$  for every  $m \geq 1$ . In this case the zeta function of  $X$  is given by*

$$Z(X, t) = \prod_{i=0}^d (1 - q^i t)^{-a_i}.$$

*Proof.* We have

$$\sum_{m \geq 1} \frac{N_m}{m} t^m = \sum_{i=0}^n a_i \sum_{m \geq 1} \frac{q^{mi}}{m} t^m = \sum_{i=0}^n -a_i \log(1 - q^i t).$$

Therefore by taking exp we get the formula in the lemma.  $\square$

**Remark 3.3.** In the context of the lemma, if  $X$  is smooth and projective, then the analogue of the Riemann hypothesis implies that  $a_i \geq 0$  for all  $i$ . In this context, we have  $b_i(X) = 0$  for  $i$  odd, and  $b_{2i}(X) = a_i$  for  $1 \leq i \leq n$ .

By combining Lemma 3.2 with Conjecture 1.4, our computations for the flag variety and the Grassmannian give the following.

**Corollary 3.4.** *The Poincaré polynomial of  $\mathrm{Fl}(n)_{\mathbf{C}}$  is given by  $\prod_{i=1}^n (1 + y^2 + \dots + y^{2i})$ , and the Poincaré polynomial of  $\mathrm{Gr}(r, n)_{\mathbf{C}}$  is given by  $\sum_{i=0}^{r(r-n)} \lambda_{n,r}(i) y^{2i}$ .*

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