LECTURE 5. WEIL COHOMOLOGY THEORIES AND THE WEIL CONJECTURES

Weil realized that the rationality and the functional equation part of the Weil conjectures would follow from the existence of a cohomology theory with suitable properties. Such a cohomology theory is nowadays called a *Weil cohomology theory*. In the first section we describe the axioms of such a cohomology theory, and derive some consequences. These will be used in the second section to deduce the rationality and the functional equation for the Hasse-Weil zeta function. In the last section, we give a brief introduction to the first Weil cohomology over fields of positive characteristic, the ℓ-adic cohomology.

1. Weil cohomology theories

In this section we work over a fixed algebraically closed field k. All varieties are defined over k. Recall that given a variety X and $r \in \mathbf{Z}_{\geq 0}$, the group of r-cycles on X is the free abelian group on the set of closed irreducible r-dimensional subvarieties of X. If V is such a variety, then we write $[V]$ for the corresponding element of the cycle group. For a closed subscheme Z of X of pure dimension r, the cycle of Z is $[Z] = \sum_{i=1}^{r} \ell(\mathcal{O}_{Z,Z_i})[Z_i]$, where the Z_i are the irreducible components of Z, and $\mathcal{O}_{Z,Z_i}$ is the zero-dimensional local ring of Z at the generic point of Z_i.

Our presentation of the formalism of Weil cohomology theories follows with small modifications and a few extra details de Jong's note [deJ1]. A Weil cohomology theory with coefficients in the characteristic zero field K is given by the following data:

(D1) A contravariant functor $X \to H^*(X) = \oplus_i H^i(X)$ from nonsingular, connected, projective varieties (over k) to graded commutative[1] K-algebras. The product of $\alpha, \beta \in H^*(X)$ is denoted by $\alpha \cup \beta$.
(D2) For every nonsingular, connected, projective algebraic variety X, a linear *trace* map $\mathrm{Tr} = \mathrm{Tr}_X \colon H^{2\dim(X)}(X) \to K$.
(D3) For every nonsingular, connected, projective algebraic variety X, and for every closed irreducible subvariety $Z \subseteq X$ of codimension c, a *cohomology class* $\mathrm{cl}(Z) \in H^{2c}(X)$.

The above data is supposed to satisfy the following set of axioms.

(A1) For every nonsingular, connected, projective variety X, all $H^i(X)$ have finite dimension over K. Furthermore, $H^i(X) = 0$ unless $0 \leq i \leq 2\dim(X)$.
(A2) (Künneth property) If X and Y are nonsingular, connected, projective varieties, and if $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ are the canonical projections, then

[1] Recall that *graded commutative* means that $\alpha\beta = (-1)^{\deg(\alpha)\deg(\beta)}\beta\alpha$ for every homogeneous elements α and β.

the K-algebra homomorphism

$$H^*(X) \otimes_K H^*(Y) \to H^*(X \times Y),\ \alpha \otimes \beta \to p_X^*(\alpha) \cup p_Y^*(\beta)$$

is an isomorphsim.

(A3) (Poincaré duality) For every nonsingular, connected, projective variety X, the trace map $\mathrm{Tr}\colon H^{2\dim(X)}(X) \to K$ is an isomorphism, and for every i with $0 \leq i \leq 2\dim(X)$, the bilinear map

$$H^i(X) \otimes_K H^{2\dim(X)-i}(X) \to K,\ \ \alpha \otimes \beta \to \mathrm{Tr}_X(\alpha \cup \beta)$$

is a perfect pairing.

(A4) (Trace maps and products) For every nonsingular, connected, projective varieties X and Y, we have

$$\mathrm{Tr}_{X\times Y}(p_X^*(\alpha) \cup p_Y^*(\beta)) = \mathrm{Tr}_X(\alpha)\mathrm{Tr}_Y(\beta)$$

for every $\alpha \in H^{2\dim(X)}(X)$ and $\beta \in H^{2\dim(Y)}(Y)$.

(A5) (Exterior product of cohomology classes) For every nonsingular, connected, projective varieties X and Y, and every closed irreducible subvarieties $Z \subseteq X$ and $W \subseteq Y$, we have

$$\mathrm{cl}(Z \times W) = p_X^*(\mathrm{cl}(Z)) \cup p_Y^*(\mathrm{cl}(W)).$$

(A6) (Push-forward of cohomology classes) For every morphism $f\colon X \to Y$ of nonsingular, connected, projective varieties, and for every irreducible closed subvariety $Z \subseteq X$, we have for every $\alpha \in H^{2\dim(Z)}(Y)$

$$\mathrm{Tr}_X(\mathrm{cl}(Z) \cup f^*(\alpha)) = \deg(Z/f(Z)) \cdot \mathrm{Tr}_Y(\mathrm{cl}(f(Z)) \cup \alpha).$$

(A7) (Pull-back of cohomology classes) Let $f\colon X \to Y$ be a morphism of nonsingular, connected, projective varieties, and $Z \subseteq Y$ an irreducible closed subvariety that satisfies the following conditions:

a) All irreducible components $W_1, \ldots, W_r$ of $f^{-1}(Z)$ have pure dimension $\dim(Z)+\dim(X) - \dim(Y)$.

b) Either f is flat in a neighborhood of Z, or Z is *generically transverse* to f, in the sense that $f^{-1}(Z)$ is generically smooth.

Under these assumptions, if $[f^{-1}(Z)] = \sum_{i=1}^r m_i W_i$, then $f^*(\mathrm{cl}(Z)) = \sum_{i=1}^r m_i \mathrm{cl}(W_i)$ (note that if Z is generically transverse to f, then $m_i = 1$ for all i).

(A8) (Case of a point) If $x = \mathrm{Spec}(k)$, then $\mathrm{cl}(x) = 1$ and $\mathrm{Tr}_x(1) = 1$.

A basic example of a Weil cohomology theory is given by singular cohomology in the case $k = \mathbf{C}$, when we may take $K = \mathbf{Q}$. In the last section we will discuss an example of a Weil cohomology theory when $\mathrm{char}(k) = p > 0$, the ℓ-adic cohomology (with $K = \mathbf{Q}_\ell$, for some $\ell \neq p$). Another example, still when $\mathrm{char}(k) > 0$, is given by crystalline cohomology (with $K = W(k)$, the ring of Witt vectors of k).

In the rest of this section we assume that we have a Weil cohomology theory for varieties over k, and deduce several consequences. In particular, we relate the Chow ring of X to $H^*(X)$. We will review below some of the basic definitions related to Chow rings. For our applications in the next section, the main result is the trace formula in Theorem 1.7 below.

Proposition 1.1. *Let X be a smooth, connected, n-dimensional projective variety.*

i) *The structural morphism $K \to H^0(X)$ is an isomorphism.*
ii) *We have $\mathrm{cl}(X) = 1 \in H^0(X)$.*
iii) *If $x \in X$ is a closed point, then $\mathrm{Tr}_X(\mathrm{cl}(x)) = 1$.*
iv) *If $f \colon X \to Y$ is a generically finite, surjective morphism of degree d between smooth, connected, projective varieties, $\mathrm{Tr}_X(f^*(\alpha)) = d \cdot \mathrm{Tr}_Y(\alpha)$ for every $\alpha \in H^{2\dim(Y)}(Y)$. In particular, if $Y = X$, then f^* acts as multiplication by d on $H^{2\dim(X)}(X)$.*

Proof. Applying condition (A3) with $i = 0$ implies that $\dim_K H^0(X) = 1$, hence the structural morphism of the K-algebra $H^*(X)$ induces an isomorphism $K \simeq H^0(X)$. Applying condition (A7) to the morphism $X \to \mathrm{Spec}\, k$, as well as condition (A8), we get $\mathrm{cl}(X) = 1 \in H^0(X)$.

Given $x \in X$, let us apply condition (A6) to the morphism $X \to \mathrm{Spec}\, k$, by taking $Z = \{x\}$ and $\alpha = 1 \in H^0(\mathrm{Spec}\, k)$. We deduce using also (A8) that $\mathrm{Tr}_X(\mathrm{cl}(x)) = 1$.

If $f \colon X \to Y$ is as in iv), let us choose a general point Q in Y. If the cycle of the fiber $f^{-1}(Q)$ is $[f^{-1}(Q)] = \sum_{i=1}^r m_i P_i$, then by hypothesis $\sum_{i=1}^r m_i = d$. Since f is flat around Q by generic flatness, condition (A7) implies

$$\mathrm{Tr}_X(f^*(\mathrm{cl}(Q))) = \mathrm{Tr}_X(\sum_i m_i \cdot \mathrm{cl}(P_i)) = d \cdot \mathrm{Tr}_Y(\mathrm{cl}(Q)).$$

Since $\mathrm{cl}(Q)$ generates $H^{2\dim(Y)}(Y)$, this proves the assertion in iv). □

We now use Poincaré duality to define push-forwards in cohomology. Let $f \colon X \to Y$ be a morphism between nonsingular, connected, projective varieties, with $\dim(X) = m$ and $\dim(Y) = n$. Given $\alpha \in H^i(X)$, there is a unique $f_*(\alpha) \in H^{2n-2m+i}(Y)$ such that

$$\mathrm{Tr}_Y(f_*(\alpha) \cup \beta) = \mathrm{Tr}_X(\alpha \cup f^*(\beta))$$

for every $\beta \in H^{2m-i}(Y)$. It is clear that f_* is K-linear. The following proposition collects the basic properties of the push-forward map.

Proposition 1.2. *Let $f \colon X \to Y$ be a morphism as above.*

i) *(Projection formula) $f_*(\alpha \cup f^*(\gamma)) = f_*(\alpha) \cup \gamma$.*
ii) *If $g \colon Y \to Z$ is another morphism, with Z smooth, connected, and projective, then $(g \circ f)_* = g_* \circ f_*$ on $H^*(X)$.*
iii) *If Z is an irreducible, closed subvariety of X, then*

$$f_*(\mathrm{cl}(Z)) = \deg(Z/f(Z))\mathrm{cl}(f(Z)).$$

Proof. Properties i) and ii) follow easily from definition, using Poincaré duality. Property iii) is a consequence of (A6). □

Proposition 1.3. *Let X and Y be nonsingular, connected, projective varieties, and $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the canonical projections. If $\alpha \in H^i(Y)$, then $p_*(q^*(\alpha)) = \mathrm{Tr}_Y(\alpha)$ if $i = 2\dim(Y)$, and $p_*(q^*(\alpha)) = 0$, otherwise.*

Proof. Note that $p_*(q^*(\alpha)) \in H^{i-2\dim(Y)}(X)$, hence it is clear that $p_*(q^*(\alpha)) = 0$ when $i \neq 2\dim(Y)$. On the other hand, if $\alpha \in H^{2\dim(Y)}(Y)$ and $\beta \in H^{2\dim(X)}(X)$, then

$$\mathrm{Tr}_X(p_*(q^*(\alpha)) \cup \beta) = \mathrm{Tr}_{X\times Y}(q^*(\alpha) \cup p^*(\beta)) = \mathrm{Tr}_Y(\alpha)\mathrm{Tr}_X(\beta),$$

where the last equality follows from condition (A4). Therefore $p_*(q^*(\alpha)) = \mathrm{Tr}_Y(\alpha)$. □

Our next goal is to show that taking the cohomology class induces a ring homomorphism from the Chow ring $A^*(X)$ to $H^{2*}(X)$, the even part of the cohomology ring. Before doing this, let us review a few facts about Chow rings. For details and proofs, we refer to [Ful, Chapters I-VIII].

Let X be an arbitrary variety over k. The Chow group $A_r(X)$ is the quotient of $Z_r(X)$ by the rational equivalence relation. Recall that this equivalence relation is generated by putting $\mathrm{div}_W(\varphi) \sim 0$, where W is an $(r+1)$-dimensional closed irreducible subvariety of X, and φ is a nonzero rational function of X. We do not give the general definition of $\mathrm{div}_W(\varphi)$, but only mention that for W normal, this is the usual definition of the principal divisor corresponding to a rational function. In particular, if φ defines a morphism $\widetilde{\varphi}\colon W \to \mathbf{P}^1$, then $\mathrm{div}_W(\varphi) = [\widetilde{\varphi}^{-1}(0)] - [\widetilde{\varphi}^{-1}(\infty)]$.

For a proper morphism $f\colon X \to Y$ one defines $f_*\colon Z_r(X) \to Z_r(Y)$ such that for an irreducible variety V of X, $f_*([V]) = \deg(V/f(V))[f(V)]$. One shows that if φ is a nonzero rational function on an $(r+1)$-dimensional irreducible closed subvariety W of X, one has $f_*(\mathrm{div}_W(\varphi)) = 0$ if $\dim(f(W) < \dim(V)$, and $f_*(\mathrm{div}_W(\varphi)) = \mathrm{div}_{f(W)}(N(\varphi))$, otherwise, where $N\colon K(f(W)) \to K(W)$ is the norm map. Therefore we get an induced morphism $f_*\colon A_r(X) \to A_r(Y)$. Note that when X is complete, the induced map $\deg\colon A_0(X) \to A_0(\mathrm{Spec}\, k) = \mathbf{Z}$ is given by taking the degree of a cycle. We extend this map by defining it to be zero on $A_i(X)$ with $i \neq 0$.

If X is nonsingular, connected, and $\dim(X) = n$, then one puts $A^i(X) = A_{n-i}(X)$, and $A^*(X) = \oplus_{i=0}^n A^i(X)$ has a structure of commutative graded ring. One denotes by $\alpha \cup \beta$ the product of $\alpha, \beta \in A^*(X)$. If X is complete, and $\alpha_1, \ldots, \alpha_r \in A^*(X)$, then the intersection number $(\alpha_1 \cdot \ldots \cdot \alpha_r)$ is given by $\deg(\alpha_1 \cup \ldots \cup \alpha_r)$.

Taking X to $A^*(X)$ gives, in fact, a contravariant functor from the category of nonsingular quasiprojective[2] varieties to that of graded rings. One defines the pull-back $f^*\colon A^*(Y) \to A^*(X)$ of a morphism $f\colon X \to Y$ of nonsingular varieties in terms of a suitable (refined) intersection product. For us, it is enough to use the following property: if Z is an irreducible subvariety of Y such that $f^{-1}(Z)$ has pure dimension $\dim(Z) + \dim(X) - \dim(Y)$, and either f is flat in a neighborhood of Z, or Z is generically transverse to f, then $f^*([Z]) = [f^{-1}(Z)]$. In fact, one can always reduce to one of these two situations: every f admits the decomposition

$$X \overset{j}{\hookrightarrow} X \times Y \overset{\mathrm{pr}_Y}{\to} Y,$$

where $j = (\mathrm{Id}_X, f)$ is the graph of f. Therefore

$$f^*([Z]) = j^*(\mathrm{pr_Y}^*([Z])) = j^*([X \times Z]).$$

[2]This assumption is only made for convenience, since we want to use the Moving Lemma, and since we are concerned with projective varieties.

Furthermore, by the Moving Lemma, $[X\times Z]$ is rationally equivalent with a sum $\sum_\ell n_\ell[W_\ell]$, with each W_ℓ generically transverse to j. Therefore $f^*([Z])=\sum_\ell n_\ell[j^{-1}(W_\ell)]$.

The product of $A^*(X)$ can be described in terms of pull-back, as follows. If Z_1 and Z_2 are irreducible closed subvarieties of X, then

$$[Z_1]\cup[Z_2]=\Delta^*([Z_1\times Z_2])\in A^*(X),$$

where $\Delta\colon X\hookrightarrow X\times X$ is the diagonal embedding. In particular, suppose that Z_1 and Z_2 are generically transverse, in the sense that all irreducible components $W_1\ldots,W_r$ of $Z_1\cap Z_2$ have dimension $\dim(Z_1)+\dim(Z_2)-\dim(X)$, and $Z_1\cap Z_2$ is generically smooth. In this case $Z_1\times Z_2$ is generically transverse to Δ, hence $[Z_1]\cup[Z_2]=\sum_{\ell=1}^r[W_\ell]$.

Suppose now that we have a Weil cohomology theory for varieties over k. If X is a smooth, connected, projective n-dimensional variety over k, taking the cohomology class induces a group homomorphism $\mathrm{cl}\colon Z_r(X)\to H^{2(n-r)}(X)$. Note that if $f\colon X\to Y$ is a morphism of such varieties, then it follows from Proposition 1.2 iii) that

$$f_*(\mathrm{cl}(\alpha))=\mathrm{cl}(f_*(\alpha)). \tag{1}$$

Lemma 1.4. *If $\alpha=\sum_{i=1}^r n_i[V_i]$ is an r-cycle that is rationally equivalent to zero, then $\sum_i n_i\mathrm{cl}(V_i)=0$ in $A^{2(n-r)}(X)$.*

Proof. We may assume that there is an irreducible $(r+1)$-dimensional subvariety W of X, and a nonzero rational function φ on W such that $\alpha=\mathrm{div}_W(\varphi)$. We have a rational map $\widetilde{\varphi}\colon W\dashrightarrow \mathbf{P}^1$ defined by φ. Let $\pi\colon W'\to W$ be a projective, generically finite morphism, with W' an integral scheme, such that $\widetilde{\varphi}\circ\pi$ is a morphism $\widetilde{\psi}$. After possibly replacing W' by a nonsingular alteration (see [deJ2]), we may assume that W' is nonsingular, connected, and projective. If $d=\deg(W'/W)$ and $\psi=f^*(\varphi)\in K(W')$, then we have the equality of cycles $f_*(\mathrm{div}(\psi))=d\cdot\mathrm{div}(\varphi)$. Since $\mathrm{char}(K)=0$, it follows from (1) that it is enough to show that $\mathrm{cl}(\mathrm{div}(\psi))=0$ in $H^*(W')$. On the other hand, by construction we have $\mathrm{div}(\psi)=[\widetilde{\psi}^{-1}(0)]-[\widetilde{\psi}^{-1}(\infty)]$, and condition (A7) implies that it is enough to show that $\mathrm{cl}(0)=\mathrm{cl}(\infty)\in H^1(\mathbf{P}^1)$. This follows from assertion iii) in Proposition 1.1. □

The above lemma implies that for every nonsingular, connected, projective n-dimensional variety X, we have a morphism of graded groups $\mathrm{cl}\colon A^*(X)\to H^{2*}(X)$.

Proposition 1.5. *The morphism $\mathrm{cl}\colon A^*(X)\to H^{2*}(X)$ is a ring homomorphism. Furthermore, it is compatible with both f^* and f_*.*

Proof. Compatibility with f_* follows from (1). We next show compatibility with f^*, where $f\colon X\to Y$ is a morphism between nonsingular, connected, projective varieties. Writing f as the composition $X\overset{j}{\hookrightarrow}X\times Y\overset{p_Y}{\to}Y$, we note that $f^*=j^*\circ p_Y^*$ both at the level of H^* and at the level of A^*. The fact that taking the cohomology class commutes with both p_Y^* and j^* is a consequence of condition (A7) (and, in the case of j^*, of the Moving Lemma).

We now show that cl is a ring homomorphism. If V and W are irreducible subvarieties of X, and $\Delta\colon X\hookrightarrow X\times X$ is the diagonal embedding, then using the compatibility with pull-back and condition (A5) we get

$$\begin{aligned}\mathrm{cl}([V]\cup[W]) &= \mathrm{cl}(\Delta^*([V\times W])) = \Delta^*(\mathrm{cl}(V\times W)) = \Delta^*(p_X^*(\mathrm{cl}(V))\cup p_Y^*(\mathrm{cl}(W)))\\ &= \Delta^*(p_X^*(\mathrm{cl}(V)))\cup\Delta^*(p_Y^*(\mathrm{cl}(W))) = \mathrm{cl}(V)\cup\mathrm{cl}(W).\end{aligned}$$

□

Corollary 1.6. *If X is a smooth, connected, projective variety, and $\alpha_i\in A^{m_i}(X)$ for $1\leq i\leq r$ are such that $m_1+\ldots+m_r=\dim(X)$, then $(\alpha_1\cdot\ldots\cdot\alpha_r)=\mathrm{Tr}_X(\mathrm{cl}(\alpha_1)\cup\ldots\cup\mathrm{cl}(\alpha_r))$.*

Proof. Since the map $\mathrm{cl}\colon A^*(X)\to H^{2*}(X)$ is a ring homomorphism, it is enough to show that for $\alpha\in Z_0(X)$, we have $\deg(\alpha)=\mathrm{Tr}_X(\mathrm{cl}(\alpha))$. By additivity, we may assume $\alpha=P$, in which case it is enough to apply assertion iii) in Proposition 1.1. □

Theorem 1.7. (*The trace formula*) *If $\varphi\colon X\to X$ is an endomorphism of the nonsingular, connected, projective variety X, and if $\Gamma_\varphi,\Delta\subset X\times X$ are the graph of φ, and respectively, the diagonal, then*

$$(\Gamma_\varphi\cdot\Delta)=\sum_{i=0}^{2\dim(X)}(-1)^i\mathrm{trace}(\varphi^*|H^i(X)).$$

In particular, if Γ_φ and Δ intersect transversely, then the above expression computes $|\{x\in X\mid\varphi(x)=x\}|$.

We will apply this result in the next section by taking φ to be the Frobenius morphism. On the other hand, by taking φ to be the identity, we obtain the following

Proposition 1.8. *If X is a smooth, connected, n-dimensional projective variety, and $\Delta\subset X\times X$ is the diagonal, then*

$$(\Delta^2)=\sum_{i=0}^{2n}(-1)^i\dim_K H^i(X).$$

We give the proof of Theorem 1.7 following [Mil, Chapter VI, §12]. We need two lemmas. With the notation in Theorem 1.7, let $\dim(X)=n$, and let $p,q\colon X\times X\to X$ denote the projections onto the first, respectively second, component.

Lemma 1.9. *If $\alpha\in H^*(X)$, then $p_*(\mathrm{cl}(\Gamma_\varphi)\cup q^*(\alpha))=\varphi^*(\alpha)$.*

Proof. Let $j\colon X\hookrightarrow X\times X$ be the embedding onto the graph of φ, so that $p\circ j=\mathrm{Id}_X$ and $q\circ j=\varphi$. Since $j_*(\mathrm{cl}(X))=\mathrm{cl}(\Gamma_\varphi)$, we deduce using the projection formula

$$p_*(\mathrm{cl}(\Gamma_\varphi)\cup q^*(\alpha))=p_*(j_*(\mathrm{cl}(X))\cup q^*(\alpha))=p_*(j_*(\mathrm{cl}(X)\cup j^*(q^*(\alpha)))=p_*(j_*(\varphi^*(\alpha)))=\varphi^*(\alpha).$$

□

Lemma 1.10. *Let (e_i^r) be a basis of $H^r(X)$ and (f_i^{2n-r}) the dual basis of $H^{2n-r}(X)$ with respect to Poincaré duality, such that $\mathrm{Tr}_X(f_\ell^{2n-r}\cup e_i^r)=\delta_{i,\ell}$. With this notation, we have*

$$\mathrm{cl}(\Gamma_\varphi)=\sum_{i,r}p^*(\varphi^*(e_i^r))\cup q^*(f_i^{2n-r})\in H^{2n}(X\times X).$$

Proof. We know by the Künneth property that we can write

$$\mathrm{cl}(\Gamma_\varphi) = \sum_{\ell,s} p^*(a_{\ell,s}) \cup q^*(f_\ell^{2n-s}),$$

for unique elements $a_{\ell,s} \in H^s(X)$. It follows from Lemma 1.9 and the projection formula that that

$$\varphi^*(e_i^r) = \sum_{\ell,s} p_*(p^*(a_{\ell,s}) \cup q^*(f_\ell^{2n-s}) \cup q^*(e_i^r)) = \sum_{\ell,s} a_{\ell,s} \cup p_*(q^*(f_\ell^{2n-s} \cup e_i^r)).$$

Lemma 1.3 implies that $p_*(q^*(f_\ell^{2n-s} \cup e_i^r)$ is zero, unless $r = s$, in which case it is equal to $\mathrm{Tr}_X(f_\ell^{2n-r} \cup e_i^r)$. By assumption, this is zero, unless $i = \ell$, in which case it is equal to 1. We conclude that $\varphi^*(e_i^r) = a_{i,r}$. □

Proof of Theorem 1.7. It follows from Lemma 1.10 that

$$\mathrm{cl}(\Gamma_\varphi) = \sum_{i,r} p^*(\varphi^*(e_i^r)) \cup q^*(f_i^{2n-r}).$$

Applying the same lemma to the identity morphism, and to the dual bases (f_ℓ^s) and $((-1)^s e_\ell^{2n-s})$, we get

$$\mathrm{cl}(\Delta) = \sum_{\ell,s} (-1)^s p^*(f_\ell^s) \cup q^*(e_\ell^{2n-s}).$$

Therefore we obtain

$$(\Gamma_\varphi \cdot \Delta) = \mathrm{Tr}_{X\times X}(\mathrm{cl}(\Gamma_\varphi) \cup \mathrm{cl}(\Delta))$$
$$= \mathrm{Tr}_{X\times X}\left(\sum_{i,j,r,s} (-1)^{s+s(2n-r)} p^*(\varphi^*(e_i^r) \cup f_\ell^s) \cup q^*(f_i^{2n-r} \cup e_\ell^{2n-s})\right)$$
$$= \sum_{i,r} \mathrm{Tr}_X(\varphi^*(e_i^r) \cup f_i^{2n-r}) \cdot \mathrm{Tr}_X(f_i^{2n-r} \cup e_i^r) = \sum_r (-1)^r \mathrm{trace}(\varphi^* | H^r(X)).$$

□

2. Rationality and the functional equation via Weil cohomology theories

In this section we assume that we have a Weil cohomology theory for varieties over $k = \overline{\mathbf{F}_p}$, and show how to get the statements of Conjectures 1.1 and 1.2 from Lecture 3 for varieties over $\mathbf{F}_{p^e}$. We start with the rationality of the zeta function. As we have seen in Lecture 2, given a variety X defined over $\mathbf{F}_q$, with $q = p^e$, we have the q-Frobenius morphism $\mathrm{Frob}_{X,q} \colon X \to X$ (a morphism over $\mathbf{F}_q$). Furthermore, if $\overline{X} = X \times_{\mathrm{Spec}\,\mathbf{F}_q} \mathrm{Spec}\, k$, then we have an endomorphism $F := \mathrm{Frob}_{\overline{X},q} = \mathrm{Frob}_{X,q} \times \mathrm{Id}$ of $\widetilde{X}$. Note that $\mathrm{Frob}_{\overline{X},q^m} = F^m$.

Theorem 2.1. *If X is a nonsingular, geometrically connected, n-dimensional projective variety over $\mathbf{F}_q$, then*

$$Z(X,t) = \frac{P_1(t)\cdot P_3(t)\cdots P_{2n-1}(t)}{P_0(t)\cdot P_2(t)\cdots P_{2n}(t)},$$

where for every i with $0 \leq i \leq 2n$ we have $P_i(t) = \det(\mathrm{Id} - tF^*|H^i(\overline{X}))$. In particular, $Z(X,t) \in \mathbf{Q}(t)$.

We will make use of the following general formula for the characteristic polynomial of a linear endomorphism.

Lemma 2.2. *For every endomorphism φ of a finite dimensional vector space V over a field K, we have*

$$\det(\mathrm{Id} - t\varphi) = \exp\left(-\sum_{m\geq 1} \mathrm{trace}(\varphi^m|V)\frac{t^m}{m}\right).$$

Proof. After replacing K by its algebraic closure $\overline{K}$, V by $\overline{V} = V \otimes_K \overline{K}$, and φ by $\varphi \otimes_K \overline{K} \colon \overline{V} \to \overline{V}$, we may assume that K is algebraically closed. After choosing a suitable basis for V, we may assume that φ is represented by an upper diagonal matrix. If the entries on the diagonal are $a_1, \ldots, a_d$, then $\det(\mathrm{Id} - t\varphi) = (1-a_1t)\cdots(1-a_dt)$. On the other hand,

$$\exp\left(-\sum_{m\geq 1} \mathrm{trace}(\varphi^m|V)\frac{t^m}{m}\right) = \exp\left(-\sum_{m\geq 1}\sum_{i=1}^{d}\frac{a_i^m t^m}{m}\right) = \exp\left(\sum_{i=1}^{d}\log(1-a_it)\right)$$

$$= \prod_{i=1}^{d}(1-a_it).$$

□

Proof of Theorem 2.1. Let $N_m = |X(\mathbf{F}_{q^m})|$. As we have seen in Lecture 2, we have $N_m = |\{x \in \overline{X} \mid F^m(x) = x\}|$. Furthermore, the graph $\Gamma_m \subset \overline{X} \times \overline{X}$ of F^m is transverse to the diagonal, hence by Theorem 1.7 we have $N_m = \sum_{i=0}^{2n}(-1)^i \mathrm{trace}((F^m)^*|H^i(\overline{X}))$. Using Lemma 2.2, we get

$$Z(X,t) = \exp\left(\sum_{m\geq 1}\sum_{i=0}^{2n}(-1)^i\mathrm{trace}((F^*)^m|H^i(\overline{X}))\frac{t^m}{m}\right) = \prod_{i=0}^{2n}\det(\mathrm{Id} - tF^*|H^i(\overline{X}))^{(-1)^{i+1}}.$$

This clearly shows that $Z(X,t)$ lies in $K(t)$. On the other hand, since $Z(X,t) \in \mathbf{Q}[\![t]\!]$, the proposition below shows that $Z(X,t)$ lies in $\mathbf{Q}(t)$. □

Proposition 2.3. *Let L be an arbitrary field, and $f = \sum_{m\geq 0} a_m t^m \in L[\![t]\!]$. We have $f \in L(t)$ if and only if there are nonnegative integers M and N such that the linear span of the vectors*

$$\{(a_i, a_{i+1}, \ldots, a_{i+N}) \in L^{\oplus(N+1)} \mid i \geq M\} \tag{2}$$

is a proper subspace of $L^{\oplus(N+1)}$. In particular, if L'/L is a field extension, then f lies in $L'(t)$ if and only if it lies in $L(t)$.

Proof. We have $f \in L(t)$ if and only if there are nonnegative integers M and N, and $c_0, \ldots, c_N \in L$, not all zero, such that $f(t) \cdot \sum_{i=0}^{N} c_i t^i$ is a polynomial of degree $< M + N$. In other words, we need

$$c_N a_i + c_{N-1} a_{i+1} + \ldots + c_0 a_{i+N} = 0 \text{ for all } i \geq M. \tag{3}$$

This condition holds precisely when the linear function $\ell(x_0, \ldots, x_N) = \sum_{j=0}^{N} c_{N-j} x_j$ vanishes on the linear span of the vectors in (2), hence the first assertion in the proposition. The second assertion follows from the fact that if $v_1, \ldots, v_r$ are elements of a vector space V over L, then $v_1, \ldots, v_r$ are linearly independent if and only if $v_1 \otimes 1, \ldots, v_r \otimes 1$ are linearly independent over L' in $V \otimes_L L'$. □

We now turn to the functional equation. We keep the same assumption and notation as in Theorem 2.1.

Theorem 2.4. *If X is a nonsingular, geometrically connected, n-dimensional projective algebraic variety over $\mathbf{F}_q$, and $E = (\Delta^2)$, where $\Delta \subset \overline{X} \times \overline{X}$ is the diagonal, we have*

$$Z(X, 1/q^n t) = \pm q^{nE/2} t^E Z(X, t).$$

The key ingredient is the following linear algebra lemma (see [Har, Lemma 4.3, App. C]).

Lemma 2.5. *Let $\varphi \colon V \times W \to K$ be a perfect pairing of vector spaces of dimension r over the field K. If $\lambda \in K \smallsetminus \{0\}$ and $f \in \mathrm{End}_K(V)$ and $g \in \mathrm{End}_K(W)$ are such that $\varphi(f(v), g(w)) = \lambda\varphi(v, w)$ for every $v \in V$, $w \in W$, then*

$$\det(\mathrm{Id} - tg|W) = \frac{(-1)^r \lambda^r t^r}{\det(f|V)} \det(\mathrm{Id} - \lambda^{-1} t^{-1} f|V) \tag{4}$$

and

$$\det(g|W) = \frac{\lambda^r}{\det(f|V)}. \tag{5}$$

Proof. After replacing K by its algebraic closure $\overline{K}$, and extending the scalars to $\overline{K}$, we may assume that K is algebraically closed. In this case we can find a basis $e_1, \ldots, e_r$ of V such that if we write $f(e_i) = \sum_{j=1}^{r} a_{i,j} e_j$, we have $a_{i,j} = 0$ for $i > j$. Let $e'_1, \ldots, e'_r$ be the basis of W such that $\varphi(e_i, e'_j) = \delta_{i,j}$ for every i and j.

Note that g is invertible: if $g(w) = 0$, then $0 = \varphi(f(v), g(w)) = \lambda\varphi(v, w)$ for every $v \in V$, hence $w = 0$. Since $\varphi(f(e_i), e'_j) = 0$ for $j < i$, we deduce that $\varphi(e_i, g^{-1}(e'_j)) = 0$. If we write $g^{-1}(e'_j) = \sum_{\ell=1}^{r} b_{j,\ell} e'_\ell$, we have $b_{j,i} = 0$ for $i > j$. Furthermore,

$$a_{j,j} = \varphi(f(e_j), e'_j) = \lambda\varphi(e_j, g^{-1}(e'_j)) = \lambda b_{j,j}.$$

Since $\det(f|V) = \prod_{i=1}^{r} a_{i,i}$ and $\det(g|W) = \prod_{j=1}^{r} b_{j,j}^{-1} = \lambda^r / \prod_{i=1}^{r} a_{i,i}$, we get (5). We also have

$$\det(\mathrm{Id} - tg|W) = \det(g|W) \cdot \det(g^{-1} - t\mathrm{Id}|W) = \frac{\lambda^r}{\det(f|V)} \cdot \prod_{j=1}^{r} (a_{j,j}\lambda^{-1} - t)$$

$$= \frac{(-1)^r\lambda^r t^r}{\det(f|V)} \cdot \prod_{j=1}^{r}(1-a_{j,j}\lambda^{-1}t^{-1}) = \frac{(-1)^r\lambda^r t^r}{\det(f|V)}\det(\mathrm{Id}-\lambda^{-1}t^{-1}f|V).$$

□

Proof of Theorem 2.4. We apply the lemma to the perfect pairing given by Poincaré duality:

$$\varphi_i\colon H^i(\overline{X})\otimes H^{2n-i}(\overline{X}) \to H^{2n}(\overline{X}) \to K,\ \varphi_i(\alpha\otimes\beta)=\mathrm{Tr}(\alpha\cup\beta).$$

Note that $F\colon \overline{X}\to\overline{X}$ is a finite morphism of degree q^n: indeed, it is enough to show that $\mathrm{Frob}_{X,q}\colon X\to X$ has this property. Arguing as in the proof of Proposition 2.4 in Lecture 2, we reduce the assertion to the case $X=\mathbf{A}^n$, when it follows from the fact that $k[x_1,\ldots,x_n]$ is free of rank q^n over $k[x_1^q,\ldots,x_n^q]$.

Proposition 1.1 implies that F^* is given by multiplication by q^n on $H^{2n}(\overline{X})$. Therefore

$$\varphi_i(F^*(\alpha),F^*(\beta)) = \mathrm{Tr}_{\overline{X}}(F^*(\alpha\cup\beta)) = \mathrm{Tr}_{\overline{X}}(q^d\alpha\cup\beta) = q^d\varphi_i(\alpha,\beta),$$

for every $\alpha\in H^i(\overline{X})$ and $\beta\in H^{2n-i}(\overline{X})$. Lemma 2.5 implies that if we put $B_i = \dim_K H^i(\overline{X})$ and $P_i(t)=\det(\mathrm{Id}-tF^*|H^i(\overline{X}))$, then

$$\det(F^*|H^{2n-i}(\overline{X})) = q^{nB_i}/\det(F^*|H^i(\overline{X}))\ \text{and} \tag{6}$$

$$P_{2n-i}(t) = \frac{(-1)^{B_i}q^{nB_i}t^{B_i}}{\det(F^*|H^i(\overline{X}))}P_i(1/q^nt). \tag{7}$$

Using (6), (7) and Theorem 2.1, as well as the fact that $E=\sum_{i=0}^{2n}(-1)^iB_i$ by Proposition 1.8, we deduce

$$Z(1/q^nt) = \prod_{i=0}^{2n}P_i(1/q^nt)^{(-1)^{i+1}} = \prod_{i=0}^{2n}P_{2n-i}(t)^{(-1)^{i+1}}\cdot\frac{(-1)^Eq^{nE}t^E}{\prod_{i=0}^{2n}\det(F^*|H^i(\overline{X}))^{(-1)^i}}$$

$$= \pm Z(X,t)\cdot\frac{q^{nE}t^E}{q^{nE/2}} = \pm q^{nE}t^EZ(X,t).$$

□

Remark 2.6. It follows from the above proof that the sign in the functional equation is $(-1)^{E+a}$, where $a=0$ if $\det(F^*\mid H^i(\overline{X}))=q^{nB_n/2}$, and $a=1$ if $\det(F^*\mid H^i(\overline{X}))=-q^{nB_n/2}$. If we write $P_n(t)=\prod_{i=1}^{B_n}(1-\alpha_it)$, an easy computation using the identity (7) for $i=n$ implies that the multiset $\{\alpha_1,\ldots,\alpha_{B_n}\}$ is invariant under $\alpha\to q^{B_n}/\alpha$, and $\prod_{i=1}^{B_n}\alpha_i=(-1)^aq^{nB_n/2}$. Therefore a has the same parity as the number of α_i equal to $-q^{n/2}$.

3. A brief introduction to ℓ-adic cohomology

In this section we give a brief overview of étale cohomology, in general, and of ℓ-adic cohomology, in particular. Needless to say, we will only describe the basic notions and results. For details and for proofs, the reader is referred to [Del1] or [Mil].

The basic idea behind étale topology is to replace the Zariski topology on an algebraic variety by a different topology. In fact, this is not a topology in the usual sense, but a *Grothendieck topology*. Sheaf theory, and in particular sheaf cohomology still make sense in this setting, and this allows the definition of ℓ-adic cohomology.

As a motivation, note that in the case of a smooth, projective, complex algebraic variety we would like to recover the singular cohomology, with suitable coefficients There are two ways of doing this algebraically. The first one consists in taking the hypercohomology of the de Rham complex. This approach, however, is known to produce pathologies in positive characteristic. The second approach consists in "refining" the Zariski topology, which as it stands, does not reflect the classical topology. The key is the notion of étale morphism. It is worth recalling that a morphism of complex algebraic varieties is étale if and only if it is a local analytic isomorphism *in the classical topology*.

Let X be a fixed Noetherian scheme. The role of the open subsets of X will be played by the category $\acute{\mathrm{E}}\mathrm{t}(X)$ of étale schemes $Y \to X$ over X. Instead of considering inclusions between open subsets, we consider morphisms in $\acute{\mathrm{E}}\mathrm{t}(X)$ (note that if Y_1 and Y_2 are étale schemes over X, any morphism $Y_1 \to Y_2$ of schemes over X is étale). The category $\acute{\mathrm{E}}\mathrm{t}(X)$ has fiber products. The role of open covers is played by *étale covers*: these are families $(U_i \xrightarrow{f_i} U)_i$ of étale schemes over X such that $U = \bigcup_i f_i(U_i)$. The set of étale covers of U is denoted by $\mathrm{Cov}(U)$.

What makes this data into a Grothendieck topology is the fact that it satisfies the following conditions:

- (C1) If $\varphi\colon U \to V$ is an isomorphism in $\acute{\mathrm{E}}\mathrm{t}(X)$, then $(\varphi) \in \mathrm{Cov}(V)$.
- (C2) If $(U_i \to U)_i \in \mathrm{Cov}(U)$ and for every i we have $(U_{i,j} \to U_i)_j \in \mathrm{Cov}(U_i)$, then $(U_{i,j} \to U)_{i,j} \in \mathrm{Cov}(U)$.
- (C3) If $(U_i \to U)_i \in \mathrm{Cov}(U)$, and $V \to U$ is a morphism in $\acute{\mathrm{E}}\mathrm{t}(X)$, then we have $(U_i \times_U V \to V)_i \in \mathrm{Cov}(V)$.

This Grothendieck topology is the *étale topology* on X.

It follows from definition that if $U \in \acute{\mathrm{E}}\mathrm{t}(X)$, and if $(U_i)_i$ is an open cover of U, then $(U_i \to U)_i$ is in $\mathrm{Cov}(U)$. Another important type of cover is the following. A finite étale morphism $V \to U$ is a *Galois cover* with group G if G acts (on the right) on V over U, and if the natural morphism

$$\bigsqcup_{g \in G} V_g \to V \times_U V,\ y \in V_g \to (y, yg)$$

is an isomorphism, where $V_g = V$ for every $g \in G$. Note that this is a G-equivariant isomorphism if we let G act on the left-hand side so that $h \in G$ takes V_g to V_{gh} via the identity map. It is a general fact that every finite étale morphism $V \to U$ can be dominated by a Galois cover $W \to U$.

Once we have a Grothendieck topology on X, we can extend the notions of presheaves and sheaves. An étale presheaf on X (say, of abelian groups) is a contravariant functor from $\acute{\mathrm{E}}\mathrm{t}(X)$ to the category of abelian groups. An étale presheaf $\mathcal{F}$ is a sheaf if for every $U \in \acute{\mathrm{E}}\mathrm{t}(X)$ and every étale cover $(U_i \to U)_i$, the following complex

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact. In particular, $\mathcal{F}$ defines a sheaf $\mathcal{F}_U$ on U, in the usual sense, for every U in $\acute{\mathrm{E}}\mathrm{t}(X)$. On the other hand, if $V \to U$ is a Galois cover in $\acute{\mathrm{E}}\mathrm{t}(X)$ with group G, then the corresponding condition on $\mathcal{F}$ is that $\mathcal{F}(U) \simeq \mathcal{F}(V)^G$ (note that G has a natural action on $\mathcal{F}$ since $\mathcal{F}$ is a presheaf). Let us consider some examples of étale sheaves.

Example 3.1. If $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X (in the usual sense), then we put for $U \xrightarrow{f} X$ in $\acute{\mathrm{E}}\mathrm{t}(X)$

$$W(\mathcal{M})(U) = \Gamma(U, f^*(\mathcal{M})).$$

It is a consequence of faithfully flat descent that $W(\mathcal{M})$ is an étale sheaf on X. Abusing notation, we usually denote $W(\mathcal{M})$ simply by $\mathcal{M}$.

Example 3.2. If A is any abelian group, then we get an étale *constant sheaf* on X that takes every $U \to X$ in $\acute{\mathrm{E}}\mathrm{t}(X)$ to $A^{\pi_0(U)}$, where $\pi_0(U)$ is the set of connected components of U. This is denoted by A_X, but whenever the scheme X is understood, we drop the subscript.

Example 3.3. Suppose that $\mathbf{G}$ is an abelian group scheme over X. We may consider $\mathbf{G}$ as an étale presheaf on X by defining for $U \to X$ in $\acute{\mathrm{E}}\mathrm{t}(X)$, $\mathbf{G}(U) = \mathrm{Hom}_X(U, \mathbf{G})$. It is another consequence of faithfully flat descent that $\mathbf{G}$ is an étale sheaf on X. For example, if $\mathbf{G} = \mathbf{G}_m = X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathbf{Z}[t, t^{-1}]$, then $\mathbf{G}_m(U)$ is the set $\mathcal{O}(U)^*$ of invertible elements in $\mathcal{O}(U)$. Another example is given by the closed subscheme

$$\mu_n = X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathbf{Z}[t]/(t^n - 1) \hookrightarrow \mathbf{G}_m.$$

In this case we have $\mu_n(U) = \{u \in \mathcal{O}_X(U) \mid u^n = 1\}$.

Example 3.4. As a last example, consider the case when $X = \mathrm{Spec}\,k$, where k is a field. Note that in this case an object in $\acute{\mathrm{E}}\mathrm{t}(X)$ is just a disjoint union of finitely many Spec K_i, where the K_i are finite, separable extensions of k. It is clear that every étale sheaf $\mathcal{F}$ over X is determined by its values $M_K := \mathcal{F}(\mathrm{Spec}\,K)$, for K/k as above. Furthermore, $G(K/k)$ has an induced action on M_K, and for every Galois extension L/K of finite, separable extensions of k, we have a functorial isomorphism $M_K \simeq (M_L)^{G(L/K)}$. Let $M := \varinjlim_{K/k} M_K$. This carries a continuous action of $G = G(k^{\mathrm{sep}}/k)$, where k^{sep} is a separable closure of k (the action being continuous means that the stabilizer of every element in M is an open subgroup of G). One can show that this defines an equivalence of categories between the category of étale sheaves on $\mathrm{Spec}\,k$ and the category of abelian groups with a continuous G-action.

Suppose now that X is an arbitrary Noetherian scheme. It is easy to see that the category $\mathbf{Psh}_{\text{ét}}(X)$ of étale presheaves on X is an abelian category. If $\mathbf{Sh}_{\text{ét}}(X)$ is the category of étale sheaves on X, then one can show that the natural inclusion $\mathbf{Psh}_{\text{ét}}(X) \hookrightarrow \mathbf{Sh}_{\text{ét}}(X)$ has a left adjoint, that takes an étale presheaf $\mathcal{F}$ to the associated étale sheaf. Using this, one can show that also $\mathbf{Sh}_{\text{ét}}(X)$ is an abelian category. We note that a complex of étale sheaves on X

$$\mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}''$$

is exact if and only if for every $U \to X$ in $\text{Ét}(X)$, every $a \in \mathcal{F}(U)$ such that $v(a) = 0$, and every $x \in U$, there is $f \colon V \to U$ in $\text{Ét}(X)$ with $x \in f(V)$, such that the image of a in $\mathcal{F}(V)$ lies in $\text{Im}(\mathcal{F}'(V) \to \mathcal{F}(V))$.

Example 3.5. Suppose that X is a scheme over $\mathbf{F}_p$, and let us assume, for simplicity, that X is integral. There is an important exact sequence of étale sheaves on X, the *Artin-Schreier sequence*, given by

$$0 \longrightarrow (\mathbf{F}_p)_X \longrightarrow \mathcal{O}_X \xrightarrow{\text{Frob}_p - \text{Id}} \mathcal{O}_X \longrightarrow 0, \tag{8}$$

where for every $U \to X$ in $\text{Ét}(X)$, we recall that $\text{Frob}_p \colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is given by $u \to u^p$. It is clear that $(\mathbf{F}_p)_X$ is the kernel of $\text{Frob}_p - \text{Id}$: this follows from the fact that for every domain A over $\mathbf{F}_p$, we have $\mathbf{F}_p = \{a \in A \mid a^p = a\}$. Note that $\text{Frob}_p - \text{Id}$ is surjective on $\mathcal{O}_X$ (for the étale topology). Indeed, given any Noetherian ring A and $a \in A$, the morphism $\varphi \colon A \to B = A[t]/(t^p - t - a)$ is étale and surjective, and there is $b = t \in B$ such that $\varphi(a) = b^p - b$.

