# SPACES OF ARCS IN BIRATIONAL GEOMETRY

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These lecture notes have been prepared for the Summer school on "Moduli spaces and arcs in algebraic geometry", Cologne, August 2007. The goal is to explain the relevance of spaces of arcs to birational geometry. In the first section we introduce the spaces of arcs and their finite-dimensional approximations, the spaces of jets. In the second section we see how to relate the spaces of arcs of birational varieties. If $f$ is a proper birational morphism between varieties $Y$ and $X$, then the induced map $f_\infty$ at the level of spaces of arcs is "almost bijective". The key result is the Birational Transformation Theorem, which shows that if $X$ and $Y$ are nonsingular, then the geometry of $f_\infty$ is governed by the order of vanishing along the discrepancy divisor $K_{Y/X}$. In the third section we give some applications to invariants of $K$-equivalent varieties, and to the description of the log canonical threshold in terms of the codimensions of certain subsets in spaces of arcs.

## 1. Introduction to jet schemes and spaces of arcs

In this section we construct the spaces of arcs and jets and prove some basic properties. The jet schemes can be thought of as higher analogues of the tangent spaces of a scheme. They parameterize maps from $\mathrm{Spec}\, k[t]/(t^{m+1})$ to our given scheme.

1.1. **Jet schemes.** Let $k$ be an algebraically closed field of arbitrary characteristic. Suppose that $X$ is a scheme of finite type over $k$ and $m$ a nonnegative integer. A scheme of finite type $X_m$ over $k$ is the $m$th *jet scheme* of $X$ if for every $k$-algebra $A$ we have a functorial bijection

$$\mathrm{Hom}(\mathrm{Spec}(A), X_m) \simeq \mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}), X). \tag{1}$$

This describes completely the functor of points of $X_m$. It follows that if $X_m$ exists, then it is unique up to a canonical isomorphism. Note that in particular, the closed points of $X_m$ are in bijection with the $k[t]/(t^{m+1})$-valued points of $X$.

Note that if the jet schemes $X_m$ and $X_p$ exist and if $m > p$, then we have a canonical projection $\pi_{m,p}\colon X_m \to X_p$. This can be defined at the level of the functor of points via (1): the corresponding map

$$\mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}), X) \to \mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{p+1}), X)$$

is induced by the truncation morphism $A[t]/(t^{m+1}) \to A[t]/(t^{p+1})$. It is clear that these morphisms are compatible whenever they are defined: $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ if $p < m < q$.

**Example 1.1.** We clearly have $X_0 = X$. For every $m$, we denote the canonical projection $\pi_{m,0}\colon X_m \to X$ by $\pi_m$.

**Proposition 1.2.** *For every $X$ as above and every nonnegative integer $m$, the $m$th jet scheme $X_m$ exists.*

Before proving the proposition we give the following lemma.

**Lemma 1.3.** *If $U \subseteq X$ is an open subset and if $X_m$ exists, then $U_m = \pi_m^{-1}(U)$.*

*Proof.* Indeed, let $A$ be a $k$-algebra and let $\iota_A \colon \mathrm{Spec}(A) \to \mathrm{Spec}\, A[t]/(t^{m+1})$ be induced by the truncation morphism. Note that a morphism $f \colon \mathrm{Spec}\, A[t]/(t^{m+1}) \to X$ factors through $U$ if and only if the composition $f \circ \iota_A$ factors through $U$ (factoring through $U$ is a set-theoretic statement). Therefore the assertion of the lemma follows from definitions. $\square$

*Proof of Proposition 1.2.* Suppose first that $X$ is affine, and consider a closed embedding $X \hookrightarrow \mathbb{A}^n$ such that $X$ is defined by the ideal $I = (f_1, \ldots, f_q)$. For every $k$-algebra $A$, giving a morphism $\mathrm{Spec}\, A[t]/(t^{m+1}) \to X$ is equivalent with giving a morphism $\phi \colon k[x_1, \ldots, x_n]/I \to A[t]/(t^{m+1})$. Moreover, such a morphism corresponds uniquely to elements $u_i = \phi(x_i) = \sum_{j=0}^m a_{i,j} t^j$ such that $f_\ell(u_1, \ldots, u_n) = 0$ for every $\ell$. We can write

$$f_\ell(u_1, \ldots, u_n) = \sum_{p=0}^m g_{\ell,p}((a_{i,j})_{i,j}) t^p,$$

for suitable polynomials $g_{\ell,p}$ depending only on the $f_\ell$. It follows that $X_m$ can be defined in $\mathbb{A}^{(m+1)n}$ by the polynomials $g_{\ell,p}$ for $\ell \leq q$ and $p \leq m$.

Suppose now that $X$ is an arbitrary scheme of finite type over $k$. Consider an affine cover $X = U_1 \cup \ldots \cup U_r$. As we have seen, we have an $m$th jet scheme $\pi_m^i \colon (U_i)_m \to U_i$ for every $i$. Moreover, by Lemma 1.3, for every $i$ and $j$ the inverse images $(\pi_m^i)^{-1}(U_i \cap U_j)$ and $(\pi_m^j)^{-1}(U_i \cap U_j)$ are both isomorphic over $X$ to $(U_i \cap U_j)_m$. Therefore they are canonically isomorphic. This shows that we may construct a scheme $X_m$ by glueing the schemes $(U_i)_m$ along the canonical isomorphisms of $(\pi_m^i)^{-1}(U_i \cap U_j)$ with $(\pi_m^j)^{-1}(U_i \cap U_j)$. Moreover, the projections $\pi_m^i$ also glue to give a morphism $\pi_m \colon X_m \to X$. It is now straightforward to check that $X_m$ has the required property. $\square$

**Remark 1.4.** It follows from the description in the above proof that for every scheme $X$, the projection $\pi_m \colon X_m \to X$ is affine.

**Example 1.5.** The first jet-scheme $X_1$ is isomorphic to the total tangent space $\mathcal{S}pec(\mathrm{Sym}(\Omega_{X/k}))$. Indeed, arguing as in the proof of Proposition 1.2, we see that it is enough to show this when $X = \mathrm{Spec}(R)$ is affine. In this case, if $A$ is a $k$-algebra, then giving a scheme morphism $f \colon \mathrm{Spec}(A) \to \mathrm{Spec}(\mathrm{Sym}(\Omega_{X/k}))$ is equivalent with giving a morphism of $k$-algebras $\phi \colon R \to A$ and a $k$-derivation $D \colon R \to A$. This is the same as giving a morphism $f \colon R \to A[t]/(t^2)$, where $f(u) = \phi(u) + tD(u)$.

Note that if $f \colon X \to Y$ is a morphism of schemes, then we get a corresponding morphism $f_m \colon X_m \to Y_m$. At the level of $A$-valued points, this takes an $A[t]/(t^{m+1})$-valued point $\gamma$ of $X$ to $f \circ \gamma$. Therefore taking $X$ to $X_m$ gives a functor from the category of schemes of finite type over $k$ to itself. Note also that the morphisms $f_m$ are compatible in the obvious sense with the projections $X_m \to X_{m-1}$ and $Y_m \to Y_{m-1}$.

**Remark 1.6.** The jet schemes of the affine space are easy to describe: we have an isomorphism $(\mathbb{A}^n)_m \simeq \mathbb{A}^{(m+1)n}$ such that the projection $\pi_m \colon (\mathbb{A}^n)_m \to (\mathbb{A}^n)_{m-1}$ corresponds to the projection onto the first $mn$ coordinates. Indeed, an $A$ valued point of $(\mathbb{A}^n)_m$ corresponds to a ring homomorphism $\phi \colon k[x_1, \ldots, x_n] \to A[t]/(t^{m+1})$, which is uniquely determined by giving each $\phi(x_i) \in A[t]/(t^{m+1}) \simeq A^{m+1}$.

**Remark 1.7.** In light of the previous remark, we see that the proof of Proposition 1.2 showed that if $i \colon X \hookrightarrow \mathbb{A}^n$ is a closed immersion, then the induced morphism $i_m \colon X_m \to (\mathbb{A}^n)_m$ is also a closed immersion. Moreover, using the description of the ideal of $X_m$ in terms of the ideal of $X$, we deduce that more generally, if $f \colon X \hookrightarrow Y$ is a closed immersion, then $f_m$ is a closed immersion, too.

The following lemma generalizes Lemma 1.3 to the case of an étale morphism.

**Lemma 1.8.** *If $f \colon X \to Y$ is an étale morphism, then for every $m$ the commutative diagram*

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d, "\pi_m^X"] & Y_m \arrow[d, "\pi_m^Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

*is Cartesian.*

*Proof.* Indeed, from the description of the $A$-valued points of $X_m$ and $Y_m$ we see that it is enough to show that for every $k$-algebra $A$ and every commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(A) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A[t]/(t^{m+1}) \arrow[r] & Y \end{tikzcd}$$

there is a unique morphism $\operatorname{Spec} A[t]/(t^{m+1}) \to X$ making the two triangles commutative. This is a consequence of the fact that $f$ is formally étale. $\square$

We will say that a morphism of schemes $g \colon V \to W$ is *locally trivial* with fiber $F$ if there is a cover by Zariski open subsets $W = U_1 \cup \ldots \cup U_r$ such that $g^{-1}(U_i) \simeq U_i \times F$, the restriction of $g$ corresponding to the projection onto the first component. The morphism $g$ will be called *piecewise trivial* with fiber $F$ if we have a cover by sets $U_i$ as above, but with $U_i$ only locally closed subsets.

**Corollary 1.9.** *If $X$ is a nonsingular variety of dimension $n$, then all projections $\pi_{m,m-1} \colon X_m \to X_{m-1}$ are locally trivial with fiber $\mathbb{A}^n$. In particular, $X_m$ is a nonsingular variety of dimension $(m+1)n$.*

*Proof.* It follows from Lemma 1.3 that it is enough to show the following: if $u_1, \ldots, u_n$ form an algebraic system of coordinates on an open subset $U$ of $X$, then $U_m \simeq U \times \mathbb{A}^{mn}$, such that $\pi_{m,m-1}$ corresponds to the projection that forgets the last $n$ components. Recall that

$u_1, \dots, u_n \in \mathcal{O}(U)$ form an algebraic system of coordinates on $U$ if $du_1, \dots, du_n$ trivialize $\Omega_X$ over $U$. Since $X$ is nonsingular, we can find such a system of coordinates in the neighborhood of every point in $X$.

An algebraic system of coordinates $u_1, \dots, u_n$ gives an étale morphism $U \to \mathbb{A}^n$. Our assertion follows now from Lemma 1.8 and the fact that it holds for the affine space. $\square$

**Example 1.10.** If $X$ is a singular curve, then $X_1$ is not irreducible. In fact, the fiber over every singular point, gives an irreducible component. This follows since the dimension of each such fiber is at least the dimension of $\overline{(X_{\text{reg}})_1}$.

1.2. **Spaces of arcs.** We consider now the projective limit of the jet schemes. Suppose that $X$ is a scheme of finite type over $k$. Since the projective system

$$\cdots \to X_m \to X_{m-1} \to \cdots \to X_0 = X$$

consists of affine morphisms, the projective limit exists in the category of schemes over $k$. It is denoted by $\mathcal{L}(X)$ and it is called the space of arcs of $X$. In general, it is not of finite type over $k$.

It comes equipped with projection morphisms $\psi_m \colon \mathcal{L}(X) \to X_m$ that are affine. In particular, we have $\psi_0 \colon \mathcal{L}(X) \to X$ and if $U \subseteq X$ is affine, then

$$\mathcal{O}(\psi_0^{-1}(U)) = \varinjlim \mathcal{O}(\pi_m^{-1}(U)).$$

It follows from the projective limit definition and the functorial description of the jet schemes that if $X$ is affine, then for every $k$-algebra $A$ we have

$$\mathrm{Hom}(\mathrm{Spec}(A), \mathcal{L}(X)) \simeq \varprojlim \mathrm{Hom}(\mathrm{Spec}(A), X_m)$$

$$\simeq \varprojlim \mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}), X) \simeq \mathrm{Hom}(\mathrm{Spec}\ A[[t]], X).$$

If we take $A = k$ and $X$ an arbitrary scheme, then every morphism Spec $k[t]/(t^{m+1}) \to X$ or Spec $k[[t]] \to X$ factors through any affine open neighborhood of the image of the closed point. It follows that for every $X$, the elements in $\mathcal{L}(X)(k)$ correspond to *arcs in* $X$, i.e. we have a bijection

$$\mathrm{Hom}(\mathrm{Spec}(k), \mathcal{L}(X)) \simeq \mathrm{Hom}(\mathrm{Spec}(k[[t]], X).$$

If $f \colon X \to Y$ is a morphism of schemes, after passing to the limit the morphisms $f_m$ we get a morphism $f_\infty \colon \mathcal{L}(X) \to \mathcal{L}(Y)$. We get in this way a functor from $k$-schemes of finite type over $k$ to arbitrary $k$-schemes (in fact, to quasicompact and quasiseparated $k$-schemes).

If $K$ is a field containing $k$, then a $K$-valued point of $\mathcal{L}(X)$ corresponds to a morphism $\gamma \colon$ Spec $K[[t]] \to X$. This determines two points on $X$: the image $\gamma(0)$ of the closed point of Spec $K[[t]]$, and the image $\gamma(\eta)$ of the generic point of Spec $K[[t]]$.

The properties we have discussed in the previous subsection for jet schemes induce corresponding properties for spaces of arcs. For example, if $f \colon X \to Y$ is an étale morphism, then we have a Cartesian diagram

$$\begin{array}{ccc}\mathcal{L}(X) & \xrightarrow{f_\infty} & \mathcal{L}(Y) \\ \Big\downarrow{\scriptstyle \psi_0^X} & & \Big\downarrow{\scriptstyle \psi_0^Y} \\ X & \xrightarrow{f} & Y.\end{array}$$

If $i\colon X \hookrightarrow Y$ is a closed immersion, then $i_\infty$ is also a closed immersion. Moreover, if $Y = \mathbb{A}^n$, then $\mathcal{L}(Y) \simeq \operatorname{Spec} k[x_1, x_2, \ldots, ]$, such that $\psi_m$ corresponds to the projection onto the first $(m+1)n$ components. As in the proof of Proposition 1.2, starting with equations for a closed subscheme $X$ of $\mathbb{A}^n$ we can write equations for $\mathcal{L}(X)$ in $\mathcal{L}(\mathbb{A}^n)$.

From now on we will ignore the scheme structure on the space of arcs of $X$. In fact, we will abuse the notation by putting $\mathcal{L}(X)$ for $\mathcal{L}(X)(k)$, the $k$-valued points of the space of arcs of $X$. This is a topological space, the topology being the restriction of the Zariski topology from the space of arcs. Note that this is the projective limit topology on $\mathcal{L}(X)$ under the identification of $\mathcal{L}(X)$ with the projective limit of (the $k$-valued points of) $X_m$.

1.3. **Complements.** If $\operatorname{char} k = 0$, equations for spaces of arcs and for jet schemes can be explicitly written down by "formally differentiating", as follows. If $S = k[x_1, \ldots, x_n]$, let us write $S_\infty = k[x_i^{(m)} \mid 1 \leq i \leq n, m \in \mathbb{N}]$, so $\operatorname{Spec}(S_\infty) = \mathcal{L}(\mathbb{A}^n)$. In practice, we simply write $x_i = x_i^{(0)}$, $x_i' = x_i^{(1)}$, and so on. Note that on $S_\infty$ we have a $k$-derivation $D$ defined by $D(x_i^{(m)}) = x_i^{(m+1)}$.

If $f \in R$, then we put $f' := D(f)$, and we define recursively $f^{(m)} := D(f^{(m-1)})$. Suppose now that $R = S/I$, where $I$ is generated by $f_1, \ldots, f_r$. We claim that if

$$R_\infty := S_\infty/(f_i, f_i', \ldots, f_i^{(m)}, \ldots \mid 1 \leq i \leq r), \tag{2}$$

then $\mathcal{L}(\operatorname{Spec}(R)) \simeq \operatorname{Spec}(R_\infty)$.

Indeed, if $A$ is a $k$-algebra, a morphism $\phi\colon k[x_1, \ldots, x_n] \to A[[t]]$ is given by

$$\phi(x_i) = \sum_{m \in \mathbb{N}} \frac{a_i^{(m)}}{m!} t^m.$$

If $f$ is an arbitrary polynomial in $k[x_1, \ldots, x_n]$, we see that

$$\phi(f) = \sum_{m \in \mathbb{N}} \frac{f^{(m)}(a, a', \ldots, a^{(m)})}{m!} t^m$$

(this can be checked noting that both sides are additive and multiplicative in $f$, which reduces the proof to $f = x_i$, when it is trivial). It follows that $\phi$ induces a morphism $R \to A[[t]]$ if and only if $f_i^{(m)}(a, a', \ldots, a^{(m)}) = 0$ for every $m$ and every $i \leq r$. This completes the proof of the above claim.

**Remark 1.11.** Note that $D$ induces a derivation $\overline{D}$ on $R_\infty$. Moreover, $(R_\infty, \overline{D})$ is universal in the following sense: we have a $k$-algebra homomorphism $j\colon R \to R_\infty$ such that if $(T, \delta)$ is another $k$-algebra with a $k$-derivation, and if $j'\colon R \to T$ is a $k$-algebra homomorphism, then there is a unique $k$-algebra homomorphism $h\colon R_\infty \to T$ such that $h \circ j = j'$

and $h$ commutes with the derivations, i.e. $\delta(h(u)) = h(D(u))$ for every $u \in R_\infty$. This is the starting point of the applications of the spaces of arcs in differential algebra, see [Bu].

Of course, if we consider finite level truncations, then we obtain equations for the jet schemes. More precisely, if we put $S_m := k[x_i^{(j)} \mid i \leq n, 0 \leq j \leq m]$ and

$$R_m := S_m/(f_i, f_i', \ldots, f_i^{(m)} \mid 1 \leq i \leq r),$$

then $\mathrm{Spec}(R_m) \simeq (\mathrm{Spec}(R))_m$. Moreover, the projections

$$(\mathrm{Spec}(R))_m \to (\mathrm{Spec}(R))_{m-1}$$

are induced by the obvious morphisms $R_{m-1} \to R_m$.

**Example 1.12.** If $X \subseteq \mathbb{A}^2$ is the cuspidal curve defined by $(u^2 - v^3)$, then $X_2$ is defined in Spec $k[u, v, u', v', u'', v'']$ by

$$(u^2 - v^3, 2uu' - 3v^2v', 2(u')^2 + 2uu'' - 6v(v')^2 - 3v^2v'').$$

**Exercise 1.13.** Show that if $f\colon X \to Y$ is a smooth, surjective morphism of schemes of finite type over $k$, then the induced morphism $f_m\colon X_m(k) \to Y_m(k)$ is surjective for every $m \in \mathbb{N}$, and a similar assertion holds for the spaces of arcs.

**Exercise 1.14.** Let $X$ be a scheme of finite type over $k$.

i) If $X$ is smooth, then all maps $X_m \to X_{m-1}$ are surjective. Is the converse true ?
ii) Show that if $\gamma_{m-1} \in X_{m-1}$ is an $(m-1)$-jet lying over $x \in X$, then the fiber $(\pi_{m,m-1})^{-1}(\gamma_{m-1})$ is either empty, or is isomorphic to $T_x(X)$.

**Exercise 1.15.** Suppose that $X \subseteq \mathbb{A}^n$ is defined by an ideal generated by homogeneous polynomials of degree $d \geq 1$. Recall that for every $m$ we denote by $\pi_m\colon X_m \to X$ the canonical projection.

i) Show that for every $m \leq d-1$ we have $\pi_m^{-1}(0) \simeq \mathbb{A}^{mn}$.
ii) Show that for every $m \geq d$ there is an isomorphism
$$\pi_m^{-1}(0) \simeq X_{m-d} \times \mathbb{A}^{n(d-1)}.$$
iii) Deduce that if $X$ is the affine cone over a smooth projective variety of dimension $r$, then for every $m \geq d$ we have
$$\dim X_m = \max\{(m+1)(r+1), \dim X_{m-d} + n(d-1)\}.$$

**Exercise 1.16.** Show that for every schemes $X$ and $Y$ and for every $m \in \mathbb{N}$, there is a canonical isomorphism $(X \times Y)_m \simeq X_m \times Y_m$ (a similar assertion holds for the spaces of arcs).

**Exercise 1.17.** Show that if $X$ is a group scheme over $k$, then $X_m$ is also a group scheme over $k$ for every $m \in \mathbb{N}$ (and a similar assertion holds for $\mathcal{L}(X)$).

**Exercise 1.18.** Show that if $G$ is a a group scheme over $k$ acting on the scheme $X$, then we have an action of $G_m$ on $X_m$ for every $m \in \mathbb{N}$ (and a similar assertion holds for the spaces of arcs).

**Exercise 1.19.** Show that if $A$ is an abelian variety, then for every $m$ we have an isomorphism (of varieties, not of group schemes)

$$A_m \simeq \pi_m^{-1}(0) \times A.$$

The goal of the next exercise is to describe the space of arcs of a toric variety, following [Is]. Suppose that $X = X(\Delta)$ is a toric variety, where $\Delta$ is a fan in a lattice $N$ (we follow the notation for toric varieties from [Fu]). Let $T = T_N$ be the torus acting on $X$, and let $D_1, \ldots, D_d$ be the prime invariant divisors on $X$. Recall that the $D_i$ are also toric varieties, hence in order to describe $\mathcal{L}(X)$ it is enough to describe

$$\mathcal{L}(X)' := \mathcal{L}(X) \smallsetminus \cup_{i=1}^{d} \mathcal{L}(D_i)$$

(we may continue by induction on dimension).

**Exercise 1.20.** With the above notation, prove the following description of $\mathcal{L}(X)'$.

i) $\mathcal{L}(X)'$ is invariant under the action of $\mathcal{L}(T)$.
ii) If $v \in N \cap |\Delta|$, then we have an arc $\gamma_v \colon \operatorname{Spec} k[[t]] \to X$ defined as follows. If $v \in N \cap |\sigma|$ for some $\sigma \in \Delta$, then $\gamma_v$ corresponds to

$$k[\sigma^\vee \cap M] \to k[[t]],\ \chi^u \to t^{\langle u,v\rangle}.$$

Show that the stabilizer of $\gamma_v$ is trivial.
iii) Show that there is a bijection between the points in $N \cap |\Delta|$ and the orbits of the $\mathcal{L}(T)$-action on $\mathcal{L}(X)'$, that takes $v$ to $\gamma_v$.

**Remark 1.21.** With the notation in the previous exercise, there is a corresponding action of $T_m$ on $X_m$ for $m \in \mathbb{N}$. However, we do not know a similar description of the orbits.

We will see in the next section that it is a typical phenomenon to have a decomposition of the space of arcs of a variety in a countable disjoint union of pieces. In this case, the interesting information is understanding the closures of these pieces. In the case of the previous exercise, this is done by the following

**Exercise 1.22.** With the notation in Exercise 1.20, show that if $v$ and $w$ are in $N \cap |\Delta|$, then $\mathcal{L}(T) \cdot \gamma_v \subseteq \overline{\mathcal{L}(T) \cdot \gamma_w}$ if and only if the following holds: if $\sigma \in \Delta$ is such that $v$ lies in the interior of $\sigma$, then both $w$ and $v - w$ lie in $\sigma$.

We describe now a similar picture for determinantal varieties. Fix positive integers $m$ and $n$, and let $r \leq \min\{m, n\}$. We consider the determinantal variety

$$X^{(r)} := \{A \in M_{m,n}(k) \mid \operatorname{rk}(A) \leq r\}.$$

Note that we have an action of $GL_m \times GL_n$ on $X^{(r)}$ given by $(P, Q) \cdot A = PAQ^{-1}$. The orbits are parameterized by $\{0, 1, \ldots, r\}$, all matrices of the same rank lying in the same orbit.

**Exercise 1.23.** Consider the induced action of $\mathcal{L}(GL_m) \times \mathcal{L}(GL_n)$ on $\mathcal{L}(X^{(r)})$. Show that the orbits are parameterized by

$$\{(a_1, \ldots, a_s) \mid s \leq r, a_1 \geq a_2 \ldots \geq a_s, a_i \in \mathbb{N}\},$$

such that the orbit corresponding to $(a_1, \ldots, a_s)$ is contained in the closure of the orbit corresponding to $(b_1, \ldots, b_{s'})$ if and only if $s \leq s'$ and $a_i \geq b_i$ for all $i \leq s$.

## 2. Spaces of arcs and birational transformations

If $f\colon X \to Y$ is a proper birational morphism, then the Valuative Criterion for Properness implies that the induced map $f_\infty\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ is "almost bijective". As we will see, this can be combined in characteristic zero with resolution of singularities in order to study the space of arcs of a singular variety by relating it to the space of arcs of a resolution of singularities. The powerful result that has many applications is the Birational Transformation Theorem: it shows that (at least when both $X$ and $Y$ are nonsingular) the geometry of $f_\infty$ is governed by the order of vanishing along the relative canonical divisor $K_{X/Y}$.

Note that since $f$ is birational there is an open subset $U$ of $Y$ such that $f^{-1}(U) \to U$ is an isomorphism. Ideally, we could find a decomposition of $Y$ in finitely many locally closed subsets $W_i$ such that over each $W_i$, $f$ is locally trivial with a suitable fiber. While this is not true in general (in the algebraic category), the Birational Transformation Theorem says that such a statement holds, in a suitable sense, for $f_\infty$ (though, as one might expect, the decomposition will have infinitely many terms).

2.1. **A theorem of Kolchin.** Let $X$ be a variety over an algebraically closed field $k$ (hence $X$ is reduced and irreducible). A key idea is that certain subsets in the space of arcs $\mathcal{L}(X)$ are "small" and they can be ignored (from the point of view of the theory of Motivic Integration these sets have measure zero, see for example [Ve]). We make an ad-hoc definition: a subset of $\mathcal{L}(X)$ is called *thin* if it is contained in $\mathcal{L}(Y)$ for a proper subscheme $Y$ of $X$. It is clear that a finite union of thin subsets is again thin. Note also that if $f\colon Y \to X$ is a morphism, and if $W \subseteq \mathcal{L}(X)$ is thin, then $f_\infty^{-1}(W)$ is also thin. The following proposition shows that a proper birational morphism induces a bijective map on the complement of suitable thin sets.

**Proposition 2.1.** *Let $f\colon Y \to X$ be a proper birational morphism of varieties over $k$. If $Z$ is a proper closed subset of $X$ such that $f$ is an isomorphism over $X \smallsetminus Z$, then the induced map*

$$\mathcal{L}(Y) \smallsetminus \mathcal{L}(f^{-1}(Z)) \to \mathcal{L}(X) \smallsetminus \mathcal{L}(Z)$$

*is bijective.*

*Proof.* Let $U = X \smallsetminus Z$. Since $f$ is proper, the Valuative Criterion for Properness implies that an arc $\gamma\colon \operatorname{Spec} k[[t]] \to X$ lies in the image of $f_\infty$ if and only if the induced morphism $\overline{\gamma}\colon \operatorname{Spec} k((t)) \to X$ can be lifted to $Y$ (moreover, if the lifting of $\overline{\gamma}$ is unique, then the lifting of $\gamma$ is also unique). On the other hand, $\gamma$ does not lie in $\mathcal{L}(Z)$ if and only if $\overline{\gamma}$ factors through $U \hookrightarrow X$. In this case, the lifting of $\overline{\gamma}$ exists and is unique since $f$ is an isomorphism over $U$. This completes the proof of the proposition. $\square$

We emphasize that despite the fact that $f_\infty$ in the above proposition is "almost bijective", it is very far from being an isomorphism. We will see in the next section how the codimension of certain subsets are changed by $f_\infty$.

We can use the above proposition to prove the following result of Kolchin.

**Theorem 2.2.** *If $X$ is a variety over an algebraically closed field $k$ of characteristic zero, then $\mathcal{L}(X)$ is irreducible.*

*Proof.* If $X$ is nonsingular, then the assertion is easy: we have seen that every $X_m$ is a nonsingular variety. Since

$$\mathcal{L}(X) = \varprojlim X_m,$$

it follows that $\mathcal{L}(X)$, too, is irreducible.

In the general case we do induction on $n = \dim(X)$, the case $n = 0$ being trivial. By Hironaka's Theorem, there is a resolution of singularities $f\colon Y \to X$. This is a proper birational morphism, with $Y$ nonsingular. Suppose that $Z$ is a proper closed subset of $X$ such that $f$ is an isomorphism over $U = X \smallsetminus Z$. It follows from Proposition 2.1 that

$$\mathcal{L}(X) = \mathcal{L}(Z) \cup \mathrm{Im}(f_\infty).$$

Moreover, the nonsingular case implies that $\mathcal{L}(Y)$, hence also $\mathrm{Im}(f_\infty)$, is irreducible. Therefore, in order to complete the proof it is enough to show that $\mathcal{L}(Z)$ is contained in the closure of $\mathrm{Im}(f_\infty)$.

Consider the irreducible decomposition $Z = Z_1 \cup \ldots \cup Z_r$, hence $\mathcal{L}(Z) = \mathcal{L}(Z_1) \cup \ldots \cup \mathcal{L}(Z_r)$. Since $f$ is surjective, for every $i$ there is an irreducible component $Y_i$ of $f^{-1}(Z_i)$ such that the induced map $Y_i \to Z_i$ is surjective. Since we are in characteristic zero, by Generic Smoothness we can find open subsets $U_i$ and $V_i$ in $Y_i$ and $Z_i$, respectively, such that the induced morphisms $g_i\colon U_i \to V_i$ are smooth and surjective. In particular, we have

$$\mathcal{L}(V_i) = \mathrm{Im}((g_i)_\infty) \subseteq \mathrm{Im}(f_\infty).$$

On the other hand, every $\mathcal{L}(Z_i)$ is irreducible by induction. Since $\mathcal{L}(V_i)$ is an open nonempty subset of $\mathcal{L}(Z_i)$, it follows that

$$\mathcal{L}(Z_i) \subseteq \overline{\mathrm{Im}(f_\infty)}$$

for every $i$. This completes the proof of the theorem. □

**Remark 2.3.** In fact, Kolchin's Theorem holds in a much more general setup, see [Kln]. Note also that we proved something slightly weaker even in our restricted setting. The result in *loc. cit.* states that the scheme $\mathcal{L}(X)$ is irreducible, while we proved only that $\mathcal{L}(X)(k)$ is irreducible. In fact, one can get the stronger statement from ours by showing that $\mathcal{L}(X)(k)$ is dense in $\mathcal{L}(X)$. In turn, this can be proved in a similar way with Theorem 2.2 above. For a different proof of (the stronger version of) Kolchin's Theorem, without using resolution of singularities, see [IK].

2.2. **The Birational Transformation Theorem.** From now on, for simplicity, we assume that $X$ is a nonsingular variety, say of dimension $n$. Recall that in this case, each projection $\pi_{m,m-1}\colon X_m \to X_{m-1}$ is locally trivial with fiber $\mathbb{A}^n$. We also make an extra assumption on the base field: we assume that $k$ is uncountable.

If $n \geq 1$, the space of arcs $\mathcal{L}(X)$ is infinite-dimensional. We will deal mostly with certain subsets of $\mathcal{L}(X)$ that come from a finite level: a *cylinder* in $\mathcal{L}(X)$ is a subset of the form $C = \psi_m^{-1}(S)$ for some $m$ and some constructible subset $S$ of $X_m$ (recall that a set is called constructible if it can be written as a finite union of locally closed subsets). It is clear that the cylinders form an algebra of sets.

A cylinder is called *closed*, *open*, *locally closed* or *irreducible* if the set $S$ can be taken with the corresponding property. Every closed cylinder has a unique decomposition in irreducible components. If $C = \psi_m^{-1}(S)$, then we put

$$\operatorname{codim}(C) := \operatorname{codim}(S, X_m) = (m+1)n - \dim(S).$$

Note that since the projections $\pi_{m,m-1}$ are locally trivial, this definition is independent of $m$.

**Remark 2.4.** Chevalley's Constructibilty Theorem implies that for every cylinder $C \subseteq \mathcal{L}(X)$, its image $\psi_m(C)$ is constructible for every $m$.

Interesting cylinders arise as follows. Suppose that $Z$ is a proper closed subscheme of $X$. We define the function

$$\operatorname{ord}_Z\colon \mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$$

to be given by the order of vanishing along $Z$. More precisely, if $\gamma\colon \operatorname{Spec} k[[t]] \to X$, then the scheme-theoretic inverse image of the ideal defining $Z$ is an ideal in $k[[t]]$ generated by $t^{\operatorname{ord}_Z(\gamma)}$ (we make the convention that if the ideal is zero, then $\operatorname{ord}_Z(\gamma) = \infty$).

The *contact locus* of order $m$ with $Z$ is the set $\operatorname{Cont}^m(Z) := \operatorname{ord}_Z^{-1}(m)$. Similarly, we put $\operatorname{Cont}^{\geq m}(Z) := \operatorname{ord}_Z^{-1}(\geq m)$. Note that we have

$$\operatorname{Cont}^{\geq m}(Z) = \psi_{m-1}^{-1}(Z_{m-1}),$$

hence $\operatorname{Cont}^{\geq m}(Z)$ is a closed cylinder. We deduce that $\operatorname{Cont}^m(Z)$ is a locally closed cylinder for every $m$. Note also that $F_Z^{-1}(\infty) = \mathcal{L}(Z)$ is thin.

The fact that $k$ is uncountable plays a role in the following lemma

**Lemma 2.5.** *If $C_1 \supseteq C_2 \supseteq \ldots$ is a sequence of nonempty cylinders, then $\cap_i C_i$ is nonempty.*

*Proof.* The proof is easy and we leave it as an exercise (see Exercise 2.14 below). □

The following result explains why the behavior on thin subsets can often be ignored.

**Proposition 2.6.** *If $C$ is a nonempty cylinder in $\mathcal{L}(X)$, then $C$ is not thin.*

*Proof.* The proof is easy, and we leave it as an exercise (see Exercise 2.15). □

**Corollary 2.7.** *If $Z$ is a proper closed subset of $X$, then*

$$\lim_{m\to\infty} \operatorname{codim} \operatorname{Cont}^{\geq m}(Z) = \infty.$$

*Proof.* If the above assertion is not true, then there are $N$ and $m_0$ such that we have $\operatorname{codim} \operatorname{Cont}^{\geq m}(Z) = N$ for every $m \geq m_0$. Note that the codimension of a closed cylinder is the minimum of the codimensions of its irreducible components. Since we have $\operatorname{Cont}^{\geq (m+1)}(Z) \subseteq \operatorname{Cont}^{\geq m}(Z)$, it follows that for every $m \geq m_0 + 1$, if $C$ is an irreducible component of $\operatorname{Cont}^{\geq m}(Z)$ having codimension $N$, then $C$ is also an irreducible component of $\operatorname{Cont}^{\geq (m-1)}(Z)$. This implies that there is a component $C$ of $\operatorname{Cont}^{\geq m}(Z)$ for every $m \geq m_0$. Hence $C$ is a cylinder contained in $\mathcal{L}(Z)$, which contradicts the proposition. □

**Remark 2.8.** In general, if $f\colon X \to Y$ is a proper birational morphism of nonsingular varieties, the image of a cylinder by $f_\infty$ is not necessarily a cylinder. However, one can give a sufficient condition for this to hold (see Exercise 2.17 below). On the other hand, for such $f$ the closure of the image of a cylinder is always a cylinder (see Exercise 2.20 below).

The Birational Transformation Theorem, due to Kontsevich, deals with a proper birational morphism $f\colon X \to Y$ between nonsingular varieties. The behavior of the corresponding morphism $f_\infty$ is governed by the *discrepancy* of $f$. This is an effective divisor on $X$ denoted by $K_{X/Y}$, defined as follows.

Let $f$ be a proper birational morphism between nonsingular varieties of dimension $n$. We have a morphism of sheaves $f^*\Omega_Y \to \Omega_X$. This is generically an isomorphism, hence we get an injective morphism of rank one locally free sheaves $f^*\Omega_Y^n \to \Omega_X^n$. This is given by multiplying with the local equation of a unique effective divisor which is $K_{X/Y}$. In other words, if we choose local algebraic coordinates $x_1, \ldots, x_n$ around $P \in X$ and $y_1, \ldots, y_n$ around $f(P) \in Y$, and if we write $f^*(y_i) = f_i(x_1, \ldots, x_n)$ for some $f_i \in k[[x_1, \ldots, x_n]]$, then a local equation for $K_{X/Y}$ around $P$ is given by the determinant of the Jacobian matrix $(\partial f_i/\partial x_j)_{i,j\leq n}$.

We can state now the Birational Transformation Theorem, due to Kontsevich. Let $f\colon X \to Y$ be a proper birational morphism between nonsingular varieties. For every $m$, we denote by $\psi_m^X$ and $\pi_{m,p}^X$ the projection morphisms corresponding to $X$, and similarly for $Y$. If $e$ is a nonnegative integer, we denote by $C^{(e)}$ the cylinder $\operatorname{Cont}^e(K_{X/Y})$.

**Theorem 2.9.** *With the above notation, let $e$ be a nonnegative integer and let $m$ be such that $m \geq 2e$.*

i) *For every $\gamma_m$, $\gamma'_m \in X_m$ such that $\gamma_m$ lies in $\psi_m^X(C^{(e)})$ and $f_m(\gamma_m) = f_m(\gamma'_m)$, we have $\pi_{m,m-e}^X(\gamma_m) = \pi_{m,m-e}^X(\gamma'_m)$.*
ii) *The induced map*
$$\psi_m^X(C^{(e)}) \to f_m(\psi_m^X(C^{(e)}))$$
*is piecewise trivial, with fiber $\mathbb{A}^e$.*

We will sketch the proof, following [Lo], in the next subsection.

**Remark 2.10.** Note that in the context of the theorem, $\psi_m^X(C^{(e)})$ is a union of fibers of $f_m$. This is a consequence of i). In particular, $f_\infty(C^{(e)})$ is a cylinder by Exercise 2.17 below.

Note that if $Z$ is the image of the exceptional locus of $f$, then the theorem gives decompositions

$$\mathcal{L}(X) \smallsetminus \mathcal{L}(f^{-1}(Z)) = \coprod_{e\in\mathbb{N}} C^{(e)},\ \mathcal{L}(Y) \smallsetminus \mathcal{L}(Z) = \coprod_{e\in\mathbb{N}} f_\infty(C^{(e)}).$$

Despite the fact that the maps

$$C^{(e)} \to f_\infty(C^{(e)})$$

are bijective, the induced map at a finite level $m \geq 2e$ is piecewise trivial with fiber $\mathbb{A}^e$.

**Example 2.11.** Suppose that $f\colon X \to \mathbb{A}^2$ is the chart of the blowing-up of the origin corresponding to $\phi\colon k[x,y] \to k[u,v]$, with $\phi(x) = u$ and $\phi(y) = uv$. Consider an $m$-jet $\gamma$ of $X$ corresponding to $u \to \alpha(t)$ and $v \to \beta(t)$. The order $e$ of $\gamma$ along $K_{X/\mathbb{A}^2}$ is the order of $\alpha(t)$, and we assume that $m \geq 2e$. The image $\delta$ of $\gamma$ corresponds to $x \to \alpha(t)$, $y \to \alpha(t)\beta(t)$.

Let us determine the fiber $f_m^{-1}(\delta)$. It consists of arcs $\gamma' = (\alpha'(t), \beta'(t)$ with $(\alpha', \alpha'\beta') = (\alpha, \alpha\beta)$. Equivalently, $\alpha' = \alpha$ and $t^e\beta = t^e\beta'$. It follows that if we write $\beta = \sum_{j=0}^m \beta_j t^j$ and $\beta' = \sum_{j=0}^m \beta'_j t^j$, the condition on $\beta'$ is that $\beta_j = \beta'_j$ for $j \leq m-e$. Since the $\beta'_j$ for $m-e+1 \leq j \leq m$ can be chosen arbitrarily, it follows that the fiber over $\delta$ is isomorphic to $\mathbb{A}^e$ (note that in this case we could have taken $m \geq e$).

**Remark 2.12.** Denef and Loeser proved in [DL] a generalization of Theorem 2.9 to the case when $Y$ is possibly singular. The role of $K_{X/Y}$ is then played by the scheme defined by the ideal $I$, such that the image of

$$f^*\Omega_Y^n \to \Omega_X^n$$

is $I \otimes \Omega_X^n$.

**Remark 2.13.** The Birational Transformation Theorem plays an important role in Motivic Integration. It translates in the Change of Variable Formula for motivic integrals (see [Ve], and also the next section).

2.3. **Complements.**

**Exercise 2.14.** Let $k$ be an uncountable algebraically closed field.

1) If $f_1, f_2, \ldots \in k[x_1, \ldots, x_n]$ are nonzero polynomials, then there is $a = (a_1, \ldots, a_n) \in k^n$ such that $f_i(a) \neq 0$ for every $i$.
2) Deduce that if $Y$ is a scheme of finite type over $k$, and if $Y_1, Y_2 \ldots$ are closed subsets of $Y$ such that $Y = \cup_i Y_i$, then there is $m$ such that $Y = Y_1 \cup \ldots \cup Y_m$.
3) Deduce that if $Y$ is a scheme of finite type over $k$ and if $W_1 \supseteq W_2 \supseteq \ldots$ are nonempty constructible subsets of $Y$, then $\cap_i W_i \neq \emptyset$.
4) Suppose now that $X$ is a nonsingular variety and $C_1 \supseteq C_2 \ldots$ are nonempty cylinders in $\mathcal{L}(X)$. Show by induction on $m$ that there are $\gamma_m \in \cap_{i\geq 1}\psi_m(C_i)$ such that $\pi_{m,m-1}(\gamma_m) = \gamma_{m-1}$ for every $m$. In particular $(\gamma_m)_m$ gives an element $\gamma \in \cap_{i\geq 1} C_i$.

**Exercise 2.15.** Prove Proposition 2.6 as follows.

i) After choosing algebraic coordinates in a neighborhood of a point where $Z \neq X$, reduce the assertion in the proposition to the following: if $f \in k[[x_1, \ldots, x_n]]$ is nonzero, if $m$ is a positive integer and if $u_1, \ldots, u_n \in k[[t]]$, then there are $v_1, \ldots, v_n \in k[[t]]$ such that $f(u_1 + t^m v_1, \ldots, u_n + t^m v_n) \neq 0$.
ii) Reduce the above assertion to the following: if $g \in k[[t]][x_1, \ldots, x_n]$ is nonzero, then there are $v_1, \ldots, v_n \in tk[[t]]$ such that $g(v_1, \ldots, v_n) \neq 0$.
iii) Prove this by induction on $n$, reducing it to the following statement: if $h \in k[[t]][x]$ is nonzero, then there is $v \in tk[[t]]$ such that $h(v) \neq 0$.

**Exercise 2.16.** Show that if $f\colon X \to Y$ is a proper birational morphism between two nonsingular varieties, then $f_m$ is surjective for every nonnegative integer $m$.

**Exercise 2.17.** Let $f\colon X \to Y$ be a proper birational morphism between nonsingular varieties. Use the previous exercise to show that if $S \subseteq X_m$ is a union of fibres of $f_m$ and if $C = (\psi_m^X)^{-1}(S)$, then $f_\infty(C) = (\psi_m^Y)^{-1}(f_m(S))$. In particular, the image of $C$ is a cylinder.

**Exercise 2.18.** Let $f\colon X \to Y$ be a proper birational morphism between nonsingular varieties. We have seen in Proposition 2.1 that the complement of $\mathrm{Im}(f_\infty)$ is thin. Use the previous exercise to show that, in fact, $f_\infty$ is surjective (note that this strengthens the assertion in Exercise 2.16).

**Exercise 2.19.** With the notation in Theorem 2.9, show that if $f$ is birational and if $C \subseteq C^{(e)}$ is a cylinder, then its image $f_\infty(C)$, too, is a cylinder.

**Exercise 2.20.** If $f\colon X \to Y$ is a proper birational morphism between nonsingular varieties and if $C \subseteq \mathcal{L}(X)$ is a cylinder, then the closure $\overline{f_\infty(C)}$ of its image is again a cylinder.

**Exercise 2.21.** Let $Y$ be a nonsingular variety, and $X \subset Y$ a nonsingular closed subvariety of codimension $r$. Show that if $\widetilde{Y} \to Y$ is the blowing-up of $Y$ along $X$, with exceptional divisor $E$, then $K_{\widetilde{Y}/Y} = (r-1)E$.

We discuss now the proof of Theorem 2.9 following [Lo]. Let $f\colon X \to Y$ be a proper birational morphism between nonsingular varieties.

**Exercise 2.22.** Let $\gamma_m \in X_m$ vanishing along $K_{X/Y}$ with order $e$, where $e \leq m$. If $\gamma_m$ lies over $x \in X$, we may consider on $k[t]/(t^{m+1})$ the structure of $\mathcal{O}_{X.x}$-module induced by $\gamma$. Show that if we consider the canonical morphism

$$\Omega_{Y,f(x)} \otimes_k k[t]/(t^{m+1}) \to \Omega_{X,x} \otimes_k k[t]/(t^{m+1}),$$

then both modules are isomorphic to $(k[t]/(t^{m+1}))^{\oplus n}$ such that the morphism is given by a diagonal matrix with entries $t^{a_1}, \ldots, t^{a_n}$ for some nonnegative integers $a_1, \ldots, a_n$ such that $\sum_{i=1}^n a_i = e$.

**Exercise 2.23.** Let $\gamma$, $\gamma' \in \mathcal{L}(X)$ and write $\gamma_m = \psi_m^X(\gamma)$ and $\gamma'_m = \psi_m^X(\gamma')$. Show that if $\gamma \in C^{(e)}$ with $m \geq 2e$ and if $f_m(\gamma_m) = f_m(\gamma'_m)$, then $\pi_{m,m-e}(\gamma_m) = \pi_{m,m-e}(\gamma'_m)$, as follows.

i) Note that it is enough to construct $\widetilde{\gamma}\in\mathcal{L}(X)$ such that $\psi^X_{m-e}(\widetilde{\gamma})=\psi^X_{m-e}(\gamma)$ and $f_\infty(\gamma')=f_\infty(\widetilde{\gamma})$. Therefore it is enough to construct by induction on $p\geq m+1$ an element $\delta_p\in X_p$ such that
$$\pi_{p,p-e-1}(\delta_p)=\pi_{p-1,p-e-1}(\delta_{p-1})\text{ and }f_p(\delta_p)=f_p(\psi^X_p(\gamma'))$$
(where $\delta_m=\gamma_m$).

ii) For the induction step, suppose that $\alpha_{p+1}\in X_{p+1}$ is an arbitrary lifting of $\delta_p$. Let $\pi_m(\gamma_m)=x$ and $y=f(x)$. Show that the set of those $\alpha'_{p+1}\in X_{p+1}$ such that
$$\pi^X_{p+1,p-e}(\alpha'_{p+1})=\pi^X_{p,p-e}(\delta_p)$$
is in bijection with $\mathrm{Der}_k(\mathcal{O}_{X,x},(t^{p-e+1})/(t^{p+2}))$, where we consider $k[t]/(t^{p+2})$ as a module over $\mathcal{O}_{X,x}$ via $\alpha_{p+1}$.

iii) Note that $f_{p+1}(\alpha_{p+1})$ is a lifting of $f_p(\delta_p)$. The other lifting $f_{p+1}(\psi^X_{p+1}(\gamma'))$ corresponds to a derivation $D\in\mathrm{Der}_k(\mathcal{O}_{Y,y},(t^{p-e+1})/(t^{p+2}))$. Moreover, we can find $\delta_{p+1}$ as required by induction if and only if $D$ lies in the image of
$$u\colon \mathrm{Der}_k(\mathcal{O}_{X,x},(t^{p-e+1})/(t^{p+2}))\to\mathrm{Der}_k(\mathcal{O}_{Y,y},(t^{p-e+1})/(t^{p+2})).$$

iv) Show that by the induction hypothesis on $\delta_p$, the class of $D$ in $\mathrm{Der}_k(\mathcal{O}_{Y,y},(t^{p-e+1})/(t^{p+2})$ lies in the image of
$$v\colon \mathrm{Der}_k(\mathcal{O}_{X,x},(t^{p-e+1})/(t^{p+1}))\to\mathrm{Der}_k(\mathcal{O}_{Y,y},(t^{p-e+1})/(t^{p+1})).$$

v) Use the previous exercise to show that the cokernels of $u$ and $v$ are naturally isomorphic, so $D$ lies in the image of $u$.

Note that the above exercise proves i) in Theorem 2.9.

**Exercise 2.24.** Let $\gamma_m\in X_m$ be an $m$-jet vanishing with order $e$ along $K_{X/Y}$, where $m\geq 2e$. Suppose that $\gamma_m$ lies over $x\in X$.

i) Show that if we consider on $k[t]/(t^{m+1})$ the $\mathcal{O}_{X,x}$-module structure given by $\gamma_m$, then we have an isomorphism
$$f_m^{-1}(f_m(\gamma_m))\simeq\mathrm{Ker}\left(\mathrm{Der}_k(\mathcal{O}_{X,x},(t^{m-e+1})/(t^{m+1}))\to\mathrm{Der}_k(\mathcal{O}_{Y,y},(t^{m-e+1})/(t^{m+1}))\right)$$
(use the fact that by the previous exercise, for every $\gamma'_m\in X_m$ such that $f_m(\gamma_m)=f_m(\gamma'_m)$, the images of $\gamma_m$ and $\gamma'_m$ in $X_{m-e}$ coincide).

ii) Use Exercise 2.22 to deduce that we have an isomorphism $f_m^{-1}(f_m(\gamma_m))\simeq\mathbb{A}^e$.

**Exercise 2.25.** Use the approach in the previous exercise to prove ii) in Theorem 2.9.

## 3. Applications of spaces of arcs

In this section we describe some applications of the Birational Transformation Theorem. We first give an application to the study of invariants of $K$-equivalent varieties. In particular, we give Kontsevich's theorem saying that two $K$-equivalent varieties have the same Hodge numbers. We next apply Theorem 2.9 applications to singularities. We show how an invariant of singularities, the log canonical threshold, that is defined in terms of log resolutions (equivalently, using divisorial valuations) can be interpreted using the codimensions of certain cylinders in the space of arcs. In the last section we sketch the construction of (Hodge realizations of) motivic integrals, and give some applications of the interpretation of invariants of singularities in terms of spaces of arcs.

In this section we need resolutions of singularities. Therefore we assume that we work over an algebraically closed field of characteristic zero.

3.1. $K$**-equivalent varieties.** Two nonsingular complete varieties $X$ and $X'$ are $K$-*equivalent* if there is a nonsingular variety $Y$ and proper birational morphisms $f\colon Y \to X$ and $f'\colon Y \to X'$ such that $K_{Y/X} = K_{Y/X'}$.

**Remark 3.1.** Note that two $K$-equivalent varieties are in particular birational. On the other hand, if two varieties $X$ and $X'$ as above are birational, then by resolving the birational map (i.e. by taking a resolution of singularities of its graph), we can always find a nonsingular variety $Z$ and proper, birational morphisms $g\colon Z \to X$ and $g'\colon Z \to X'$.

**Remark 3.2.** In fact, in the definition of $K$-equivalence one can put the weaker condition that $K_{Y/X}$ is linearly equivalent with $K_{Y/X'}$ (or even that $K_{Y/X}$ and $K_{Y/X'}$ are numerically equivalent), see Exercise 3.10 below. In particular, if $X$ and $X'$ are birational Calabi-Yau varieties (by which we mean that $\Omega_X^n \simeq \mathcal{O}_X$ and similarly for $X'$), then $X$ and $X'$ are $K$-equivalent.

**Remark 3.3.** The existence of non-isomorphic, but $K$-equivalent varieties is a phenomenon that appears in dimension at least three. Suppose, for example that $X$ is a nonsingular complete threefold and that $C \simeq \mathbb{P}^1$ is a curve in $X$ with $N_{C/X} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. If $\pi\colon W = \mathrm{Bl}_C X \to X$ is the blowing-up of $X$ along $C$, with exceptional divisor $E$, then $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$, with $\pi|_E$ corresponding to the first projection. One can show that the second projection can be extended to a morphism $W \to X'$ that is again an isomorphism outside $E$ and such that $X'$ is again nonsingular. Moreover, the image of $E$ in $X'$ is a curve $C' \simeq \mathbb{P}^1$ such that $N_{C'/X'} \simeq \mathcal{O}_{C'}(1) \oplus \mathcal{O}_{C'}(1)$. The birational transformation $X \dashrightarrow X'$ is a *standard flop*. Since $K_{W/X} = E = K_{W/X'}$, we see that $X$ and $X'$ are equivalent.

Note that in this example we have decompositions $X = (X \smallsetminus C) \coprod C$ and $X' = (X' \smallsetminus C') \coprod C'$ such that $X \smallsetminus C \simeq X' \smallsetminus C'$ and $C \simeq C'$. Whenever we deal with invariants that are additive with respect to such locally closed decompositions, we can conclude that $X$ and $X'$ have the same invariants. However, it is not known whether for two arbitrary $K$-equivalent varieties $X$ and $X'$ there are decompositions

$$X = X_1 \coprod \ldots \coprod X_r,\ X' = X_1' \coprod \ldots \coprod X_r',$$

where the $X_i$ and the $X_i'$ are locally closed and $X_i \simeq X_i'$ for every $i$. However, the Birational Transformation Theorem provides a decomposition with similar properties if instead of considering the varieties themselves we consider instead the corresponding spaces of arcs (however, this is an infinite decomposition, so one needs to develop a suitable formalism).

As an example, we will prove the following theorem of Kontsevich [Kon].

**Theorem 3.4.** *If $Y$ and $Y'$ are $K$-equivalent smooth projective varieties, then $Y$ and $Y'$ have the same Hodge numbers: $h^{p,q}(Y) = h^{p,q}(Y')$ for every $p$ and $q$.*

Recall that if $Y$ is a smooth projective variety, then $h^{p,q}(Y) := \dim_k H^q(Y, \Omega_Y^p)$. These numbers can be put together in the *Hodge polynomial*

$$E(Y; u, v) := \sum_{p,q=0}^{\dim(Y)} (-1)^{p+q} h^{p,q}(Y) u^p v^q.$$

A fundamental property of this invariant is that it can be extended as an additive invariant to all varieties over $k$. In other words, one can define for every variety $Y$ a polynomial $E(Y; u, v) \in \mathbb{Z}[u, v]$ called the *Hodge-Deligne polynomial* of $Y$ such that

1) If $Y$ is smooth and projective, then this is the Hodge polynomial of $Y$.
2) If $Z$ is a closed subvariety of $X$, then $E(Y) = E(Z) + E(Y \smallsetminus Z)$.

It is easy to see that if such a polynomial exists, then it is unique. In fact, it can be computed by induction on dimension, by reducing first to the affine case, and then compactifying to a projective variety and using resolution of singularities. This computation also shows that the degree of $E(X)$ in each of $u$ and $v$ is $\dim(X)$, and the total degree is $2\dim(X)$. Existence over $\mathbb{C}$ was proved by Deligne [De] using the mixed Hodge structure on the cohomology with compact support. A more elementary proof, valid over an arbitrary field of characteristic zero, can be obtained using a theorem of Bittner [Bi].

Another property of the Hodge-Deligne polynomial is that $E(X \times Y) = E(X) \cdot E(Y)$ for every varieties $X$ and $Y$ (one can reduce this to the case of smooth projective varieties, when it follows from definition and the Künneth Formula). Together with additivity, this implies that if $X \to Y$ is piecewise trivial with fiber $Z$, then $E(X) = E(Y) \cdot E(Z)$.

**Example 3.5.** Since the Hodge polynomial of $\mathbb{P}^1$ is $1 + uv$, it follows that $E(\mathbb{A}^1) = (1 + uv) - 1 = uv$. Therefore $E(\mathbb{A}^n) = (uv)^n$.

Suppose now that $X$ is a nonsingular variety of dimension $n$ and $C = (\psi_m^X)^{-1}(S)$ is a locally closed cylinder in $\mathcal{L}(X)$. If we put $E(C; u, v) := E(S; u, v) \cdot (uv)^{-mn}$, then this is well-defined since all projections $\pi_{m+1,m}$ are locally trivial with fiber $\mathbb{A}^n$. It is clear that $E$ is additive in the obvious sense also on cylinders. Note that. for example, $E(\mathcal{L}(X); u, v) = E(X; u, v)$. If $C$ is an arbitrary cylinder, we may define $E(C)$ by additivity, by writing $C$ as a disjoint union of locally closed cylinders. It is clear that $E(C; u, v)$ is a Laurent polynomial of degree $-\operatorname{codim}(C)$ in each of $u$ and $v$.

*Proof of Theorem 3.4.* Note first that after suitably extending the ground field, we may assume that it is uncountable, hence we may apply the results in the previous section. Suppose that $f \colon X \to Y$ is a proper birational morphism of nonsingular varieties. Recall that if we denote by $Z$ the image of the exceptional locus, then the Birational Transformation Theorem, gives decompositions

$$\mathcal{L}(X) \smallsetminus \mathcal{L}(f^{-1}(Z)) = \coprod_{e \in \mathbb{N}} C^{(e)}, \ \mathcal{L}(Y) \smallsetminus \mathcal{L}(Z) = \coprod_{e \in \mathbb{N}} f_\infty(C^{(e)}).$$

We remark that by the Birational Transformation Theorem, we have

$$E(C^{(e)};u,v)=E(f_\infty(C^{(e)});u,v)\cdot(uv)^e$$

for every $e\in\mathbb{N}$. We claim that

$$E(X)=\sum_{e\in\mathbb{N}}E(C^{(e)}),\ E(Y)=\sum_{e\in\mathbb{N}}E(f_\infty(C^{(e)})),$$

where the sums are in the ring of Laurent power series in $u^{-1}$ and $v^{-1}$. Note that together with the previous remark, this implies the assertion of the theorem by the definition of $K$-equivalence. On the other hand, the claim follows from the following more general statement. ☐

**Lemma 3.6.** *Suppose that $X$ is a nonsingular variety and $C$ is a cylinder in $\mathcal{L}(X)$. If $C_1,C_2,\ldots$ are disjoint cylinders contained in $C$ such that $C\smallsetminus(C_1\coprod C_2\ldots)$ is thin, then $E(C)=\sum_{m\in\mathbb{N}}E(C_m)$.*

*Proof.* Let $Z$ be a proper closed subset of $X$ such that $C\smallsetminus\coprod_{m\geq 1}C_m\subseteq\mathcal{L}(Z)$. We need to show that for every $N$, there is $m_0$ such that

$$E(C)-\sum_{i=1}^{m}E(C_i)=E(C\smallsetminus(C_1\coprod\ldots\coprod C_m))$$

has degree $\leq -N$ in both $u$ and $v$.

By Corollary 2.7 we know that if $p\gg 0$, then codim $\mathrm{Cont}^{\geq p}(Z)\geq N$. On the other hand, we have

$$C\subseteq\mathrm{Cont}^{\geq p}(Z)\cup C_1\cup C_2\cup\ldots,$$

hence by Lemma 2.5 there is $m_0$ such that

$$C\subseteq\mathrm{Cont}^{\geq p}(Z)\cup C_1\cup\ldots\cup C_{m_0}.$$

In particular, for every $m\geq m_0$ we have $C\smallsetminus(C_1\coprod\ldots\coprod C_m)\subseteq\mathrm{Cont}^{\geq p}(Z)$. Therefore $E(C\smallsetminus(C_1\coprod\ldots\coprod C_m))$ has degree in each of $u$ and $v$ bounded above by

$$-\operatorname{codim}(C\smallsetminus(C_1\cup\ldots\cup C_m))\leq-\operatorname{codim}\mathrm{Cont}^{\geq p}(Z)\leq -N,$$

which completes the proof. ