LECTURE 6. FULTON'S TRACE FORMULA FOR COHERENT SHEAF COHOMOLOGY

Our goal in this lecture is to give a proof, following [Ful2], of a trace formula for the Frobenius action on the cohomology of the structure sheaf.

1. The statement of the main theorem

Suppose that X is a scheme over the finite field $k = \mathbf{F}_q$. Recall that we have the q-Frobenius morphism $F = \mathrm{Frob}_{X,q} \colon X \to X$, whose corresponding morphism of sheaves $\mathcal{O}_X \to F_*(\mathcal{O}_X) = \mathcal{O}_X$ is given by $u \to u^q$. This is an $\mathbf{F}_q$-linear morphism, and therefore we get induced $\mathbf{F}_q$-linear actions $F \colon H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$.

Theorem 1.1. *If X is a projective scheme over a finite field $\mathbf{F}_q$, then*

$$|X(\mathbf{F}_q)| \bmod p = \sum_{i=0}^{\dim(X)} (-1)^i \mathrm{trace}(F|H^i(X, \mathcal{O}_X)). \tag{1}$$

Remark 1.2. Note that we have $|X(\mathbf{F}_q)| = |X_{\mathrm{red}}(\mathbf{F}_q)|$. However, it is not a priori clear that the term on the right-hand side of (1) only depends on the reduced scheme structure of X.

Remark 1.3. Given X as in the above theorem, let $X_m = X \times_{\mathrm{Spec}\,\mathbf{F}_q} \mathrm{Spec}\,\mathbf{F}_{q^m}$. Note that $\mathrm{Frob}_{X_m,q^m} = \mathrm{Frob}_{X,q}^m \times \mathrm{Id}$, and we have a canonical isomorphism $H^i(X_m, \mathcal{O}_{X_m}) \simeq H^i(X, \mathcal{O}_X) \otimes_{\mathbf{F}_q} \mathbf{F}_{q^m}$. By applying the theorem for X_m, we get

$$|X(\mathbf{F}_{q^m})| \bmod p = \sum_{i=0}^{\dim(X)} (-1)^i \mathrm{trace}(F^m|H^i(X, \mathcal{O}_X)).$$

Recall from Lecture 2 that we may identify $X(\mathbf{F}_q)$ with the closed points $x \in X$ with $k(x) = \mathbf{F}_q$. In what follows we will often make this identification without any further comment.

A stronger congruence formula was proved be Deligne [Del2] and Katz [Katz]. In fact, we will also prove a strengthening of the above statement, but in a different direction. The first extension is to sheaves with a Frobenius action.

A coherent F-module on X is a coherent sheaf $\mathcal{M}$ on X, together with a *Frobenius action* on $\mathcal{M}$, that is, a morphism of sheaves of $\mathcal{O}_X$-modules $F_{\mathcal{M}} \colon \mathcal{M} \to F_*(\mathcal{M})$. In other words, $F_{\mathcal{M}}$ is a morphism of sheaves of $\mathbf{F}_q$-vector spaces $\mathcal{O}_X \to \mathcal{O}_X$ such that $F_{\mathcal{M}}(am) = a^q F_{\mathcal{M}}(m)$ for every $a \in \mathcal{O}_X(U)$ and $m \in \mathcal{M}(U)$, where U is any open subset of X. As above, since $F_{\mathcal{M}}$ is $\mathbf{F}_q$-linear, it follows that it induces $\mathbf{F}_q$-linear maps on cohomology that, abusing notation, we write $F_{\mathcal{M}} \colon H^i(X, \mathcal{M}) \to H^i(X, \mathcal{M})$. Despite the fact that $F_{\mathcal{M}}$ is not

$\mathcal{O}_X$-linear, for every $x \in X(\mathbf{F}_q)$ we get an $\mathbf{F}_q$-linear endomorphism of $\mathcal{M}(x) := \mathcal{M}_x \otimes k(x)$, that we denote by $F_{\mathcal{M}}(x)$.

Theorem 1.4. *If X is a projective scheme over $\mathbf{F}_q$, and $(\mathcal{M}, F_{\mathcal{M}})$ is a coherent F-module on X, we have*

$$\sum_{x \in X(\mathbf{F}_q)} \mathrm{trace}(F_{\mathcal{M}}(x)) = \sum_{i=0}^{\dim(X)} (-1)^i \mathrm{trace}(F_{\mathcal{M}} | H^i(X, \mathcal{M})). \tag{2}$$

An obvious example of a coherent F-module on X is given by $(\mathcal{O}_X, F)$. Note that if $x \in X(\mathbf{F}_q)$, then $F(x)$ is the identity on $\mathcal{O}_X(x) = \mathbf{F}_q$. Therefore the result in Theorem 1.1 is a special case of the one in Theorem 1.4.

