LECTURE 8. THE GROTHENDIECK RING OF VARIETIES AND KAPRANOV'S MOTIVIC ZETA FUNCTION

In this lecture we give an introduction to the Grothendieck ring of algebraic varieties, and discuss Kapranov's lifting of the Hasse-Weil zeta function to this Grothendieck ring. One interesting feature is that this makes sense over an arbitrary field. We will prove the rationality of Kapranov's zeta function for curves by a variant of the argument used in Lecture 4 for the Hasse-Weil zeta function. We will end by discussing the results of Larsen and Lunts on Kapranov zeta functions of algebraic surfaces.

1. THE GROTHENDIECK RING OF ALGEBRAIC VARIETIES

In this section we recall the definition and the basic properties of the Grothendieck ring of algebraic varieties. Let k be an arbitrary field. The *Grothendieck group $K_0(\mathrm{Var}/k)$ of varieties over k* is the quotient of the free abelian group on the set of isomorphism clases of varieties over k, by relations of the form

$$[X] = [Y] + [X \smallsetminus Y],$$

where Y is a closed subvariety of the variety X (here $[X]$ denotes the image of the variety X in $K_0(\mathrm{Var}/k)$). Note that the above relation implies $[\emptyset] = 0$.

In fact, $K_0(\mathrm{Var}/k)$ is a commutative ring, with the product given by

$$[X] \cdot [Y] = [(X \times Y)_{\mathrm{red}}],$$

where the product on the right is understood to be over $\mathrm{Spec}\, k$. It is clear that this induces a bilinear map $K_0(\mathrm{Var}/k) \times K_0(\mathrm{Var}/k) \to K_0(\mathrm{Var}/k)$ that is commutative and associative, and has unit $\mathrm{Spec}\, k$.

The class of $\mathbf{A}_k^1$ in $K_0(\mathrm{Var}/k)$ is denoted by $\mathbf{L}$. Therefore $[\mathbf{A}_k^n] = \mathbf{L}^n$. The usual decomposition $\mathbf{P}_k^n = \mathbf{P}_k^{n-1} \sqcup \mathbf{A}_k^n$ implies by induction on n that $[\mathbf{P}_k^n] = 1 + \mathbf{L} + \ldots + \mathbf{L}^n$.

Proposition 1.1. *Suppose that X is a variety over k, and we have a decomposition $X = Y_1 \sqcup \ldots \sqcup Y_r$, where all Y_i are locally closed subvarieties of X. In this case $[X] = [Y_1] + \ldots + [Y_r]$.*

Proof. We argue by induction on $\dim(X)$ (the case $\dim(X) = 0$ being trivial), and then by induction on the number of irreducible components of X of maximal dimension. Let Z be an irreducible component of X of maximal dimension, and η_Z its generic point. If i is such that $\eta_Z \in Y_i$, then $Z \subseteq \overline{Y_i}$, and since Y_i is open in $\overline{Y_i}$, it follows that there is an open subset U of X contained in $Y_i \cap Z$ (for example, we may take to be the complement in $Y_i \cap Z$ of all irreducible components of X different from Z). By definition, we have

(1) $$[Y_i] = [U] + [Y_i \smallsetminus U] \text{ and } [X] = [U] + [X \smallsetminus U].$$

On the other hand, either $\dim(X \smallsetminus U) < \dim(X)$, or $\dim(X \smallsetminus U) = \dim(X)$ and $X \smallsetminus U$ has fewer irreducible components of maximal dimension than X does. Applying the induction hypothesis to the decomposition $X \smallsetminus U = (Y_i \smallsetminus U) \sqcup \bigsqcup_{j \neq i} Y_j$, we have

$$(2) \qquad\qquad [X \smallsetminus U] = [Y_i \smallsetminus U] + \sum_{j \neq i} [Y_j].$$

By combining (1) and (2), we get the formula in the proposition. $\qquad\square$

Given a variety X over k, we want to define the class in $K_0(\mathrm{Var}/k)$ of a constructible subset of X. This is achieved using the following easy lemma.

Lemma 1.2. *Any constructible subset W of a variety X over k can be written as a finite disjoint union of locally closed subsets.*

Proof. We prove this by induction on $d = \dim(\overline{W})$, the case $d = 0$ being trivial. Let us write $W = W_1 \cup \ldots \cup W_r$, with all W_i locally closed, hence $\overline{W} = \overline{W_1} \cup \ldots \cup \overline{W_r}$. After replacing each W_i by its irreducible decomposition, we may assume that all W_i are irreducible. After renumbering, we may assume that $\overline{W_1}, \ldots, \overline{W_s}$ are the irreducible components of $\overline{W}$. Since each W_i is open in $\overline{W_i}$, the set $U = \bigcup_{i=1}^{s} \left(W_i \smallsetminus \bigcup_{j \neq i} \overline{W_j} \right)$ is open and dense in $\overline{W}$, and it is contained in W. If $V = W \smallsetminus U$, then V is constructible, and $\dim(\overline{V}) < \dim(\overline{W})$, hence by induction we have a decomposition $V = V_1 \sqcup \ldots \sqcup V_s$, with each V_i locally closed in X. Therefore we have a decomposition $W = U \sqcup V_1 \sqcup \ldots \sqcup V_s$, as required. $\qquad\square$

Suppose now that X is a variety over k, and W is a constructible subset of X. By the above lemma, there is a disjoint decomposition $W = W_1 \sqcup \ldots \sqcup W_r$, with each W_i locally closed in X. We put $[W] := \sum_{i=1}^{r} [W_i]$.

Proposition 1.3. *With the above notation, the following hold:*

i) *The definition of $[W]$, for W constructible in X, is independent of the disjoint decomposition.*
ii) *If $W_1, \ldots, W_s$ are disjoint constructible subsets of X, and $W = \bigcup_i W_i$, then $[W] = \sum_{i=1}^{s} [W_i]$.*

Proof. Suppose that we have two decompositions into locally closed subsets

$$W = W_1 \sqcup \ldots \sqcup W_r \text{ and } W = W' \sqcup \ldots \sqcup W'_s.$$

Let us also consider the decomposition $W = \bigsqcup_{i,j} (W_i \cap W'_j)$. It follows from Proposition 1.1 that $[W_i] = \sum_{j=1}^{s} [W_i \cap W'_j]$ for every i, and $[W'_j] = \sum_{i=1}^{r} [W_i \cap W'_j]$ for every j. Therefore

$$\sum_{i=1}^{r} [W_i] = \sum_{i=1}^{r} \sum_{j=1}^{s} [W_i \cap W'_j] = \sum_{j=1}^{s} \sum_{i=1}^{r} [W_i \cap W'_j] = \sum_{j=1}^{s} [W'_j].$$

This proves i). The assertion in ii) follows from i): if we consider disjoint unions $W_i = W_{i,1} \sqcup \ldots \sqcup W_{i,m_i}$ for every i, with each $W_{i,j}$ locally closed in X, then $W = \bigsqcup_{i,j} W_{i,j}$, and

$$[W] = \sum_{i,j} [W_{i,j}] = \sum_i [W_i].$$

$\square$

A morphism $f \colon X \to Y$ is *piecewise trivial*, with fiber F, if there is a decomposition $Y = Y_1 \sqcup \ldots \sqcup Y_r$, with all Y_i locally closed in Y, such that $f^{-1}(Y_i) \simeq Y_i \times F$ for all i.

Proposition 1.4. *If $f \colon X \to Y$ is piecewise trivial with fiber F, then $[X] = [Y] \cdot [F]$ in $K_0(\mathrm{Var}/k)$.*

Proof. By assumption, there is a decomposition $Y = Y_1 \sqcup \ldots \sqcup Y_r$ into locally closed subsets such that $[f^{-1}(Y_i)] = [F] \cdot [Y_i]$. By Proposition 1.1 we have $[X] = \sum_i [f^{-1}(Y_i)]$ and $[Y] = \sum_i [Y_i]$, hence we get the assertion in the proposition. $\square$

Example 1.5. It is clear that if E is a vector bundle on Y of rank n, then $E \to Y$ is piecewise trivial with fiber $\mathbf{A}_k^n$ and $\mathbf{P}(E) \to Y$ is piecewise trivial with fiber $\mathbf{P}_k^{n-1}$. Therefore $[E] = [Y] \cdot \mathbf{L}^n$ and $[\mathbf{P}(E)] = [Y](1 + \mathbf{L} + \ldots + \mathbf{L}^{n-1})$.

The following lemma is an immediate consequence of the definitions.

Lemma 1.6. *If k'/k is a field extension, then we have a ring homomorphism $K_0(\mathrm{Var}/k) \to K_0(\mathrm{Var}/k')$, that takes $[X]$ to $[(X \times_k k')_{\mathrm{red}}]$ for every variety X over k.*

An Euler-Poincaré characteristic for varieties over k is a map χ that associates to a variety X over k an element $\chi(X)$ in a group A, such that if Y is a closed subvariety of X, we have $\chi(X) = \chi(Y) + \chi(X \smallsetminus Y)$. Note that the map taking X to $[X] \in K_0(\mathrm{Var}/k)$ is the *universal Euler-Poincaré characteristic*: every Euler-Poincaré characteristic as above is induced by a unique group homomorphism $\chi \colon K_0(\mathrm{Var}/k) \to A$. If A is a ring, then the Euler-Poincaré characteristic is called *multiplicative* if χ is a ring homomorphism.

Example 1.7. If k is a finite field, then for every finite extension K/k we have a multiplicative Euler-Poicaré characteristic with values in $\mathbf{Z}$, that takes X to $|X(K)|$. One can put all these together in a group homomorphism

$$K_0(\mathrm{Var}/k) \to (1 + t\mathbf{Z}[\![t]\!], \cdot), \ [X] \to Z(X, t).$$

Example 1.8. If $k = \mathbf{C}$, then we have a multiplicative Euler-Poincaré characteristic that associates to X the usual Euler-Poincaré characteristic for singular cohomology $\chi_{\mathrm{top}}(X) = \sum_{i \geq 0} (-1)^i \dim_{\mathbf{Q}} H^i(X^{\mathrm{an}}, \mathbf{Q})$ (compare with the more refined invariant in Example 1.13 below). The fact that $\chi_{\mathrm{top}}(X)$ gives an Euler-Poincaré characteristic is a consequence of the fact that $\chi_{\mathrm{top}}(X)$ is also equal to the Euler-Poincaré characteristic for compactly supported cohomology $\chi_{\mathrm{top}}^c(X) := \sum_{i \geq 0} (-1)^i \dim_{\mathbf{Q}} H_c^i(X^{\mathrm{an}}, \mathbf{Q})$ (see [Ful, p. 141-142]). Indeed, if Y is a closed subvariety of the complex variety X, and $U = X \smallsetminus Y$, then there is a long exact sequence for cohomology with compact supports

$$\ldots \to H_c^i(U^{\mathrm{an}}, \mathbf{Q}) \to H_c^i(X^{\mathrm{an}}, \mathbf{Q}) \to H_c^i(Y^{\mathrm{an}}, \mathbf{Q}) \to H_c^{i+1}(U^{\mathrm{an}}, \mathbf{Q}) \to \ldots,$$

which implies $\chi_{\mathrm{top}}^c(X) = \chi_{\mathrm{top}}^c(U) + \chi_{\mathrm{top}}^c(Y)$.

The most convenient way of constructing Euler-Poincaré characteristics when the ground field is algebrically closed of characteristic zero involves a presentation of $K_0(\mathrm{Var}/k)$ due to Bittner [Bit]. The following lemma is elementary (and we have seen some of its avatars before).

Lemma 1.9. *If* $\mathrm{char}(k) = 0$, *then* $K_0(\mathrm{Var}/k)$ *is generated by classes of nonsingular, connected, projective varieties over* k. *More precisely, given any irreducible variety* X *of dimension* n, *there is a nonsingular, irreducible, projective variety* Y *that is birational to* X *such that* $[X] - [Y] = \sum_{i=1}^{N} m_i [W_i]$, *for some smooth, projective, irreducible varieties* W_i *of dimension* $< n$, *and some* $m_i \in \mathbf{Z}$.

Proof. Note first that the second assertion implies the first. Indeed, it is enough to show by induction on n that for every n-dimensional variety W over k, we have $[W] \in K_0'$, where K_0' is the subgroup of $K_0(\mathrm{Var}/k)$ generated by classes of nonsingular, connected, projective varieties. The assertion is clear if $n = 0$. For the induction step, given W with irreducible components $W_1, \ldots, W_r$, let $U_i = W_i \smallsetminus \bigcup_{j \neq i} W_j$, and $U = \bigcup_{i=1}^{r} U_i$. Since $\dim(W \smallsetminus U) < n$, it follows by induction that $[W \smallsetminus U] \in K_0'$, and since $[U] = \sum_{i=1}^{r} [U_i]$ we see that it is enough to show that every $[U_i]$ lies in K_0'. This is a consequence of the second assertion in the lemma.

We now prove the second assertion in the lemma by induction on $n = \dim(X)$. Let X' be an irreducible projective variety that is birational to X. By Hironaka's theorem on resolution of singularities, there is a birational morphism $f \colon Y \to X'$, with Y nonsingular, connected, and projective. Since X and Y are birational, we can find isomorphic open subsets $U \subseteq X$ and $V \subseteq Y$, so that we have

$$(3) \qquad\qquad [X] - [Y] = [X \smallsetminus U] - [Y \smallsetminus V],$$

and $\dim(X \smallsetminus V), \dim(Y \smallsetminus U) < n$. Arguing as above, we see that the induction hypothesis implies that both $[X \smallsetminus U]$ and $[Y \smallsetminus V]$ can be written as linear combinations of classes of nonsingular, irreducible, projective varieties of dimension $< n$, with integer coefficients. Using (3), we obtain the assertion in the lemma about X. $\square$

Bittner's theorem shows that with respect to the system of generators described in the lemma, the relations are generated by the ones coming from blow-ups with smooth centers.

Theorem 1.10. ([Bit]) *Let* k *be an algebraically closed field of characteristic zero. The kernel of the natural morphism from the free abelian group on isomorphism classes of smooth, connected, projective varieties over* k *to* $K_0(\mathrm{Var}/k)$ *is generated by the following elements:*

   i) $[\emptyset]$
  ii) $([\mathrm{Bl}_Y X] - [E]) - ([X] - [Y])$,

with X *and* Y *are smooth, connected, projective varieties, with* Y *a subvariety of* X, *and where* $\mathrm{Bl}_Y X$ *is the blow-up of* X *along* Y, *with exceptional divisor* E.

We do not give the proof here, but only mention that the main ingredient is the following Weak Factorization Theorem of Abramovich, Karu, Matsuki, and Włodarczyk.

Theorem 1.11. ([AKMW]) *If k is an algebraically closed field of characteristic zero, then every birational map between two smooth projective varieties over k can be realized as a composition of blow-ups and blow-downs of smooth irreducible centers on smooth projective varieties.*

Example 1.12. Let us show that if k is algebraically closed, of characteristic zero, then there is a (unique) Euler-Poincaré characteristic Q with values in $\mathbf{Z}[t]$ such that for every smooth projective variety X, we have

$$Q(X,t) = \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, \mathcal{O}_X) t^i.$$

By Theorem 1.10, it is enough to show that if X and Y are smooth, connected, projective varieties, with Y a closed subvariety of X, and if W is the blow-up of X along Y, with exceptional divisor E, then $Q(W,t) - Q(E,t) = Q(X,t) - Q(Y,t)$. If $p\colon W \to X$ and $q\colon E \to Y$ are the natural projections, then $R^i p_*(\mathcal{O}_W) = 0$, and $R^i q_*(\mathcal{O}_E) = 0$ for all $i > 0$, while $p_*(\mathcal{O}_W) = \mathcal{O}_X$ and $q_*(\mathcal{O}_E) = \mathcal{O}_Y$. We thus have isomorphisms

$$H^j(X, \mathcal{O}_X) \simeq H^j(W, \mathcal{O}_W), \;\; H^j(Y, \mathcal{O}_Y) \simeq H^j(E, \mathcal{O}_E)$$

for all $j \geq 0$, which imply $Q(W,t) = Q(X,t)$ and $Q(E,t) = Q(Y,t)$.

Example 1.13. A more refined example of an Euler-Poincaré characteristic is given by the Hodge-Deligne polynomial of algebraic varieties. This is an Euler-Poincaré characteristic of varieties over an algebraically closed field k of characteristic zero that takes values in $\mathbf{Z}[u,v]$, such that for a smooth projective variety X, $E(X,u,v)$ is the Hodge polynomial

$$\sum_{p,q=0}^{\dim(X)} (-1)^{p+q} h^{p,q}(X) u^p v^q,$$

where $h^{p,q}(X) = h^q(X, \Omega_X^p)$. Note that with the notation in the previous example, we have $Q(X,t) = E(X,0,t)$. The original definition of the Hodge-Deligne polynomial (over $\mathbf{C}$) uses the mixed Hodge structure on the singular cohomology with compact supports of complex algebraic varieties. It would be nice to give an elementary argument using Theorem 1.10, as in the previous example.

The polynomial $P_{\mathrm{vir}}(X,t) := E(X,t,t)$ is the *virtual Poincaré polynomial* of X. Note that if $k = \mathbf{C}$, then this polynomial is characterized by the fact that it induces a group homomorphism $K_0(\mathrm{Var}/\mathbf{C}) \to \mathbf{Z}[t]$, and if X is a smooth projective variety, then $P_{\mathrm{vir}}(X,t)$ is the usual Poincaré polynomial of X, given by $\sum_{i \geq 0} (-1)^i \dim_{\mathbf{Q}} H^i(X, \mathbf{Q}) t^i$ (this is a consequence of the Hodge decomposition for smooth projective varieties). In particular, we see that $P_X(1) = \chi_{\mathrm{top}}(X)$.

Exercise 1.14. *Use the Künneth formula to show that the Hodge-Deligne polynomial is a multiplicative Euler-Poincaré characteristic.*

Example 1.15. If $X = \mathbf{P}^1$, then $h^{0,0}(X) = h^{1,1}(X) = 1$ and $h^{1,0}(X) = h^{0,1}(X) = 0$, hence $E(\mathbf{P}^1, u, v) = 1 + uv$, and therefore $E(\mathbf{A}^1, u, v) = E(\mathbf{P}^1, u, v) - E(\operatorname{Spec} k, u, v) = uv$. It follows from the previous exercise that $E(\mathbf{A}^n, u, v) = (uv)^n$.

Remark 1.16. Recall that if X is a smooth projective complex variety, then we have the following symmetry of the Hodge numbers: $h^{p,q}(X) = h^{q,p}(X)$. This implies that $E(Y, u, v) = E(Y, v, u)$ for every variety over an algebraically closed field of characteristic zero.

Exercise 1.17. *Let k be an algebraically closed field of characteristic zero. Show that if X is a variety iver k, then $E(X, u, v)$ is a polynomial of degree $2\dim(X)$, and the term of maximal degree is $m(uv)^{\dim(X)}$, where m is the number of irreducible components of X of maximal dimension.*

Exercise 1.18. *Show that if X and Y are varieties over a field k such that $[X] = [Y]$ in $K_0(\operatorname{Var}/k)$, then $\dim(X) = \dim(Y)$. Hint: in characteristic zero, one can use the previous exercise; in positive characteristic, reduce to the case $k = \overline{\mathbf{F}}_p$, and then use the Lang-Weil estimates (in fact, the characteristic zero case can also be reduced to positive characteristic).*

As an application of Bittner's result, we give a proof of a result of Larsen and Lunts [LL2] (see also [Sa]), relating the Grothendieck group of varieties with stable birational geometry.

We keep the assumption that k is an algebraically closed field of characteristic zero. Recall that two irreducible varieties X and Y are *stably birational* if $X \times \mathbf{P}^m$ and $Y \times \mathbf{P}^n$ are birational for some $m, n \geq 0$.

Let SB/k denote the set of stably birational equivalence classes of irreducible algebraic varieties over k. We denote the class of X in SB/k by $\langle X \rangle$. Note that SB/k is a commutative semigroup, with multiplication induced by $\langle X \rangle \cdot \langle Y \rangle = \langle X \times Y \rangle$. Of course, the identity element is $\operatorname{Spec} k$.

Let us consider the semigroup algebra $\mathbf{Z}[\operatorname{SB}/k]$ associated to the semigroup SB/k.

Proposition 1.19. *There is a unique ring homomorphism $\Phi \colon K_0(\operatorname{Var}/k) \to \mathbf{Z}[\operatorname{SB}/k]$ such that $\Phi([X]) = \langle X \rangle$ for every smooth, connected, projective variety X over k.*

Proof. Uniqueness is a consequence of Lemma 1.9. In order to prove the existence of a group homomorphism Φ as in the proposition, we apply Theorem 1.10. This shows that it is enough to check that whenever X and Y are smooth, connected, projective varieties, with Y a closed subvariety of X, we have

$$\langle \operatorname{Bl}_Y(X) \rangle - \langle E \rangle = \langle X \rangle - \langle Y \rangle,$$

where $\operatorname{Bl}_Y X$ is the blow-up of X along Y, and E is the exceptional divisor. In fact, we have $\langle X \rangle = \langle \operatorname{Bl}_Y(X) \rangle$ since X and $\operatorname{Bl}_Y(X)$ are birational, and $\langle Y \rangle = \langle E \rangle$, since E is birational to $Y \times \mathbf{P}_k^{r-1}$, where $r = \operatorname{codim}_X(Y)$.

In order to check that Φ is a ring homomorphism, it is enough to show that $\Phi(uv) = \Phi(u)\Phi(v)$, where u and v vary over a system of group generators of $K_0(\operatorname{Var}/k)$.

By Lemma 1.9, we may take this system to consist of classes of smooth, connected, projective varieties, in which case the assertion is clear. $\square$

Since $\langle \mathbf{P}_k^1 \rangle = \langle \operatorname{Spec} k \rangle$, it follows that $\Phi(\mathbf{L}) = 0$, hence Φ induces a ring homomorphism

$$\overline{\Phi} \colon K_0(\operatorname{Var}/k)/(\mathbf{L}) \to \mathbf{Z}[\mathrm{SB}].$$

Theorem 1.20. ([LL2]) *The above ring homomorphism $\overline{\Phi}$ is an isomorphism.*

Proof. The key point is to show that we can define a map $\mathrm{SB}/k \to K_0(\operatorname{Var}/k)/(\mathbf{L})$ such that whenever X is a smooth, connected, projective variety, $\langle X \rangle$ is mapped to $[X] \bmod (\mathbf{L})$. Note first that by Hironaka's theorem on resolution of singularities, for every irreducible variety Y over k, there is a nonsingular, irreducible, projective variety X that is isomorphic to Y. In particular $\langle X \rangle = \langle Y \rangle$. We claim that if X_1 and X_2 are stably birational nonsingular, irreducible, projective varieties, then $[X] - [Y] \in (\mathbf{L})$.

Suppose that $X_1 \times \mathbf{P}^m$ and $X_2 \times \mathbf{P}^n$ are birational. It follows from Theorem 1.11 that $X_1 \times \mathbf{P}^m$ and $X_2 \times \mathbf{P}^n$ are connected by a chain of blow-ups and blow-downs with smooth centers. Note that

$$[X_1] - [X_1 \times \mathbf{P}^m] = -[X_1] \cdot \mathbf{L}(1 + \mathbf{L} + \ldots + \mathbf{L}^{m-1}) \in (\mathbf{L}).$$

Similarly, we have $[X_2] - [X_2 \times \mathbf{P}^n] \in (\mathbf{L})$. Therefore in order to prove our claim, it is enough to show the following: if Z and W are smooth, connected, projective varieties, with Z a closed subvariety of W, then $[\mathrm{Bl}_Z W] - [W] \in (\mathbf{L})$, where $\mathrm{Bl}_Z(W)$ is the blow-up of W along Z. Let $r = \operatorname{codim}_W(Z)$, and let E be the exceptional divisor, so $E \simeq \mathbf{P}(N)$, where N is the normal bundle of Z in W. Our assertion now follows from

$$[\mathrm{Bl}_Z(W)] - [W] = [E] - [Z] = [E \cdot \mathbf{P}^{r-1}] - [E] = [E] \cdot \mathbf{L}(1 + \mathbf{L} + \ldots + \mathbf{L}^{r-2}).$$

We thus get a group homomorphism $\Psi \colon \mathbf{Z}[\mathrm{SB}/k] \to K_0(\operatorname{Var}/k)/(\mathbf{L})$ such that $\Psi(\langle X \rangle) = [X] \bmod (\mathbf{L})$ for every smooth, connected, projective variety X. It is clear that $\overline{\Phi}$ and Ψ are inverse maps, which proves the theorem. $\square$

We end this section by mentioning the following result of Poonen [Po]:

Theorem 1.21. *If k is a field of characteristic zero, then $K_0(\operatorname{Var}/k)$ is not a domain.*

Sketch of proof. Let $\overline{k}$ denote an algebraic closure of k. We denote by $\mathrm{AV}/\overline{k}$ the semigroup of isomorphism classes of abelian varieties over $\overline{k}$ (with the product given again by Cartezian product). Note that we have a morphism of semigroups $\mathrm{SB}/\overline{k} \to \mathrm{AB}/\overline{k}$, that takes $\langle X \rangle$ to $\mathrm{Alb}(X)$ for every smooth, connected, projective variety X over $\overline{k}$, where $\mathrm{Alb}(X)$ is the Albanese variety of X. Indeed, arguing as in the proof of Theorem 1.20, we see that it is enough to show that $\mathrm{Alb}(X) = \mathrm{Alb}(X \times \mathbf{P}^n)$ and $\mathrm{Alb}(X') = \mathrm{Alb}(X)$ if $X' \to X$ is the blow-up of the smooth, connected, projective variety X along a smooth closed subvariety. Both assertions follow from the fact that any rational map $\mathbf{P}^m \dashrightarrow A$, where A is an abelian variety, is constant. Therefore we have ring homomorphisms

$$K_0(\operatorname{Var}/k) \to K_0(\operatorname{Var}/\overline{k}) \to \mathbf{Z}[\mathrm{SB}/\overline{k}] \to \mathbf{Z}[\mathrm{AV}/\overline{k}].$$

The technical result in [Po] says that there are abelian varieties A and B over k such that $A \times A \simeq B \times B$, but $A_{\overline{k}} \not\simeq B_{\overline{k}}$. In this case $([A] - [B])([A] + [B]) = 0$ in $K_0(\mathrm{Var}/k)$. However, both $[A] - [B]$ and $[A] + [B]$ are nonzero in $K_0(\mathrm{Var}/k)$, since their images in $\mathbf{Z}[\mathrm{AV}/\overline{k}]$ are nonzero. Hence $K_0(\mathrm{Var}/k)$ is not a domain. $\qquad\square$

Remark 1.22. Note that the zero-divisors constructed in the proof of the above theorem are nonzero in $K_0(\mathrm{Var}/\overline{k})/(\mathbf{L})$. This suggests that the localization $K_0(\mathrm{Var}/k)[\mathbf{L}^{-1}]$ might still be a domain, but this is an open question.

2. Symmetric products of varieties and Kapranov's motivic zeta function

We begin by recalling the definition of the symmetric products of an algebraic variety. For simplicity we work over a perfect field k. Let X be a quasiprojective variety over k. For every $n \geq 1$, we have a natural action of the symmetric group S_n on the product X^n. Since X^n is again quasiprojective, by the results in the Appendix, we may construct the quotient by the action of S_n. This is the *symmetric product* $\mathrm{Sym}^n(X)$. We make the convention that $\mathrm{Sym}^0(X)$ is $\mathrm{Spec}\, k$. Note that since k is perfect, X^n is reduced, hence $\mathrm{Sym}^n(X)$ is reduced too.

Example 2.1. For every $n \geq 1$, there is an isomorphism $\mathrm{Sym}^n(\mathbf{A}_k^1) \simeq \mathbf{A}_k^n$. Indeed, the ring of symmetric polynomials $k[x_1, \ldots, x_n]^{S_n} \subseteq k[x_1, \ldots, x_n]$ is generated as a k-algebra by the elementary symmetric functions $e_1, \ldots, e_n$. Note that since $\dim(k[x_1, \ldots, x_n]^{S_n}) = n$, the polynomials $e_1, \ldots, e_n$ are algebraically independent over k, hence $\mathrm{Sym}^n(\mathbf{A}_k^1) \simeq \mathbf{A}_k^n$.

Remark 2.2. Note that by Remark 1.5, for every field extension K/k (say, with K perfect), we have $\mathrm{Sym}^n(X) \times_k K \simeq \mathrm{Sym}^n(X \times_k K)$. In particular, if K is algebraically closed, then $\mathrm{Sym}^n(X)(K)$ is in bijection with the set of effective zero-cycles on $X \times_k K$ of degree n.

In order to define Kapranov's motivic zeta function [Kap], we need some preparations. We will work with the quotient $\widetilde{K}_0(\mathrm{Var}/k)$ of $K_0(\mathrm{Var}/k)$ by the subgroup generated by the relations $[X] - [Y]$, where we have a radicial surjective morphism $X \to Y$ of varieties over k. See Appendix, §3, for a review of radicial morphisms. Note that in fact $\widetilde{K}_0(\mathrm{Var}/k)$ is a quotient ring of $K_0(\mathrm{Var}/k)$: this follows from the fact that if $f \colon X \to Y$ is surjective and radicial, then for every variety Z, the morphism $f \times \mathrm{Id}_Z \colon X \times Z \to Y \times Z$ is surjective and radicial (since $f \times \mathrm{Id}_Z$ is the base-change of f with respect to the projection $Y \times Z \to Y$).

Proposition 2.3. *If* $\mathrm{char}(k) = 0$, *then the canonical morphism* $K_0(\mathrm{Var}/k) \to \widetilde{K}_0(\mathrm{Var}/k)$ *is an isomorphism.*

Proof. This is a consequence of the fact that if $\mathrm{char}(k) = 0$ and $f \colon X \to Y$ is radicial and surjective, then f is a piecewise isomorphism (see Appendix, Proposition 3.7), hence $[X] = [Y]$ in $K_0(\mathrm{Var}/k)$. $\qquad\square$

Proposition 2.4. *If* $k = \mathbf{F}_q$ *is a finite field, then the ring homomorphism* $K_0(\mathrm{Var}/k) \to \mathbf{Z}$ *given by* $[X] \to |X(\mathbf{F}_{q^e})|$ *factors through* $\widetilde{K}_0(\mathrm{Var}/k)$.

Proof. We need to show that if $f\colon X \to Y$ is a radicial, surjective morphism of varieties over $\mathbf{F}_q$, then $|X(\mathbf{F}_{q^e})| = |Y(\mathbf{F}_{q^e})|$. This is a consequence of the fact that f gives a bijection between the closed points of X and Y, such that for every $x \in X_{\mathrm{cl}}$ we have $k(f(x)) = k(x)$ (for this it is enough to note that $k(f(x))$ is a finite field, hence perfect, and therefore it has no nontrivial purely inseparable extensions). $\qquad\square$

The next proposition shows that the Grothendieck group of varieties over k can be described in terms of quasiprojective varieties. Let $K_0^{\mathrm{qpr}}(\mathrm{Var}/k)$ be the quotient of the free abelian group on the set of isomorphism classes of quasiprojective varieties over k, modulo the relations

$$[X] = [Y] + [X \smallsetminus Y],$$

where X is a quasiprojective variety and Y is a closed subvariety of X. It is clear that we have a group homomorphism $\Phi\colon K_0^{\mathrm{qpr}}(\mathrm{Var}/k) \to K_0(\mathrm{Var}/k)$, such that $\Phi([X]) = [X]$. We similarly define $\widetilde{K}^{\mathrm{qpr}}(\mathrm{Var}/k)$ as the quotient of $K_0^{\mathrm{qpr}}(\mathrm{Var}/k)$ by the relations $[X] - [Y]$, where we have a surjective, radicial morphism of quasiprojective varieties $f\colon X \to Y$. We have a corresponding group homomorphism $\widetilde{\Phi}\colon \widetilde{K}_0^{\mathrm{qpr}}(\mathrm{Var}/k) \to \widetilde{K}_0(\mathrm{Var}/k)$.

Proposition 2.5. *Both Φ and $\widetilde{\Phi}$ are isomorphisms.*

Proof. Let us define an inverse homomorphism $\Psi\colon K_0(\mathrm{Var}/k) \to K_0^{\mathrm{qpr}}(\mathrm{Var}/k)$. Given a variety X over k, we consider a disjoint decomposition $X = V_1 \sqcup \ldots \sqcup V_r$, where each V_i is quasiprojective and locally closed in X (for example, we may even take the V_i to be affine). In this case, we define $\Psi([X]) = \sum_{i=1}^r [V_i] \in K_0^{\mathrm{qpr}}(\mathrm{Var}/k)$.

We need to show that the definition is independent of the decomposition we choose. Suppose that $X = W_1 \sqcup \ldots \sqcup W_s$ is another such decomposition. We get a corresponding decomposition $X = \bigsqcup_{i,j}(V_i \sqcup W_j)$. We have an obvious analogue of Proposition 1.1 for $K_0^{\mathrm{qpr}}(\mathrm{Var}/k)$, hence

$$[V_i] = \sum_{j=1}^s [V_i \cap W_j] \text{ and } [W_j] = \sum_{i=1}^r [V_i \cap W_j] \text{ in } K_0^{\mathrm{qpr}}(\mathrm{Var}/k).$$

This gives the following equalities in $K_0^{\mathrm{qpr}}(\mathrm{Var}/k)$:

$$\sum_{i=1}^r [V_i] = \sum_{i=1}^r \sum_{j=1}^s [V_i \cap W_j] = \sum_{j=1}^s \sum_{i=1}^r [V_i \cap W_j] = \sum_{j=1}^s [W_j].$$

Therefore $\Psi([X])$ is well-defined.

Suppose now that Y is a closed subvariety of X, and consider a decomposition $X = V_1 \sqcup \ldots \sqcup V_r$ for X as above. If $U = X \smallsetminus Y$, we get corresponding decompositions

$$Y = \bigsqcup_{i=1}^r (V_i \cap Y), \quad U = \bigsqcup_{i=1}^r (V_i \cap U),$$

from which we get that $\Psi([X]) = \Psi([Y]) + \Psi([U])$. Therefore Ψ gives a group homomorphism $K_0(\mathrm{Var}/k) \to K_0^{\mathrm{qpr}}(\mathrm{Var}/k)$, and it is clear that Φ and Ψ are inverse to each other.

In order to show that Ψ induces an inverse to $\widetilde{\Phi}$, it is enough to show that if $f\colon X \to Y$ is a surjective, radicial morphism, then there is a disjoint decomposition $Y = V_1 \sqcup \ldots \sqcup V_r$ such that all V_i and $f^{-1}(V_i)$ are quasiprojective (note that each $f^{-1}(V_i) \to V_i$ is automatically radicial and surjective). Arguing by Noetherian induction, it is enough to show that there is an affine open subset $V \subseteq Y$ such that $f^{-1}(V)$ is affine. If $Y_1, \ldots, Y_m$ are the irreducible components of Y, we may replace Y by $Y_1 \smallsetminus \bigcup_{i \geq 2} Y_i$, and therefore assume that Y is irreducible. Since f is bijective, there is only one irreducible component of X that dominates Y, hence after restricting to a suitable open subset of Y, we may assume that both X and Y are irreducible. In this case there is an open subset V of Y such that $f^{-1}(V)_{\mathrm{red}} \to V_{\mathrm{red}}$ is a finite morphism (see [Har, Exercise II.3.7]). We may assume that V is affine, in which case $f^{-1}(V)_{\mathrm{red}}$ is affine, hence $f^{-1}(V)$ is affine by [Har, Exercise III.3.1]. This completes the proof of the proposition. $\qquad\square$

For every quasiprojective variety over X, the *Kapranov zeta function* of X is

$$Z_{\mathrm{mot}}(X,t) = \sum_{n \geq 0} [\mathrm{Sym}^n(X)] t^n \in 1 + t \cdot \widetilde{K}_0(\mathrm{Var}/k)[t].$$

Proposition 2.6. *The map* $[X] \to Z_{\mathrm{mot}}(X,t)$, *for* X *quasiprojective, defines a group homomorphism*

$$K_0(\mathrm{Var}/k) \to (1 + t\widetilde{K}_0(\mathrm{Var}/k)[\![t]\!], \cdot),$$

which factors through $\widetilde{K}_0(\mathrm{Var}/k)$.

The key ingredient is provided by the following lemma.

Lemma 2.7. *If* X *is a quasiprojective variety, and* $Y \hookrightarrow X$ *is a closed subvariety with complement* U, *then*

$$[\mathrm{Sym}^n(X)] = \sum_{i+j=n} [\mathrm{Sym}^i(Y)] \cdot [\mathrm{Sym}^j(U)] \text{ in } \widetilde{K}_0(\mathrm{Var}/k).$$

Proof. For nonnegative i and j with $i + j = n$, we denote by $W^{i,j}$ the locally closed subset of X^n given by $\bigcup_{g \in S_n} (Y^i \times U^j)g$. The $W^{i,j}$ give a disjoint decomposition of X^n by locally closed subvarieties preserved by the S_n-action (in order to show that these sets are disjoint and cover X^n, it is enough to consider the $\overline{k}$-rational points, where $\overline{k}$ is an algebraic closure of k). If $\pi\colon X^n \to \mathrm{Sym}^n(X)$ is the quotient morphism, it follows that the locally closed subvarieties $\pi(W^{i,j})$ give a disjoint decomposition of $\mathrm{Sym}^n(X)$ in locally closed subsets, hence

$$(4) \qquad\qquad [\mathrm{Sym}^n(X)] = \sum_{i+j=n} [\pi(W^{i,j})] \text{ in } K_0(\mathrm{Var}/k).$$

For every pair (i,j) as above, consider the open subset $Y^i \times U^j$ of $W^{i,j}$. For every $g, h \in S_n$, the subsets $(Y^i \times U^j)g$ and $(Y^i \times U^j)h$ of $W^{i,j}$ are either equal, or disjoint. Note also that the subgroup H consisting of all $g \in G$ such that $(Y^i \times U^j)g = Y^i \times U^j$ is equal to $S_i \times S_j \subseteq S_n$. We may therefore apply Propositions 1.8 and 1.7 in the Appendix to conclude that we have an isomorphism

$$W^{i,j}/S_n \simeq \mathrm{Sym}^i(Y) \times \mathrm{Sym}^j(U).$$

On the other hand, Proposition 4.1 in the Appendix implies that the induced morphism $W^{i,j}/S_n \to \pi(W^{i,j})$ is radicial and surjective, hence

$$[\pi(W^{i,j})] = [(W^{i,j}/S_n)] = [\mathrm{Sym}^i(Y)] \cdot [\mathrm{Sym}^j(U)] \text{ in } \widetilde{K}_0(\mathrm{Var}/k).$$

Using this and (4), we obtain the statement in the lemma. $\qquad\square$

Proof of Proposition 2.6. It follows from the lemma that if X is a quasiprojective variety, Y is a closed subvariety of X, and $U = X \smallsetminus Y$, then

$$Z_{\mathrm{mot}}(X, t) = \sum_{n \geq 0}[\mathrm{Sym}^n(X)]t^n = \sum_{n \geq 0}\sum_{i+j=n}[\mathrm{Sym}^i(Y)] \cdot [\mathrm{Sym}^j(U)]t^{i+j} = Z_{\mathrm{mot}}(Y, t) \cdot Z_{\mathrm{mot}}(U, t).$$

In light of Proposition 2.5, this proves the first assertion in the proposition. For the second assertion, it is enough to show that if $f \colon X \to Y$ is a surjective, radicial morphism of varieties over k, then the induced morphism $\mathrm{Sym}^n(f) \colon \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(Y)$ is radicial and surjective for every $n \geq 1$. It is easy to see that the surjectivity of f implies that $X^n \to Y^n$ is surjective, and since $Y^n \to \mathrm{Sym}^n(Y)$ is surjective, we deduce that $\mathrm{Sym}^n(f)$ is surjective. In order to show that $\mathrm{Sym}^n(f)$ is radicial, it is enough to prove the injectivity of

$$(5) \qquad\qquad \mathrm{Hom}(\mathrm{Spec}\, K, \mathrm{Sym}^n(X)) \to \mathrm{Hom}(\mathrm{Spec}\, K, \mathrm{Sym}^n(Y))$$

for every algebraically closed extension K of k. Using Remark 2.2, we may identify $\mathrm{Hom}(\mathrm{Spec}\, K, \mathrm{Sym}^n(X))$ with the quotient of $X(K)^n$ by the action of S_n. A similar description holds for $\mathrm{Hom}(\mathrm{Spec}\, K, \mathrm{Sym}^n(Y))$, and the injectivity of $X(K) \to Y(K)$ implies the injectivity of (5). This completes the proof of the proposition. $\qquad\square$

Remark 2.8. If X is not necessarily perfect, then we may still define the motivic zeta function of a quasiprojective variety X by considering the reduced scheme corresponding to X^n/S_n. All results in this section carry through in that setting. We preferred to make the assumption that k is perfect in order to simplify the exposition, since we are mostly interested in the case when k is either a finite field, or it has characteristic zero.

As a consequence of Proposition 2.6, we can define $Z_{\mathrm{mot}}(X, t)$ for a variety over k that is not necessarily quasiprojective. Indeed, we just apply the morphism in that proposition to $[X] \in \widetilde{K}_0(\mathrm{Var}/k)$.

As we have seen in Proposition 2.4, when $k = \mathbf{F}_q$ is a finite field, we have a specialization map $\widetilde{K}_0(\mathrm{Var}/k) \to \mathbf{Z}$ given by counting the number of $\mathbf{F}_q$-rational points. The following proposition shows that if we apply this specialization to Kapranov's motivic zeta function, we recover the Hasse-Weil zeta function.

Proposition 2.9. *If k is a finite field, and X is a variety over k, then the image of $Z_{\mathrm{mot}}(X, t)$ in $1 + t\mathbf{Z}[\![t]\!]$ is equal to $Z(X, t)$.*

Proof. We may clearly assume that X is quasiprojective. By Remark 3.4 in Lecture 2, it is enough to show that for every $n \geq 1$, the number of effective 0-cycles on X of degree n is equal to $|\mathrm{Sym}^n(X)(k)|$. We have $\mathrm{Sym}^n(X)_{\overline{k}} \simeq \mathrm{Sym}^n(X_{\overline{k}})$ by Remark 2.2. Note that if $g \in G = G(\overline{k}/k)$ acts on $X_{\overline{k}}$ by σ, then g acts on $\mathrm{Sym}^n(X)_{\overline{k}}$ by $\mathrm{Sym}^n(\sigma)$. We can identify

$\mathrm{Sym}^n(X)(k)$ with the points of $\mathrm{Sym}^n(X)(\overline{k}) = X(\overline{k})^n/S_n$ that are fixed by all $g \in G$. An element of $X(\overline{k})^n/S_n$ corresponds to an effective 0-cycle of degree n on $X_{\widetilde{k}}$, and this is fixed by every $g \in G$ if and only if it corresponds to an effective cycle of degree n on X (see Proposition 2.5 in the Appendix). This completes the proof of the proposition. $\square$

Proposition 2.10. *If X is a variety over k, then $Z_{\mathrm{mot}}(X \times \mathbf{A}_k^n, t) = Z_{\mathrm{mot}}(X, \mathbf{L}^n t)$, where $\mathbf{L} = [\mathbf{A}_k^1]$.*

Proof. We only sketch the argument, which is due to Totaro [Go, Lemma 4.4]. It is enough to prove the assertion when X is quasiprojective. Arguing by induction on n, it follows that it is enough to prove the case $n = 1$. We need to show that for every $n \geq 1$, we have $[\mathrm{Sym}^n(X \times \mathbf{A}_k^1)] = [\mathrm{Sym}^n(X)] \cdot \mathbf{L}^n$ in $\widetilde{K}_0(\mathrm{Var}/k)$.