Example 3.6. Suppose now that X is a scheme over a field k, and n is a positive integer, not divisible by $\text{char}(k)$. In this case we have an exact sequence of étale sheaves, the *Kummer sequence*

$$0 \to \mu_\ell \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 0.$$

In order to see that the morphism $\mathbf{G}_m \to \mathbf{G}_m$ that takes $u \to u^n$ is surjective, it is enough to note that for every k-algebra A, and every $a \in A$, the natural morphism $\varphi \colon A \to B = A[t]/(t^n - a)$ is étale and surjective, and there is $b = t \in B$, such that $\varphi(a) = t^n$.

Note that if k is separably closed, then it is clear that for every k-algebra that is an integral domain, we have $\{u \in A \mid u^n = 1\} \subseteq k$. Suppose, for simplicity, that X is an integral scheme. In this case, the choice of a primitive n^{th} root of 1 gives an isomorphism $\mu_n \simeq (\mathbf{Z}/n\mathbf{Z})_X$.

If $f \colon X \to Y$ is a morphism of Noetherian schemes, for every $U \to Y$ in $\text{Ét}(Y)$, we have $X \times_Y U \to X$ in $\text{Ét}(X)$. Furthermore, if $(U_i \to U)_i$ is an étale cover of U, then $(X \times_Y U_i \to X \times_Y U)_i$ is an étale cover of $X \times_Y U$. Using this, it is easy to see that we have a functor $f_* \colon \mathbf{Sh}_{\text{ét}}(X) \to \mathbf{Sh}_{\text{ét}}(Y)$, such that $f_*(\mathcal{F})(U) = \mathcal{F}(X \times_Y U)$. This is a left exact functor, and one can show that it has a left adjoint, denoted by f^*. For example, we have $f^*(A_Y) \simeq A_X$.

One can show that the category $\mathbf{S}h_{\text{ét}}(X)$ has enough injectives. In particular, for every $U \to X$ in $\text{Ét}(X)$ we can consider the right derived functors of the left exact functor $\mathcal{F} \to \mathcal{F}(U)$. These are written as $H^i_{\text{ét}}(U, \mathcal{F})$, for $i \geq 0$.

Example 3.7. If $\mathcal{M}$ is a quasi-coherent sheaf on X, and $W(\mathcal{M})$ is the corresponding étale sheaf associated to $\mathcal{M}$ as in Example 3.1, then one can show that there are canonical isomorphisms $H^i(X, \mathcal{M}) \simeq H^i_{\text{ét}}(X, W(\mathcal{M}))$.

Example 3.8. Let $X = \operatorname{Spec} k$, where k is a field. If we identify an étale sheaf on X with an abelian group M with a continuous G-action, where $G = G(k^{\text{sep}}/k)$, then the functor of taking global sections for the sheaf gets identified to the functor $M \to M^G$. Therefore its derived functors are given precisely by the Galois cohomology functors.

Example 3.9. One can show that as in the case of the Zariski topology, there is an isomorphism $H^1_{\text{ét}}(X, \mathbf{G}_m) \simeq \operatorname{Pic}(X)$. Suppose now that X is an integral scheme over a separably closed field k, and n is a positive integer that is not divisible by $\operatorname{char}(k)$. It follows from Example 3.6 that we have an exact sequence

$$\Gamma(X, \mathcal{O}_X)^* \xrightarrow{\alpha} \Gamma(X, \mathcal{O}_X)^* \to H^1_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z}) \to \operatorname{Pic}(X) \xrightarrow{\beta} \operatorname{Pic}(X),$$

where both α and β are given by taking the n^{th}-power.

Example 3.10. Suppose that X is an integral scheme over $\mathbf{F}_p$. The Artin-Schreier exact sequence from Example 3.5, together with the assertion in Example 3.7 implies that we have a long exact sequence of cohomology

$$(9) \qquad \cdots \longrightarrow H^i_{\text{ét}}(X, \mathbf{F}_p) \longrightarrow H^i(X, \mathcal{O}_X) \xrightarrow{\text{Id}-\text{Frob}_p} H^i(X, \mathcal{O}_X) \longrightarrow H^{i+1}_{\text{ét}}(X, \mathbf{F}_p) \longrightarrow \cdots.$$

Suppose now that X is complete over a field k, so each $H^i(X, \mathcal{O}_X)$ is finite-dimensional over k. Since $\text{Frob}_p \colon H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$ has the property that $\text{Frob}_p(au) = a^p \text{Frob}_p(u)$, it is a general fact that if k is algebraically closed, the morphism $\text{Id} - \text{Frob}_p$ is always surjective. It follows that under this assumption, the above long exact sequence breaks into short exact sequences

$$(10) \qquad 0 \longrightarrow H^i_{\text{ét}}(X, \mathbf{F}_p) \longrightarrow H^i(X, \mathcal{O}_X) \xrightarrow{\text{Id}-\text{Frob}_p} H^i(X, \mathcal{O}_X) \longrightarrow 0.$$

It turns out that it is particularly interesting to compute the étale cohomology of schemes with coefficients in finite abelian groups. The basic computation is that of the étale cohomology groups of a curve. In fact, the proofs of the fundamental results about étale cohomology are reduced to the case of curves via involved *dévissage* arguments.

Theorem 3.11. *Let X be a smooth, connected, projective curve, over an algebraically closed field k. If n is a positive integer that is not divisible by* $\operatorname{char}(k)$, *then there are canonical isomorphisms*

$$H^0_{\acute{e}t}(X, \mu_n) \simeq \mu_n(\operatorname{Spec} k),$$
$$H^1_{\acute{e}t}(X, \mu_n) \simeq \{L \in \operatorname{Pic}(X) \mid L^n \simeq \mathcal{O}_X\},$$
$$H^2_{\acute{e}t}(X, \mu_n) \simeq \mathbf{Z}/n\mathbf{Z},$$

while $H^i_{\acute{e}t}(X, \mu_n) = 0$ for $i > 2$.

The key point is to show that $H^i_{\text{ét}}(X, \mathbf{G}_m) = 0$ for $i \geq 2$. This is deduced from a theorem of Tsen, saying that every nonconstant homogeneous polynomial $f \in k[x_1, \ldots, x_n]$ of degree $< n$ has a nontrivial zero. The assertions in the above theorem then follow from the long exact sequence in cohomology corresponding to the Kummer exact sequence. Note that $\mu_n(\operatorname{Spec} k)$ is non-canonically isomorphic to $\mathbf{Z}/n\mathbf{Z}$. Furthermore, every $L \in \operatorname{Pic}(X)$ such that $L^n \simeq \mathcal{O}_X$ lies in $\operatorname{Pic}^0(X)$. Since $\operatorname{Pic}^0(X)$ consists of the k-rational points of a g-dimensional abelian variety over k (where g is the genus of X), it follows that

$$\{L \in \operatorname{Pic}(X) \mid L^n \simeq \mathcal{O}_X\} \simeq (\mathbf{Z}/n\mathbf{Z})^{2g}$$

(see [Mum, p. 60]). Furthermore, multiplication by n is surjective on $\operatorname{Pic}^0(X)$ (see [Mum, p. 40]), which gives the isomorphism $H^2_{\text{ét}}(X, \mu_n) \simeq \mathbf{Z}/n\mathbf{Z}$ in the theorem.

Remark 3.12. When the characteristic of k divides n, the ranks of the cohomology groups $H^i_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z})$ do not behave as expected. Suppose, for example, that X is an elliptic curve defined over an algebraically closed field of characteristic $p > 0$. It follows from the exact sequence (10) that $\dim_{\mathbf{F}_p} H^1_{\text{ét}}(X, \mathbf{Z}/p\mathbf{Z}) \leq 1$ (compare with the fact that for a prime $\ell \neq p$, we have $\dim_{\mathbf{F}_\ell} H^1_{\text{ét}}(X, \mathbf{Z}/p\mathbf{Z}) = 2$). On the other hand, this étale cohomology group detects interesting information about the elliptic curve: it follows from the exact sequence (10) that $\dim_{\mathbf{F}_p} H^1_{\text{ét}}(X, \mathbf{Z}/p\mathbf{Z}) = 1$ if and only if the Frobenius action on $H^1(X, \mathcal{O}_X)$ is nonzero (in this case one says that X is *ordinary*; otherwise, it is *supersingular*).

The ℓ-adic cohomology groups are defined as follows. Let k be an algebraically closed field, and let ℓ be a prime different from $\operatorname{char}(k)$ (in case this is positive). For every $i \geq 0$, and every $m \geq 1$, consider the $\mathbf{Z}/\ell^m\mathbf{Z}$-module $H^i_{\text{ét}}(X, \mathbf{Z}/\ell^m\mathbf{Z})$. We have obvious maps

$$H^i_{\text{ét}}(X, \mathbf{Z}/\ell^{m+1}\mathbf{Z}) \to H^i_{\text{ét}}(X, \mathbf{Z}/\ell^m\mathbf{Z})$$

and one puts

$$H^i_{\text{ét}}(X, \mathbf{Z}_\ell) := \varprojlim_m H^i_{\text{ét}}(X, \mathbf{Z}/\ell^m\mathbf{Z}).$$

This has a natural structure of $\mathbf{Z}_\ell$-module, where $\mathbf{Z}_\ell$ is the ring of ℓ-adic integers, and one defines

$$H^i_{\text{ét}}(X, \mathbf{Q}_\ell) := H^i_{\text{ét}}(X, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

It is worth pointing out that taking cohomology does *not* commute with projective limits, hence $H^i_{\text{ét}}(X, \mathbf{Z}_\ell)$ is *not* the étale cohomology group of X with coefficients in the constant sheaf $\mathbf{Z}_\ell$. It follows from the fundamental theorems on étale cohomology that when restricting to smooth, connected, projective varieties over k, one gets in this way a Weil cohomology theory with coefficients in $\mathbf{Q}_\ell$ (see [Mil, Chapter VI]).

In particular, if X is a smooth, geometrically connected, n-dimensional projective variety over $\mathbf{F}_q$, with $q = p^e$, let ℓ be a prime different from p. Theorem 2.1 gives the following expression for the zeta function of X:

$$Z(X, t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)}, \tag{11}$$

where $P_i(t) = \det\left(\mathrm{Id} - tF^* | H^i_{\text{ét}}(X \times_k \overline{k}, \mathbf{Q}_\ell)\right)$.

Furthermore, general results about étale cohomology give a proof for the Weil conjecture relating the zeta function $Z(X, t)$ with the Betti numbers for singular cohomology (see Conjecture 4 in Lecture 3). Note first that we have already seen the first assertion in this conjecture: with the above notation, Proposition 1.8 gives $\sum_{i\geq 0}(-1)^i\deg(P_i)=(\Delta^2)$, where $\Delta\subseteq X\times X$ is the diagonal. Suppose now that $\widetilde{X}$ is a smooth projective scheme over a finitely generated $\mathbf{Z}$-subalgebra R of $\mathbf{C}$, and $P\in\mathrm{Spec}(R)$ is such that $R/P=\mathbf{F}_q$, and $\widetilde{X}\times_{\mathrm{Spec}\,R}\mathrm{Spec}\,\mathbf{F}_q=X$. It is a consequence of the smooth base change theorem (see [Mil, Corollary VI.4.2]) that there are isomorphisms

$$H^i_{\text{ét}}(X,\mathbf{Z}/\ell^m\mathbf{Z})\simeq H^i_{\text{ét}}(\widetilde{X}\times_{\mathrm{Spec}\,R}\mathrm{Spec}\,\mathbf{C},\mathbf{Z}/\ell^m\mathbf{Z}).$$

Furthermore, a comparison theorem between singular and étale cohomology (see [Mil, Theorem III.3.12]) implies that the étale and singular cohomology groups of smooth complex varieties, with coefficients in finite abelian groups are isomorphic. In particular,

$$H^i_{\text{ét}}(\widetilde{X}\times_{\mathrm{Spec}\,R}\mathrm{Spec}\,\mathbf{C},\mathbf{Z}/\ell^m\mathbf{Z})\simeq H^i(\widetilde{X}(\mathbf{C})^{\mathrm{an}},\mathbf{Z}/\ell^m\mathbf{Z}).$$

After taking the projective limit over $m\geq 1$, and tensoring with $\mathbf{Q}_\ell$, we get

$$H^i_{\text{ét}}(X,\mathbf{Q}_\ell)\simeq H^i(\widetilde{X}(\mathbf{C})^{\mathrm{an}},\mathbf{Q}_\ell).$$

Therefore we have $\deg(P_i)=\dim_{\mathbf{Q}}H^i(\widetilde{X}(\mathbf{C})^{\mathrm{an}},\mathbf{Q})$, proving the fourth of the Weil conjectures.

The fundamental result of Deligne [Del1], settling the hardest of the Weil conjectures, the analogue of the Riemann Hypothesis, is the following.

Theorem 3.13. *If X is a smooth, geometrically connected, projective variety over $\mathbf{F}_q$, and $F=\mathrm{Frob}_{\overline{X},q}$ is the induced Frobenius morphism on $\overline{X}=X\times_k\overline{k}$, then*

$$\det\left(\mathrm{Id}-tF^*|H^i_{\acute{e}t}(\overline{X},\mathbf{Q}_\ell)\right)=\prod_i(1-\alpha_i t)\in\mathbf{Z}[t],$$

and for every choice of an isomorphism $\mathbf{Q}_\ell\simeq\mathbf{C}$, we have $|\alpha_i|=q^{i/2}$ for all i.

We mention that one can give a proof for the rationality of the zeta function for arbitrary varieties in the setting of ℓ-adic cohomology. The point is that if k is an algebraically closed field of characteristic p, and ℓ is a prime different from p, then for every separated variety over k one can define ℓ-adic cohomology groups with compact supports $H^i_c(Y,\mathbf{Q}_\ell)$. These are always finite-dimensional $\mathbf{Q}_\ell$-vector spaces, and they are zero unless $0\leq i\leq 2\dim(Y)$.

If X is a separated variety over a finite field $\mathbf{F}_q$, and if $F=\mathrm{Frob}_{\overline{X},q}$ is the Frobenius endomorphism of $\overline{X}=X\times_k\overline{k}$, then one has the following formula for the zeta function of X (see [Mil, Theorem VI.13.1]):

$$Z(X,t)=\prod_{i\geq 0}\det(F^*\mid H^i_c(\overline{X},\mathbf{Q}_\ell)^{(-1)^{i+1}}.$$

In particular, it follows that $Z(X,t)$ is a rational function, and this implies the rationality of the zeta function for every variety over $\mathbf{F}_q$.

References

[deJ1] A. J. de Jong, Weil cohomology theories, note available at *http://www.math.columbia.edu/˜ dejong/seminar/note_on_weil_cohomology.pdf*. 1

[deJ2] A. J. de Jong, Smoothness, semi-stability and alterations, Inst. Hautes Études Sci. Publ. Math. **83** (1996), 51–93. 5

[Del1] P. Deligne, Cohomologie étale: les points de départ, in *Cohomologie Étale* (SGA 4 1/2), Lecture Notes in Mathematics 569, 4–75, Springer-Verlag, Berlin-Heidelberg-New York, 1977. 11, 16

[Del2] P. Deligne, La conjecture de Weil, I, Publ. Math. IHES **43** (1974), 273–307.

[Ful] W. Fulton, *Intersection theory*, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 2, Springer-Verlag, Berlin, 1998. 4

[Har] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. 9

[Mil] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, 33, Princeton University Press, Princeton, N.J., 1980. 6, 11, 15, 16

[Mum] D. Mumford, *Abelian varieties*, with appendices by C. P. Ramanujam and Y. I. Manin, corrected reprint of the second (1974) edition, Tata Institute of Fundamental Research Studies in Mathematics, 5, published for the Tata Institute of Fundamental Research, Bombay, by Hindustan Book Agency, New Delhi, 2008. 15