In fact, Theorem 1.4 will follow from a result describing the Grothendieck group of coherent F-modues. Given a scheme X of finite type over $\mathbf{F}_q$, consider the category $\mathcal{C}oh_F(X)$ consisting of coherent F-modules. A morphism $(\mathcal{M}, F_{\mathcal{M}}) \to (\mathcal{M}', F_{\mathcal{M}'})$ in this category is a morphism $f\colon \mathcal{M} \to \mathcal{M}'$ of coherent sheaves, such that $f \circ F_{\mathcal{M}} = F_{\mathcal{M}'} \circ f$. It is easy to see that if f is a morphism of coherent F-modules, then $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ have induced Frobenius actions that makes them coherent F-modules. We thus see that $\mathcal{C}oh_F(X)$ is an abelian category. Whenever the Frobenius action is understood, we simply write $\mathcal{M}$ instead of $(\mathcal{M}, F_{\mathcal{M}})$.

The Grothendieck group $K_\bullet^F(X)$ of coherent F-modules is the quotient of the free abelian group on isomorphism classes of coherent F-modules $(\mathcal{M}, F_{\mathcal{M}})$ as above, by the following type of relations:

(A) $(\mathcal{M}, F_{\mathcal{M}}) = (\mathcal{M}', F_{\mathcal{M}}) + (\mathcal{M}'', F_{\mathcal{M}''})$, for every exact sequence
$$0 \to (\mathcal{M}', F_{\mathcal{M}'}) \to (\mathcal{M}, F_{\mathcal{M}'}) \to (\mathcal{M}'', F_{\mathcal{M}''}) \to 0.$$

(B) $(\mathcal{M}, F_1 + F_2) = (\mathcal{M}, F_1) + (\mathcal{M}, F_2)$ for every morphisms of $\mathcal{O}_X$-modules $F_1, F_2\colon \mathcal{M} \to F_*(\mathcal{M})$, where $\mathcal{M}$ is a coherent sheaf on X.

Given a coherent F-module $(\mathcal{M}, F_{\mathcal{M}})$, we denote by $[\mathcal{M}, F_{\mathcal{M}}]$ its class in the Grothendieck group. Note that $K_\bullet^F(X)$ is, in fact, an $\mathbf{F}_q$-vector space, with $\lambda \cdot [\mathcal{M}, F_{\mathcal{M}}] = [\mathcal{M}, \lambda F_{\mathcal{M}}]$.

Lemma 1.5. *We have an isomorphism $K_\bullet^F(\mathrm{Spec}\, \mathbf{F}_q) \simeq \mathbf{F}_q$ of $\mathbf{F}_q$-vector spaces, given by*

$$[\mathcal{M}, F_{\mathcal{M}}] \to \mathrm{trace}(F_{\mathcal{M}}(x)),$$

where x is the unique point of $\mathrm{Spec}\, \mathbf{F}_q$.

Proof. Note that $\mathcal{C}oh_F(\mathrm{Spec}\, \mathbf{F}_q)$ is the category of pairs (V, φ), where V is a finite-dimensional vector space over $\mathbf{F}_q$, and φ is a linear endomorphism. Since $\mathrm{trace}(\varphi_1 + \varphi_2) = \mathrm{trace}(\varphi_1) + \mathrm{trace}(\varphi_2)$, and given an exact sequence $0 \to (V', \varphi') \to (V, \varphi) \to (V'', \varphi'') \to 0$ we have $\mathrm{trace}(\varphi) = \mathrm{trace}(\varphi') + \mathrm{trace}(\varphi'')$, taking (V, φ) to $\mathrm{trace}(\varphi)$ gives a morphism of $\mathbf{F}_q$-vector spaces $u\colon K_\bullet^F(\mathrm{Spec}\, \mathbf{F}_q) \to \mathbf{F}_q$. We have a map w in the opposite direction that takes $a \in \mathbf{F}_q$ to $[\mathbf{F}_q, a \cdot \mathrm{Id}]$. It is clear that $u \circ w$ is the identity. In order to show that u and w are inverse isomorphisms, it is enough to show that w is surjective. The fact that $[V, \varphi]$ lies in the image of w follows easily by induction on $\dim(V)$, since whenever

$\dim(V) \geq 2$, φ can be written as a sum of maps, each of which has an invariant proper nonzero subspace. $\square$

If $f\colon X \to Y$ is a proper morphism, note that the higher direct images induce functors $R^i f_*\colon \mathcal{C}oh_F(X) \to \mathcal{C}oh_F(Y)$. Indeed, if $U \subseteq Y$ is an affine open subset of Y, and $(\mathcal{M}, F_{\mathcal{M}}) \in \mathcal{C}oh_F(X)$, then $H^i(f^{-1}(U), \mathcal{M})$ has an endomorphism induced by $F_{\mathcal{M}}$, and these endomorphisms glue together to give the Frobenius action on $R^i f_*(\mathcal{M})$. As a consequence, we get a morphism of $\mathbf{F}_q$-vector spaces $f_*\colon K_\bullet^F(X) \to K_\bullet^F(Y)$ given by $f_*([\mathcal{M}]) = \sum_{i\geq 0}(-1)^i[R^i f_*(\mathcal{M})]$. Note that this is well-defined: if

$$0 \to (\mathcal{M}', F_{\mathcal{M}'}) \to (\mathcal{M}, F_{\mathcal{M}}) \to (\mathcal{M}'', F_{\mathcal{M}''}) \to 0$$

is an exact sequence of coherent F-modules, then the long exact sequence in cohomology

$$\ldots \to R^i f_*(\mathcal{M}') \to R^i f_*(\mathcal{M}) \to R^i f_*(\mathcal{M}'') \to R^{i+1} f_*(\mathcal{M}') \to \ldots$$

is compatible with the Frobenius actions, and therefore we get

$$\sum_{i\geq 0}(-1)^i[R^i f_*(\mathcal{M})] = \sum_{i\geq 0}(-1)^i[R^i f_*(\mathcal{M}')] + \sum_{i\geq 0}(-1)^i[R^i f_*(\mathcal{M}'')] \text{ in } K_\bullet^F(Y).$$

The compatibility with the type (B) relations is straightforward, hence $f_*\colon K_\bullet^F(X) \to K_\bullet^F(Y)$ is well-defined.

Exercise 1.6. *Use the Leray spectral sequence to show that if $g\colon Y \to Z$ is another proper morphism, then we have $(g \circ f)_* = g_* \circ f_*\colon K_\bullet^F(X) \to K_\bullet^F(Z)$.*

In fact, we will only use the assertion in the above exercise when f is a closed immersion, in which case everything is clear since $R^i g_* \circ f_* = R^i(g \circ f)_*$ for all $i \geq 0$, and $R^j f_* = 0$ for all $j \geq 1$. The proof of the next lemma is straightforward.

Lemma 1.7. *If X is the disjoint union of the subschemes $X_1, \ldots, X_r$, then the inclusions $X_i \hookrightarrow X$ induce an isomorphism*

$$\bigoplus_{i=1}^r K_\bullet^F(X_i) \simeq K_\bullet^F(X).$$

The following is the main result of this lecture. For a scheme X, we consider $X(\mathbf{F}_q)$ as a closed subscheme of X, with the reduced scheme structure. Note that by Lemmas 1.5 and 1.7, we have an isomorphism $K_\bullet^F(X(\mathbf{F}_q)) \simeq \oplus_{x\in X(\mathbf{F}_q)}\mathbf{F}_q(x)$, and we denote by $\langle x\rangle \in K_\bullet^F(X(\mathbf{F}_q))$ the element corresponding to $1 \in \mathbf{F}_q(x)$.

Theorem 1.8. (*Localization Theorem*) *For every projective scheme X over $\mathbf{F}_q$, the inclusion $\iota\colon X(\mathbf{F}_q) \hookrightarrow X$ induces an isomorphism $K_\bullet^F(X(\mathbf{F}_q)) \simeq K_\bullet^F(X)$. Its inverse is given by $t\colon K_\bullet^F(X) \to K_\bullet^F(X(\mathbf{F}_q))$,*

$$t([\mathcal{M}, F_{\mathcal{M}}]) = \sum_{x\in X(\mathbf{F}_q)} \operatorname{trace}(F_{\mathcal{M}}(x))\langle x\rangle.$$

Let us see that this gives Theorem 1.4.

Proof of Theorem 1.4. Consider the structure morphism $f\colon X \to \operatorname{Spec}\mathbf{F}_q$. Let $\langle \mathrm{pt}\rangle$ denote the element of $K_\bullet^F(\operatorname{Spec}\mathbf{F}_q)$ that corresponds to $1 \in \mathbf{F}_q$ via the isomorphism given by Lemma 1.5. By definition, for every $[\mathcal{M}, F_{\mathcal{M}}] \in K_\bullet^F(X)$, we have

$$f_*([\mathcal{M}, F_{\mathcal{M}}]) = \left(\sum_{i=0}^{\dim(X)} (-1)^i \mathrm{trace}(F_{\mathcal{M}}|H^i(X, \mathcal{M}))\right)\langle \mathrm{pt}\rangle.$$

On the other hand, if we apply the isomorphism t in Theorem 1.8, we have

$$u := t([\mathcal{M}, F_{\mathcal{M}}]) = \sum_{x\in X(\mathbf{F}_q)} \mathrm{trace}(F_{\mathcal{M}}(x))\langle x\rangle.$$

If $\iota\colon X(\mathbf{F}_q) \to X$ is the inclusion, then it is clear that

$$f_*\left(\iota_*\left(\sum_{x\in X(\mathbf{F}_q)} m_x\langle x\rangle\right)\right) = \left(\sum_{x\in X(\mathbf{F}_q)} m_x\right)\langle \mathrm{pt}\rangle.$$

In particular, we have $f_*\circ\iota_*(u) = \left(\sum_{x\in X(\mathbf{F}_q)} \mathrm{trace}(F_{\mathcal{M}}(x))\right)\langle \mathrm{pt}\rangle$. Since t and ι are inverse to each other, the assertion in Theorem 1.4 follows. □

Remark 1.9. In fact, Theorem 1.8 is proved in [Ful2] also for arbitrary schemes of finite type over $\mathbf{F}_q$. In particular, Theorems 1.1 and 1.4 also hold if X is only assumed to be complete.

2. The proof of the Localization Theorem

We start with a few lemmas.

Lemma 2.1. *For every scheme X, and every coherent sheaf on X with Frobenius action $(\mathcal{M}, F_{\mathcal{M}})$ such that $F_{\mathcal{M}}$ is nilpotent, we have $[\mathcal{M}, F_{\mathcal{M}}] = 0$ in $K_\bullet^F(X)$.*

Proof. We prove the assertion by induction on m such that $\varphi^m = 0$. If $m = 1$, it is enough to use relation (B) in the definition of $K_\bullet^F(X)$, that gives $[\mathcal{M}, 0] = [\mathcal{M}, 0] + [\mathcal{M}, 0]$. If $m \geq 2$, and $\mathcal{M}' = \mathrm{Ker}(F_{\mathcal{M}})$, then $\mathcal{M}'$ is a coherent $\mathcal{O}_X$-submodule of $\mathcal{M}$, and we have an exact sequence of coherent sheaves with Frobenius action

$$0 \to (\mathcal{M}', F_{\mathcal{M}'}) \to (\mathcal{M}, F_{\mathcal{M}}) \to (\mathcal{M}'', F_{\mathcal{M}''}) \to 0.$$

This gives $[\mathcal{M}, F_{\mathcal{M}}] = [\mathcal{M}', F_{\mathcal{M}'}] + [\mathcal{M}'', F_{\mathcal{M}''}]$. Since $F_{\mathcal{M}'} = 0$ and $F_{\mathcal{M}''}^{m-1} = 0$, it follows by the induction hypothesis that $[\mathcal{M}', F_{\mathcal{M}'}] = 0 = [\mathcal{M}'', F_{\mathcal{M}''}]$. Therefore $[\mathcal{M}, F_{\mathcal{M}}] = 0$. □

Lemma 2.2. *If $j\colon X \hookrightarrow Y$ is a closed embedding, then we have a morphism of $\mathbf{F}_q$-vector spaces $j^*\colon K_\bullet^F(Y) \to K_\bullet^F(X)$ given by $j^*([\mathcal{M}, F_{\mathcal{M}}]) = [\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{O}_X, \overline{F_{\mathcal{M}}}]$, where $\overline{F_{\mathcal{M}}}$ is the Frobenius action induced by $F_{\mathcal{M}}$ on $\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{O}_X$. In particular, the composition $j^*\circ j_*$ is the identity on $K_\bullet^F(X)$.*

Proof. Let $\mathcal{I}$ be the ideal defining X in Y. Since $F_{\mathcal{M}}(\mathcal{I}\mathcal{M}) \subseteq \mathcal{I}^q\mathcal{M}$, it follows that $F_{\mathcal{M}}$ indeed induces a Frobenius action $\overline{F_{\mathcal{M}}}$ on $\mathcal{M}/\mathcal{I}\mathcal{M}$. We have $\overline{F_1 + F_2} = \overline{F_1} + \overline{F_2}$, hence in order to show that we have an induced morphism $K_\bullet^F(Y) \to K_\bullet^F(X)$, we only need to show that if

$$0 \to (\mathcal{M}', F_{\mathcal{M}'}) \to (\mathcal{M}, F_{\mathcal{M}}) \to (\mathcal{M}'', F_{\mathcal{M}''}) \to 0$$

is an exact sequence of coherent F-modules on Y, then

$$[\mathcal{M}/\mathcal{I}\mathcal{M}] = [\mathcal{M}'/\mathcal{I}\mathcal{M}'] + [\mathcal{M}''/\mathcal{I}\mathcal{M}'']$$

in $K_\bullet^F(X)$. Note that we have an exact sequence of coherent F-modules on X

$$0 \to \mathcal{M}'/\mathcal{M}' \cap \mathcal{I}\mathcal{M} \to \mathcal{M}/\mathcal{I}\mathcal{M} \to \mathcal{M}''/\mathcal{I}\mathcal{M}'' \to 0,$$

and a surjection $\mathcal{M}'/\mathcal{I}\mathcal{M}' \to \mathcal{M}'/\mathcal{M}' \cap \mathcal{I}\mathcal{M}$, with kernel $\mathcal{M}' \cap \mathcal{I}\mathcal{M}/\mathcal{I}\mathcal{M}'$. In light of Lemma 2.1, it is enough to show that the Frobenius action on $\mathcal{M}' \cap \mathcal{I}\mathcal{M}/\mathcal{I}\mathcal{M}'$ is nilpotent. Since $F_{\mathcal{M}}^m(\mathcal{I}\mathcal{M}) \subseteq \mathcal{I}^{q^m}(\mathcal{M})$, we see that $\mathcal{M}' \cap F_{\mathcal{M}}^m(\mathcal{I}\mathcal{M}) \subseteq \mathcal{I}\mathcal{M}'$ for $m \gg 0$ by Artin-Rees. This shows that j^* is well-defined, and the fact that $j^* \circ j_*$ is the identity follows from definition. □

Note that if X is any scheme, and we consider $j\colon X(\mathbf{F}_q) \hookrightarrow X$, then j^* is the morphism t in Theorem 1.8. Since $j^* \circ j_*$ is the identity, in order to prove Theorem 1.8 for a projective scheme X, it is enough to show that $j_* \circ j^*$ is the identity on $K_\bullet^F(X)$. In fact, it is enough to show that j_* is surjective.

Lemma 2.3. *If $(\mathcal{M}, \varphi)$ is a coherent $\mathcal{O}_X$-module with a Frobenius action, and $\mathcal{M}$ decomposes as $\mathcal{M} = \mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_r$, and if $\varphi_{i,j}$ is the composition $\mathcal{M}_i \to \mathcal{M} \xrightarrow{\varphi} \mathcal{M} \to \mathcal{M}_j$, then $[\mathcal{M}, \varphi] = \sum_{i=1}^r [\mathcal{M}_i, \varphi_{i,i}]$ in $K_\bullet^F(X)$.*

Proof. Let $\widetilde{\varphi}_{i,j}\colon \mathcal{M} \to \mathcal{M}$ be the map induced by $\varphi_{i,j}$, so that $\varphi = \sum_{i,j} \widetilde{\varphi}_{i,j}$. By condition (B) we have $[\mathcal{M}, \varphi] = \sum_{i,j} [\mathcal{M}, \widetilde{\varphi}_{i,j}]$. For every $i \neq j$ we have $\widetilde{\varphi}_{i,j}^2 = 0$, hence $[\mathcal{M}, \widetilde{\varphi}_{i,j}] = 0$ by Lemma 2.1. Therefore

$$[\mathcal{M}, \varphi] = \sum_{i=1}^r [\mathcal{M}, \widetilde{\varphi}_{i,i}] = \sum_{i=1}^r [\mathcal{M}_i, \varphi_{i,i}],$$

by condition (A). □

The key ingredient in the proof of Theorem 1.8 is provided by the case $X = \mathbf{P}^n_{\mathbf{F}_q}$. We now turn to the description of $K_\bullet^F(\mathbf{P}^n_{\mathbf{F}_q})$. We will use the Serre correspondence between coherent sheaves on $\mathbf{P}^n_{\mathbf{F}_q}$ and finitely generated graded modules over $S = \mathbf{F}_q[x_0, \ldots, x_n]$.

Suppose that $\mathcal{M}$ is a coherent sheaf on $\mathbf{P}^n_{\mathbf{F}_q}$ with a Frobenius action $F_{\mathcal{M}}\colon \mathcal{M} \to F_*(\mathcal{M})$. This induces for every i a morphism

$$\mathcal{M}(i) \to F_*(\mathcal{M}) \otimes \mathcal{O}(i) \to F_*(\mathcal{M}(qi)),$$

where we used the projection formula, and the fact that for every line bundle L we have $F^*(L) \simeq L^q$. It follows that if $M = \Gamma_*(\mathcal{M}) := \oplus_{i \geq 0} \Gamma(\mathbf{P}^n_{\mathbf{F}_q}, \mathcal{M}(i))$, then we get a *graded Frobenius action* on M: this is an $\mathbf{F}_q$-linear map $F_M\colon M \to M$ such that $F_M(M_i) \subseteq M_{qi}$ and $F_M(au) = a^q F_M(u)$ for $a \in S$ and $u \in M$.

Conversely, given a finitely generated graded S-module M with a graded Frobenius action F_M, we get an induced coherent F-module structure on $\widetilde{M}$, as follows. If $U_i \subset \mathbf{P}^n_{\mathbf{F}_q}$

is the open subset defined by $x_i \neq 0$, then $\Gamma(U_i, \widetilde{M}) = (M_{x_i})_0$, and $F_{\widetilde{M}}\left(\frac{u}{x_i^N}\right) = \frac{F_M(u)}{x_i^{qN}}$ for every $u \in M_N$. It is straightforward to check that this gives a Frobenius action on $\widetilde{M}$. If $(\mathcal{M}, F_{\mathcal{M}})$ is a coherent F-module and $M = \Gamma_*(\mathcal{M})$, with the graded Frobenius action described above, then we have an isomorphism of graded F-modules $\mathcal{F} \simeq \widetilde{M}$.

If $M = S(-i)$, then giving a graded Frobenius action F_M on M, is equivalent to giving $f = F_M(1) \in S_{(q-1)i}$. In particular, if $i < 0$, then the only graded Frobenius action on $S(-i)$ is the zero one. For an arbitrary finitely generated graded S-module M, we consider a graded free resolution of M

$$0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0,$$

where each F_j is a direct sum of of S-modules of the form $S(-b_{i,j})$, with $b_{i,j} \in \mathbf{Z}$. If we have a graded Frobenius action on M, then we can put graded Frobenius actions on each F_i, such that the above exact sequence is compatible with the graded Frobenius actions. In particular, we get $[\widetilde{M}] = \sum_{i=0}^{n}(-1)^i[\widetilde{F}_i]$ in $K_\bullet^F(\mathbf{P}^n_{\mathbf{F}_q})$. It follows from the above discussion and Lemma 2.3 that $K_\bullet^F(\mathbf{P}^n_{\mathbf{F}_q})$ is generated (as an $\mathbf{F}_q$-vector space) by $[\mathcal{O}(-i), x_0^{a_0} \cdots x_n^{a_n}]$, where $a_\ell \geq 0$ and $\sum_{\ell=0}^{n} a_\ell = i(q-1)$.

Proposition 2.4. *The $\mathbf{F}_q$-vector space $K_\bullet^F(\mathbf{P}^n_{\mathbf{F}_q})$ is generated by $[\mathcal{O}(-i), x_0^{a_0} \cdots x_n^{a_n}]$, with $0 \leq a_\ell \leq q-1$ for all ℓ, with some $a_\ell < q-1$, and where $\sum_{\ell=0}^{n} a_\ell = i(q-1)$.*

Proof. Let us show first that $K_\bullet^F(\mathbf{P}^n_{\mathbf{F}_q})$ is generated by $[\mathcal{O}(-i), x_0^{a_0} \cdots x_n^{a_n}]$, with $0 \leq a_\ell \leq q-1$ for all ℓ, and with $\sum_{\ell=0}^{n} a_\ell = i(q-1)$. In light of the discussion preceding the proposition, it is enough to show the following: if u is a monomial such that $u = x_i^q w$, then $[\mathcal{O}(-i), u] = [\mathcal{O}(-i+1), x_i w]$ in $K_\bullet^F(\mathbf{P}^n_{\mathbf{F}_q})$. If H is the hyperplane in $\mathbf{P}^n_{\mathbf{F}_q}$ defined by $(x_i = 0)$, we have an exact sequence of coherent sheaves with Frobenius action

$$0 \to \mathcal{O}(-i) \xrightarrow{\cdot x_i} \mathcal{O}(-i+1) \to \mathcal{O}_H(-i+1) \to 0,$$

where the Frobenius actions on $\mathcal{O}(-i)$ and $\mathcal{O}(-i+1)$ are defined by u and $x_i w$, respectively. Since $x_i w$ restricts to zero on H, it follows that the Frobenius action on $\mathcal{O}_H(-i+1)$ is zero, and we conclude from the above exact sequence that $[\mathcal{O}(-i), u] = [\mathcal{O}(-i+1), x_i w]$.

In order to complete the proof of the proposition, it is enough to show that we can write $[\mathcal{O}(-(n+1)), (x_0 \cdots x_n)^{q-1}]$ in terms of the remaining elements of the above system of generators. In order to do this, let us consider the Koszul complex on $\mathbf{P}^n_{\mathbf{F}_q}$ corresponding to the global sections $x_0, \ldots, x_n$ of $\mathcal{O}(1)$:

$$0 \to \mathcal{E}_{n+1} \to \ldots \to \mathcal{E}_1 = \mathcal{O}(-1)^{\oplus(n+1)} \xrightarrow{h} \mathcal{E}_0 = \mathcal{O}_{\mathbf{P}^n_{\mathbf{F}_q}} \to 0,$$

where $h = (x_0, \ldots, x_n)$. Using the above decomposition $\mathcal{E}_1 = L_0 \oplus \ldots \oplus L_n$, then

$$\mathcal{E}_r = \bigoplus_{0 \leq i_1 < \ldots < i_r \leq n} (L_{i_1} \otimes \ldots \otimes L_{i_r}) \simeq \mathcal{O}(-r)^{\binom{n+1}{r}}.$$

If on the factor $L_{i_1} \otimes \ldots \otimes L_{i_r}$ of $\mathcal{E}_r$ we consider the F-module structure given by the monomial $x_{i_1}^{q-1} \cdots x_{i_r}^{q-1}$, then the above complex becomes a complex of F-modules. We deduce that in $K^F(\mathbf{P}^n_{\mathbf{F}_q})$ we have the following relation:

$$\sum_{r=0}^{n+1}(-1)^r \sum_{0\leq i_1<\ldots<i_r\leq n} [\mathcal{O}(-r), x_{i_1}^{q-1}\cdots x_{i_r}^{q-1}] = 0,$$

which completes the proof of the proposition. □

Corollary 2.5. *The assertion in Theorem 1.8 holds when* $X = \mathbf{P}^n_{\mathbf{F}_q}$.

Proof. As we have seen, if $j\colon \mathbf{P}^n(\mathbf{F}_q) \hookrightarrow \mathbf{P}^n_{\mathbf{F}_q}$ is the inclusion, then j_* is injective, and it is enough to show that it is surjective. Proposition 2.4 implies that $\dim_{\mathbf{F}_q} K^F_\bullet(\mathbf{P}^n_{\mathbf{F}_q}) \leq \alpha_n - 1$, where

$$\alpha_n := |\{(a_0,\ldots,a_n) \mid 0\leq a_i\leq q-1,\, (q-1)\,\text{divides}\sum_{i=0}^n a_i\}|.$$

Suppose we have $(a_0,\ldots,a_{n-1})$ with $0\leq a_\ell\leq q-1$ for $0\leq \ell\leq n-1$, and we want to choose a_n with $0\leq a_n\leq q-1$ such that $\sum_{\ell=0}^n a_\ell$ is divisible by $(q-1)$. If $\sum_{\ell=0}^{n-1} a_\ell$ is divisible by $(q-1)$, then we may take $a_n=0$ or $a_n=q-1$; if $\sum_{\ell=0}^{n-1} a_\ell$ is not divisible by $(q-1)$, then we have precisely one choice for a_n. Therefore $\alpha_n = 2\alpha_{n-1} + (q^n - \alpha_{n-1}) = q^n + \alpha_{n-1}$. Since $\alpha_0 = 2$, we conclude that $\alpha_n = (1+q+\ldots+q^n)+1$.

Therefore $\dim_{\mathbf{F}_q} K^F_\bullet(\mathbf{P}^n_{\mathbf{F}_q}) \leq |\mathbf{P}^n(\mathbf{F}_q)| = \dim_{\mathbf{F}_q} K^F_\bullet(\mathbf{P}^n(\mathbf{F}_q))$. Since j_* is injective, it follows that j_* is also surjective, completing the proof. □

Proof of Theorem 1.8. Let us fix a closed immersion $j\colon X\hookrightarrow Y = \mathbf{P}^n_{\mathbf{F}_q}$. By Corollary 2.5, it is enough to show that if Theorem 1.8 holds for Y, then it also holds for X.

Consider the following commutative diagram:

$$\begin{tikzcd} X(\mathbf{F}_q) \arrow[r, "j'"] \arrow[d, "\iota"'] & Y(\mathbf{F}_q) \arrow[d, "\iota'"] \\ X \arrow[r, "j"] & Y \end{tikzcd} \tag{3}$$

in which all maps are closed immersions. As we have already mentioned, in order to prove Theorem 1.8 for X, it is enough to show that $\iota_*\circ\iota^*$ is the identity on $K^F_\bullet(X)$. Since the theorem holds for Y, we know that $\iota'_*\circ(\iota')^*$ is the identity on $K^F_\bullet(Y)$.

Note that $j'_*\circ\iota^* = (\iota')^*\circ j_*$: this is an immediate consequence of the definitions. Therefore

$$j_*\circ\iota_*\circ\iota^* = (\iota')_*\circ j'_*\circ\iota^* = \iota'_*\circ(\iota')^*\circ j_* = j_*. \tag{4}$$

On the other hand, Lemma 2.2 implies that $j^*\circ j_*$ is the identity on $K^F_\bullet(X)$. In particular, j_* is injective. We conclude from (4) that $\iota_*\circ\iota^*$ is the identity on $K^F_\bullet(X)$, and this completes the proof of the theorem. □

3. Supersingular Calabi-Yau hypersurfaces

As an application of Theorem 1.1, we discuss a characterization of supersingular Calabi-Yau hypersurfaces. More generally, we prove the following

Proposition 3.1. *Let $f \in \mathbf{F}_q[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $n+1$, with $n \geq 2$, defining the hypersurface $Z \subset \mathbf{P}^n$. The following are equivalent:*

i) *The action induced by the Frobenius morphism on $H^{n-1}(Z, \mathcal{O}_Z)$ is bijective (equivalently, it is nonzero).*
ii) $|Z(\mathbf{F}_q)| \not\equiv 1 \pmod{p}$.
iii) *The coefficient of $(x_0 \cdots x_n)^{q-1}$ in f^{q-1} is nonzero.*
iv) *The coefficient of $(x_0 \cdots x_n)^{p-1}$ in f^{p-1} is nonzero.*

If Z as above is a smooth hypersurface, then it is *ordinary* if it satisfies the above equivalent conditions. Otherwise, it is *supersingular.*

Proof. Since Z is a hypersurface of degree $(n+1)$ in $\mathbf{P}^n$, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-n-1) \xrightarrow{\cdot f} \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_Z \to 0. \tag{5}$$

This gives $H^i(Z, \mathcal{O}_Z) = 0$ for $1 \leq i \leq n-2$, and $H^0(Z, \mathcal{O}_Z) \simeq \mathbf{F}_q \simeq H^{n-1}(Z, \mathcal{O}_Z)$. Frobenius acts on $H^0(Z, \mathcal{O}_Z)$ as the identity, and if it acts as multiplication by $\lambda \in \mathbf{F}_q$ on $H^{n-1}(Z, \mathcal{O}_Z)$, then Theorem 1.1 gives

$$|Z(\mathbf{F}_q)| \bmod p = 1 + (-1)^{n-1}\lambda.$$

Therefore $\lambda = 0$ if and only if $|Z(\mathbf{F}_q)| \equiv 1 \pmod{p}$. This proves i)$\Leftrightarrow$ii).

In order to prove that iii) and iv) are equivalent, note first that for every $r \geq 1$, we may uniquely write

$$f^{p^r-1} = c_r (x_0 \cdots x_n)^{p^r-1} + u_r, \tag{6}$$

where $u_r \in (x_0^{p^r}, \ldots, x_n^{p^r})$. If we raise to the p^{th}-power in (6), we get

$$f^{p^{r+1}-p} = c_r^p (x_0 \cdots x_n)^{p^{r+1}-p} + u_r^p.$$

Since $u_r^p \in (x_0^{p^{r+1}}, \ldots, x_n^{p^{r+1}})$ and

$$(x_0 \cdots x_n)^{p^{r+1}-p} \cdot (x_0^p, \ldots, x_n^p) \subseteq (x_0^{p^{r+1}}, \ldots, x_n^{p^{r+1}}),$$

we deduce that

$$f^{p^{r+1}} - c_r^p c_1 (x_0 \cdots x_n)^{p^{r+1}-1} \in (x_0^{p^{r+1}}, \ldots, x_n^{p^{r+1}}).$$

Therefore $c_{r+1} = c_r^p c_1$, which immediately gives that $c_r = c_1^{1+p+\ldots p^{r-1}}$ for every $r \geq 1$. In particular, if $q = p^e$, we see that $c_1 \neq 0$ if and only if $c_e \neq 0$, hence iii)$\Leftrightarrow$iv).

In order to prove the equivalence of i) and iii), we consider the explicit description of the Frobenius action F on $H^{n-1}(Z, \mathcal{O}_Z)$ via the isomorphism $\delta\colon H^{n-1}(Z, \mathcal{O}_Z) \to H^n(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-n-1))$ induced by (5). We compute the cohomology of $\mathcal{O}_{\mathbf{P}^n}(-n-1)$ and of $\mathcal{O}_Z$ as Cech cohomology with respect to the affine cover of $\mathbf{P}^n$ by the open subsets $(x_i \neq 0)$. Recall that

$$H^n(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-n-1)) \simeq (S_{x_0\cdots x_n})_{-n-1} / \sum_{i=0}^{n} (S_{x_0\cdots \widehat{x_i}\cdots x_n})_{-n-1} = \mathbf{F}_q \cdot \frac{1}{x_0\cdots x_n}.$$

Suppose that $u \in H^{n-1}(Z, \mathcal{O}_Z)$ is represented by the Cech cocycle $h = (\overline{h_0}, \ldots, \overline{h_n}) \in \oplus_{i=0}^n ((S/f)_{x_0\cdots\widehat{x_i}\cdots x_n})_0$. If $h_i \in (S_{x_0\cdots\widehat{x_i}\cdots x_n})_0$ is a lift of $\overline{h_i}$, then $\delta(u)$ is represented by the unique $w \in (S_{x_0\ldots x_n})_{-n-1}$ such that $fw = \sum_{i=0}^n (-1)^i h_i$.

On the other hand, $F(u)$ is represented by $(\overline{h_0}^q, \ldots, \overline{h_n}^q)$. Since we have $f(f^{q-1}w^q) = \sum_{i=0}^n (-1)^i h_i^q$, it follows that via the isomorphism δ, we can describe F as the linear map on $(S_{x_0\cdots x_n})_{-n-1} / \sum_{i=0}^n (S_{x_0\cdots\widehat{x_i}\cdots x_n})_{-n-1}$ induced by $w \to f^{q-1}w^q$. This map multiplies the class of $\frac{1}{x_0\cdots x_n}$ in this quotient by the coefficient of $(x_0\cdots x_n)^{q-1}$ in f^{q-1}. This completes the proof of ii)$\Leftrightarrow$iii), hence the proof of the proposition. $\square$

Remark 3.2. In the context of Proposition 3.1, note that if $\mathrm{trace}(F \mid H^{n-1}(Z, \mathcal{O}_Z)) = 1 + (-1)^{n-1}a$, then for every $r \geq 1$ we have $1 + (-1)^{n-1}a^r = |Z(\mathbf{F}_{q^r})| \bmod p$. This is a consequence of Theorem 1.1 (see also Remark 1.3).

Exercise 3.3. *Give a direct proof for the implication* ii) $\Leftrightarrow$ iii) *in Proposition 3.1 by computing* $\sum_{a \in \mathbf{F}_q^{n+1}} f(a)^{q-1}$ *(see* [Knu]*).*

Exercise 3.4. *Show that if X is an elliptic curve (that is, X is a smooth, geometrically connected, projective curve of genus* 1*) over $\mathbf{F}_p$, with $p \neq 2, 3$, then X is supersingular if and only if $|X(\mathbf{F}_p)| = p + 1$.*

Exercise 3.5. *Let $Z \subset \mathbf{P}^n_{\mathbf{F}_q}$ be a complete intersection subscheme of codimension r, defined by $(F_1, \ldots, F_r)$. Let $d_i = \deg(F_i)$, and assume that $\sum_i d_i = n + 1$. Show that the following are equivalent:*

i) *The action induced by the Frobenius morphism on the cohomology group $H^{n-r}(Z, \mathcal{O}_Z)$ is bijective (equivalently, it is nonzero).*
ii) $|Z(\mathbf{F}_q)| \not\equiv 1$ *(mod p).*
iii) *The polynomial $F_1 \cdots F_r$ satisfies the equivalent conditions in Proposition 3.1.*

References

[Del2] P. Deligne, Fonctions L modulo ℓ^n et modulo p, in *Cohomologie Étale* (SGA 4 1/2), Lecture Notes in Mathematics 569, 110–128, Springer-Verlag, Berlin-Heidelberg-New York, 1977. 1

[Ful2] W. Fulton, A fixed point formula for varieties over finite fields, Math. Scand. **42** (1978), 189–196. 1, 4

[Katz] N. Katz, Une formule de congruence pour la fonction ζ, Exposé XXII, in *Groupes de Monodromie en Géometrie Algébrique* (SGA 7 II), Lecture Notes in Mathematics 340, 401–438, Springer-Verlag, Berlin-Heidelberg-New York, 1973. 1

[Knu] A. Knutson, Frobenius splitting, point counting, and degenerations, arXiv:0911.4941. 9