We start by describing a general decomposition into locally closed subsets of $\mathrm{Sym}^n(X)$. For every $r \geq 1$, we denote by $(X^r)^\circ$ the complement in X^r of the union of the (big) diagonals (when $r = 1$, this is simply X). Given positive integers $d_1, \ldots, d_r$ with $d_1 \leq \ldots \leq d_r$ and $\sum_{i=1}^r d_i = n$, consider the locally closed embedding $(X^r)^\circ \hookrightarrow X^n$ given by $\Delta_{d_1} \times \ldots \times \Delta_{d_r}$, where $\Delta_i \colon X \to X^i$ is the diagonal embedding. We denote the image of $(X^r)^\circ$ by $X_{d_1,\ldots,d_r}$. It is clear that for every $\sigma, \tau \in S_n$, the subsets $X_{d_1,\ldots,d_r}\sigma$ and $X_{d_1,\ldots,d_r}\tau$ are either disjoint, or equal. We may thus apply Propositions 1.8 and 4.1 in the Appendix to deduce that if $H = \{g \in G \mid X_{d_1,\ldots,d_r}g = X_{d_1,\ldots,d_r}\}$, then $X_{d_1,\ldots,d_r}/H$ has a radicial morphism onto its image in $\mathrm{Sym}^n(X)$, that we denote by $\widetilde{X}_{d_1,\ldots,d_r}$. It is clear that when we consider all tuples $(d_1, \ldots, d_r)$ as above, the $\widetilde{X}_{d_1,\ldots,d_r}$ give a partition of $\mathrm{Sym}^n(X)$ into locally closed subsets (consider, for example, the $\overline{k}$-valued points, where $\overline{k}$ is an algebraic closure of k).

Suppose that $m_1 < m_2 < \ldots < m_s$ are such that the first ℓ_1 of the d_i are equal to m_1, the next ℓ_2 of the d_i are equal to m_2, and so on. In this case $H = H_1 \rtimes H_2$, where $H_1 = \prod_{i=1}^r S_{d_i}$ and $H_2 = \prod_{j=1}^s S_{\ell_j}$. Each S_{d_i} acts by permuting the entries of X^n in the slots $d_1 + \ldots + d_{i-1} + 1, \ldots, d_1 + \ldots + d_i$, while S_{ℓ_j} permutes the ℓ_j sets of m_j entries of X^n. Note that H_1 acts trivially on $X_{d_1,\ldots,d_r}$, hence $X_{d_1,\ldots,d_r}/H = X_{d_1,\ldots,d_r}/H_2$.

We now consider the inverse image $W_{d_1,\ldots,d_r} = X_{d_1,\ldots,d_r} \times \mathbf{A}_k^n$ of $X_{d_1,\ldots,d_r}$ in $(X \times \mathbf{A}_k^1)^n$, as well as its image $\widetilde{W}_{d_1,\ldots,d_r}$ in $\mathrm{Sym}^n(X \times \mathbf{A}_k^1)$. As above, we have a surjective, radicial morphism $W_{d_1,\ldots,d_r}/H \to \widetilde{W}_{d_1,\ldots,d_r}$. In order to complete the proof of the proposition, it is enough to show that $[W_{d_1,\ldots,d_r}/H] = [(X_{d_1,\ldots,d_r}/H) \times \mathbf{A}_k^n]$ in $\widetilde{K}_0(\mathrm{Var}/k)$.

It follows from Proposition 1.10 in the Appendix that $W_{d_1,\ldots,d_r}/H \simeq (W_{d_1,\ldots,d_r}/H_1)/H_2$. On the other hand, Proposition 1.7 in the Appendix and Example 2.1 give an isomorphism $W_{d_1,\ldots,d_r}/H_1 \simeq X_{d_1,\ldots,d_r} \times \prod_{i=1}^r \mathbf{A}_k^{d_i} = X_{d_1,\ldots,d_r} \times \prod_{j=1}^s (\mathbf{A}_k^{m_j})^{\ell_j} = X_{d_1,\ldots,d_r} \times \mathbf{A}_k^n$. One can show that since H_2 acts without fixed points on $X_{d_1,\ldots,d_r}$, the projection $\pi \colon X_{d_1,\ldots,d_r} \to X_{d_1,\ldots,d_r}/H_2$ is étale, and we have a Cartezian diagram

$$
\begin{array}{ccc}
X_{d_1,\ldots,d_r} \times \mathbf{A}_k^n & \longrightarrow & (W_{d_1,\ldots,d_r}/H_1)/H_2 \\
\downarrow & & \downarrow{\varphi} \\
X_{d_1,\ldots,d_r} & \longrightarrow & X_{d_1,\ldots,d_r}/H_2.
\end{array}
$$

One can show using this that φ has a structure of rank n vector bundle locally trivial in the étale topology, and by Hilbert's Theorem 90 [Se, p. 1.24], this is locally trivial also in the Zariski topology. This gives $[W_{d_1,\dots,d_r}/H] = [X_{d_1,\dots,d_r}/H_2] \cdot \mathbf{L}^n$ in $K_0(\mathrm{Var}/k)$. $\qquad\square$

3. Rationality of the Kapranov zeta function for curves

Our goal in this section is to prove a result of Kapranov [Kap], extending the rationality of the Hasse-Weil zeta function for smooth, geometrically connected, projective curves defined over finite fields to motivic zeta functions.

Since the Kapranov zeta function does not have coefficients in a field, there are (at least) two possible notions of rationality that can be considered. If R is a commutative ring and $f \in R[\![t]\!]$, we say that f is *rational* if there are polynomials $u, v \in R[t]$, with v invertible in $R[\![t]\!]$ such that $f(t) = \frac{u(t)}{v(t)}$. We say that f is *pointwise rational* if for every morphism $R \to K$, where K is a field, the image of f in $K[\![t]\!]$ is rational. It is clear that a rational formal power series is also pointwise rational. The formal power series we will consider satisfy $f(0) = 1$, hence the image in every $K[\![t]\!]$ as above is nonzero. Of course, when R is a field, then the two notions of rationality coincide.

Theorem 3.1. *Let k be a perfect field. If X is a smooth, geometrically connected, projective curve of genus g over k, then $Z_{\mathrm{mot}}(X,t)$ is a rational function. If k is finite or algebraically closed, then*

$$Z_{\mathrm{mot}}(X,t) = \frac{f(t)}{(1-t)(1-\mathbf{L}t)},$$

for a polynomial f of degree $\leq 2g$ with coefficients in $\widetilde{K}_0(\mathrm{Var}/k)$.

Proof. Recall that for every $d \geq 0$ we have a morphism $\mathrm{Sym}^d(X) \to \mathrm{Pic}^d(X)$. This can be defined using the universal property of $\mathrm{Pic}^d(X)$, but let us describe it at the level of $\overline{k}$-valued points, where $\overline{k}$ is an algebraic closure of k. A $\overline{k}$-valued point of $\mathrm{Sym}^d(X)$ corresponds to an effective divisor D on $X_{\overline{k}}$ of degree d. On the other hand, a $\overline{k}$-valued point of $\mathrm{Pic}^d(X)$ corresponds to a line bundle on $X_{\overline{k}}$ of degree d, and the above map takes D to $\mathcal{O}_X(D)$. If $d \geq 2g-1$, then the fiber of $\mathrm{Sym}^d(X)_{\overline{k}} \to \mathrm{Pic}^d(X)_{\overline{k}}$ over L is naturally isomorphic to the linear system $|L| \simeq \mathbf{P}_{\overline{k}}^{d-g}$. In fact, there is an isomorphism $\mathrm{Sym}^d(X) \simeq \mathbf{P}(E)$, where E is a vector bundle on $\mathrm{Pic}^d(X)$ of rank $d-g+1$.

Let e be the smallest positive integer such that there is a line bundle of degree e on X. In this case we have $\mathrm{Pic}^d(X) \simeq \mathrm{Pic}^{d'}(X)$ if $d \equiv d' \pmod{e}$. It follows from definition and the above discussion that

$$Z_{\mathrm{mot}}(X,t) = \sum_{d \geq 0}[\mathrm{Sym}^d(X)]t^d = \sum_{0 \leq d \leq 2g-2}[\mathrm{Sym}^d(X)]t^d + \sum_{d \geq \min\{2g-1,0\}}[\mathrm{Pic}^d(X)] \cdot [\mathbf{P}_k^{d-g}]t^d.$$

For $0 \leq i \leq e-1$, let d_i be the smallest $d \geq \min\{2g-1,0\}$ that is congruent to i modulo e. We have

$$S_i := \sum_{j \geq 0}[\mathrm{Pic}^i(X)][\mathbf{P}_k^{d_i+je-g}]t^{d_i+je} = [\mathrm{Pic}^i(X)]\sum_{j \geq 0}\frac{\mathbf{L}^{d_i+je-g+1}-1}{\mathbf{L}-1}t^{d_i+je}$$

$$= [\mathrm{Pic}^i(X)] \left(\frac{\mathbf{L}^{d_i-g+1}t^{d_i}}{(\mathbf{L}-1)(1-\mathbf{L}^e t^e)} - \frac{t^{d_i}}{(\mathbf{L}-1)(1-t^e)} \right)$$

$$= \frac{[\mathrm{Pic}^i(X)]}{(1-t^e)(1-\mathbf{L}^e t^e)} \cdot t^{d_i} \left(\frac{\mathbf{L}^{d_i-g+1}-1}{\mathbf{L}-1} + t^e \frac{\mathbf{L}^e - \mathbf{L}^{d_i-g}}{\mathbf{L}-1} \right).$$

Since the expression in parenthesis is a polynomial with coefficients in $\widetilde{K}_0(\mathrm{Var}/k)$, we see that each S_i is a rational function. We have

$$Z_{\mathrm{mot}}(X,t) = \sum_{0 \le d \le 2g-2} [\mathrm{Sym}^d(X)]t^d + \sum_{0 \le i \le e-1} S_i,$$

hence $Z_{\mathrm{mot}}(X,t)$ is a rational function.

Suppose now that k is either algebraically closed, or a finite field. In this case $e = 1$ (this is clear if k is algebraically closed, and it was proved in Lecture 4 when k is finite), and we get the more precise formula in the statement of the theorem. $\qquad\square$

4. Kapranov zeta function of complex surfaces

In this section we assume that k is an algebraically closed field, and consider the rationality of $Z_{\mathrm{mot}}(X,t)$ when $\dim(X) = 2$, following [LL1] and [LL2].

Proposition 4.1. *If X is a variety over k with $\dim(X) \le 1$, then $Z_{\mathrm{mot}}(X,t)$ is rational.*

Proof. The assertion is clearly true when X is a point, since

$$Z(\mathrm{Spec}\,k,t) = \sum_{n \ge 0} t^n = \frac{1}{1-t},$$

and for a smooth, connected, projective curve it follows from Theorem 3.1. It is easy to deduce the general case in the proposition by taking closures of affine curves in suitable projective spaces, and normalizations of irreducible projective curves. Since we have already given several such arguments, we leave the details as an exercise for the reader. $\qquad\square$

Given a variety X of dimension 2, we consider a decomposition of $X = X_1 \sqcup \ldots \sqcup X_r$, with each X_i irreducible and quasiprojective. Since $Z_{\mathrm{mot}}(X,t) = \prod_{i=1}^r Z_{\mathrm{mot}}(X_i,t)$, we reduce studying the rationality or pointwise rationality of $Z_{\mathrm{mot}}(X,t)$ to that of all $Z_{\mathrm{mot}}(X_i,t)$.

Proposition 4.2. *If X and Y are birational irreducible varieties over k of dimension two, then $Z_{\mathrm{mot}}(X,t)$ is rational (pointwise rational) if and only if $Z_{\mathrm{mot}}(Y,t)$ has the same property.*

Proof. By assumption, there are isomorphic open subsets $U \subseteq X$ and $V \subseteq Y$. We thus have

$$Z_{\mathrm{mot}}(X,t) = Z_{\mathrm{mot}}(Y,t) \frac{Z_{\mathrm{mot}}(X \smallsetminus U, t)}{Z_{\mathrm{mot}}(Y \smallsetminus V)},$$

and both $Z_{\mathrm{mot}}(X \smallsetminus U, t)$ and $Z_{\mathrm{mot}}(Y \smallsetminus V, t)$ are rational by Proposition 4.1. Therefore $Z_{\mathrm{mot}}(X,t)$ is rational (pointwise rational) if and only if $Z_{\mathrm{mot}}(Y,t)$ is. $\qquad\square$

If X is an arbitrary irreducible surface, there is a smooth, connected, projective surface Y such that X is birational to Y. Indeed, resolution of singularities for surfaces holds over fields of arbitrary characteristic.

Therefore from now on we concentrate on smooth, connected, projective surfaces. Let X be such a surface. We start by recalling a fundamental result from classification of surfaces. We refer to [Beau] for the case of complex surfaces, and to [Bad] for the general case. Recall that the Kodaira dimension of X is said to be negative if $H^0(X, \mathcal{O}(mK_X)) = 0$ for al $m \geq 1$. This is a birational property of X. Given any X, there is a morphism $\pi \colon X \to Y$ that is a composition of blow-ups of points on smooth surfaces such that Y is *minimal*, that is, it admits no birational morphism $Y \to Z$, where Z is a smooth surface. By Castelnuovo's criterion for contractibility, this is the case if and only if Y contains no smooth curve $C \simeq \mathbf{P}^1$ with $(C^2) = -1$. A fundamental result in the classification of surfaces says that if X (hence also Y) has negative Kodaira dimension, then Y is birational to $C \times \mathbf{P}^1$, for some smooth curve C.

Proposition 4.3. *If X is a smooth, connected, projective surface of negative Kodaira dimension, then $Z_{\mathrm{mot}}(X, t)$ is a rational power series.*

Proof. It follows from the above discussion that X is birational to $C \times \mathbf{P}^1$ for a smooth curve C, hence by Proposition 4.2 it is enough to show that $Z_{\mathrm{mot}}(C \times \mathbf{A}^1, t)$ is rational. This follows from Proposition 4.1, since $Z_{\mathrm{mot}}(C \times \mathbf{A}^1, t) = Z_{\mathrm{mot}}(C, \mathbf{L}t)$ by Proposition 2.10. $\square$

The following theorem, the main result of [LL1], gives the converse in the case of complex surfaces.

Theorem 4.4. *If X is a smooth, connected, projective complex surface such that $Z_{\mathrm{mot}}(X, t)$ is pointwise rational, then X has negative Kodaira dimension.*

We will not discuss the proof of this result, but in what follows we will sketch the proof of the following earlier, more special result of Larsen and Lunts [LL2]. Recall that if X is a smooth projective variety, its geometric genus is $p_g(X) = h^0(X, \omega_X)$.

Proposition 4.5. *If X is a smooth, connected, projective surface with $p_g(X) \geq 2$, then $Z_{\mathrm{mot}}(X, t)$ is not pointwise rational.*

We start by describing the group homomorphism $K_0(\mathrm{Var}/\mathbf{C}) \to K$ that is used in the proof of Proposition 4.5. Let S denote the multiplicative semigroup of polynomials $h \in \mathbf{Z}[t]$ with $h(0) = 1$. Since the only invertible element in S is 1, and $\mathbf{Z}[t]$ is a factorial ring, every element in S can be written uniquely as $h_1^{n_1} \cdots h_r^{n_r}$, where the h_i are elements in S that generate prime ideals in $\mathbf{Z}[t]$. It follows that the semigroup algebra $\mathbf{Z}[S]$ is a polynomial ring in infinitely many variables. In particular, it is a domain, and we take K to be the fraction field of $\mathbf{Z}[S]$. In order to avoid confusion, we denote by $\varphi(h)$ the element in $\mathbf{Z}[S]$ corresponding to $h \in S$, hence $\varphi(g)\varphi(h) = \varphi(gh)$.

We now define a group homomorphism $\mathrm{SB}/\mathbf{C} \to S$ by taking $\langle X \rangle$, for X smooth, connected, and projective, to $R(X, t) := E(X, t, 0) = \sum_{i=0}^{\dim(X)} (-1)^i h^0(X, \Omega_X^i) t^i \in S$. It is an easy consequence of the Künneth theorem (see Exercise 1.14) that $R(X \times Y, t) =$

$R(X,t) \cdot R(Y,t)$. Note that $R(\mathbf{P}^n, t) = 1$ for all $n \geq 0$. Indeed, using the Hodge symmetry we have $h^0(\mathbf{P}^n, \Omega^i_{\mathbf{P}^n}) = h^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) = 0$ (exercise: give a direct proof using the description of $\Omega_{\mathbf{P}^n}$ provided by the Euler exact sequence). We deduce that $R(X \times \mathbf{P}^n, t) = R(X,t)$. Furthermore, if X and Y are smooth, projective birational varieties, then $h^0(X, \Omega^i_X) = h^0(Y, \Omega^i_Y)$ for all i (see [Har, Theorem II.8.19], whose proof extends to the case $i < \dim(X)$). We conclude that we have a well-defined semigroup homomorphism $\mathrm{SB}/\mathbf{C} \to S$ that takes $\langle X \rangle$ to $R(X,t)$ for every X smooth, connected, and projective. This induces a ring homomorphism $\mathbf{Z}[\mathrm{SB}/\mathbf{C}] \to \mathbf{Z}[S]$.

By Theorem 1.20, we have an isomorphism $K_0(\mathrm{Var}/\mathbf{C})/(\mathbf{L}) \to \mathbf{Z}[\mathrm{SB}/\mathbf{C}]$. We thus have a ring homomorphism $K_0(\mathrm{Var}/\mathbf{C}) \to \mathbf{Z}[S] \hookrightarrow K$, that we denote by μ, which takes $[X]$ to $\varphi(R(X,t))$ for every smooth, connected, projective variety X. Therefore if $X_1, \ldots, X_r$ are such varieties, then $\mu(\sum_{i=1}^r n_i[X_i]) = \sum_{i=1}^r n_i \varphi(E(X_i, t, 0))$.

We emphasize that μ is different from the Euler-Poincaré characteristic that takes X to $E(X, t, 0)$, which takes values in $\mathbf{Z}[t]$. We will see in Lemma 4.6 below that μ recovers more information than this latter Euler-Poincaré characteristic.

Note that if X is a smooth, connected, n-dimensional projective variety, then the degree of $R(X,t)$ is $\leq n$, and the coefficient of t^n is $(-1)^n p_g(X)$. When X is an arbitrary irreducible variety, we will denote by $p_g(X)$ the geometric genus of every smooth, irreducible, projective variety Y that is birational to X. As we have mentioned, this is independent of the choice of Y.

Lemma 4.6. *Suppose that $Y, X_1, \ldots, X_r$ are irreducible varieties of the same dimension, and $n_1, \ldots, n_r$ are integers such that $\mu(Y) = \sum_{i=1}^r n_i \mu(X_i)$. If $p_g(Y) \neq 0$, then there is i such that $p_g(X) = p_g(Y_i)$.*

Proof. It follows from Lemma 1.9 that we can find a smooth, connected, projective variety Y' that is birational to Y, such that $[Y] - [Y']$ is a linear combination, with integer coefficients, of classes of smooth, irreducible, projective varieties of dimension smaller than $n = \dim(Y)$. Applying this also to the X_i, we conclude that we may assume that Y and all X_i are smooth, connected, and projective, and that we have smooth, connected, projective varieties $X'_1, \ldots, X'_s$ of dimension less than n, and $n'_1, \ldots, n'_s \in \mathbf{Z}$ such that

$$\mu(Y) = \sum_{i=1}^r n_i \mu(X_i) + \sum_{j=1}^s n'_j \mu(X'_j).$$

By assumption, $\mu(Y)$ has degree n, while each $\mu(X'_j)$ has degree $< n$, hence $\mu(Y) \neq \mu(X'_j)$ for every j. This implies that there is i such that $\mu(Y) = \mu(X_i)$, and we get, in particular, $p_g(Y) = p_g(X_i)$. $\qquad\square$

The key technical ingredient in the proof of Proposition 4.5 is the computation of the geometric genera for the symmetric powers of a smooth, connected, projective complex surface X. It is shown in [LL2] that if $p_g(X) = r$, then

(6)
$$p_g(\mathrm{Sym}^n(X)) = \binom{n + r - 1}{r - 1}.$$

Note that $\mathrm{Sym}^n(X)$ has a resolution of singularities given by the Hilbert scheme of n points on X. This is a projective scheme $X^{[n]}$ that parametrizes 0-dimensional subschemes of X of length n. It is a result of Fogarty that for a smooth, connected surface X, the Hilbert scheme $X^{[n]}$ is smooth and connected. Furthermore, there is a morphism $X^{[n]} \to \mathrm{Sym}^n(X)$ that takes a scheme Z supported at the points $x_1, \ldots, x_m$ to $\sum_{i=1}^m \ell(\mathcal{O}_{Z,x_i}) x_i$. This gives an isomorphism onto the image on the open subset parametrizing reduced subschemes. Therefore $X^{[n]}$ gives a resolution of singularities of $\mathrm{Sym}^n(X)$, hence $p_g(\mathrm{Sym}^n(X)) = p_g(X^{[n]})$. The above formula for $p_g(\mathrm{Sym}^n(X))$ is then deduced from results of Göttsche and Soergel [GS] on the Hodge structure on the cohomology of $X^{[n]}$.

Proof of Proposition 4.5. Suppose by way of contradiction that $h = \sum_{n \geq 0} \mu_n t^n \in K[\![t]\!]$ is a rational function, where $\mu_n = \mu(\mathrm{Sym}^n(X))$. Therefore we may write

$$h = \frac{a_0 + a_1 t + \ldots + a_e t^e}{b_0 + b_1 t + \ldots + b_m t^m},$$

for some $a_i, b_j \in K$, with not all b_j zero. This implies that $\mu_d b_m + \mu_{d+1} b_{m-1} + \ldots + \mu_{d+m} b_0 = 0$ for all $d \geq \min\{0, e - m + 1\}$. Since some b_j is nonzero, by considering these relations for $d, d+1, \ldots, d+m$, we conclude that $D := \det(\mu_{d+i+j})_{0 \leq i \leq m} = 0$. By expanding this determinant, we obtain a relation

$$(7) \qquad \mu\left(\prod_{i=0}^m \mathrm{Sym}^{d+2i}(X)\right) = \sum_{\sigma \in S_{m+1} \smallsetminus \{1\}} -\mathrm{sign}(\sigma) \mu\left(\prod_{i=0}^m \mathrm{Sym}^{d+\sigma(i)+i}(X)\right),$$

where we consider S_{m+1} to be the group of permutations of $\{0, 1, \ldots, m\}$.

Note that for every $\sigma \in S_{m+1}$, the variety $\prod_{i=0}^m \mathrm{Sym}^{d+\sigma(i)+i}(X)$ has dimension equal to $2(m+1)(d+m)$, and geometric genus $\prod_{i=0}^m \binom{d+\sigma(i)+i+r-1}{r-1}$ (see formula (6)). We deduce from (7) and Lemma 4.6 that there is a permutation $\sigma \in S_{m+1}$ different from the identity such that

$$(8) \qquad \prod_{i=0}^m \binom{d+\sigma(i)+i+r-1}{r-1} = \prod_{i=0}^m \binom{d+2i+r-1}{r-1}.$$

Since $r \geq 2$, for every $\sigma \in S_{m+1}$ different from the identity, the following polynomial in d

$$P_\sigma(d) = \prod_{i=0}^m \binom{d+\sigma(i)+i+r-1}{r-1} - \prod_{i=0}^m \binom{d+2i+r-1}{r-1}$$

is not zero, hence it does not vanish for $d \gg 0$. Indeed, if i_0 is the largest i such that $\sigma(i) \neq i$, then we can write

$$P_\sigma(d) = \prod_{i=i_0+1}^m \binom{d+2i+r-1}{r-1} \cdot (Q_1(d) - Q_2(d)),$$

and the linear polynomial $d + 2i_0 + r - 1$ divides $Q_2(d)$, but it does not divide $Q_1(d)$. Since we have only finitely many permutations to consider (note that m is fixed), we conclude that by taking $d \gg 0$, we obtain a contradiction. $\square$

Remark 4.7. The Euler-Poincaré characteristic constructed above, that makes $Z_{\mathrm{mot}}(X, t)$ not pointwise rational, vanishes on $\mathbf{L}$. It would be interesting to find such an Euler-Poincaré characteristic that is nonzero on $\mathbf{L}$ (hence factors through $K_0(\mathrm{Var}/\mathbf{C})[\mathbf{L}^{-1}]$).

Remark 4.8. It is interesting to compare Theorem 4.4 on the rationality of $Z_{\mathrm{mot}}(X, t)$ with Mumford's theorem on the finiteness of the Chow group $A^2(X)_0$ of rational equivalence classes of 0-cycles on X of degree zero. It is shown in [Mum] that if X is a smooth, connected, projective complex surface with $p_g(X) \neq 0$, then $A^2(X)_0$ is infinitely dimensional in a suitable sense (in particular, it can not be parametrized by the points of an algebraic variety). This can also be interpreted as a statement about the growth of the symmetric products $\mathrm{Sym}^n(X)$, when n goes to infinity. On the other hand, it was conjectured by Bloch that the converse is also true, namely that if $p_g(X) = 0$, then $A^2(X)_0$ is finite-dimensional. While this is still a conjecture, it is known to hold for surfaces of Kodaira dimension ≤ 1. In particular, we see that for any surface X of Kodaira dimension 0 or 1 with $p_g(X) = 0$, we have $A^0(X)_0$ finite dimensional, but $Z_{\mathrm{mot}}(X, t)$ is not rational.

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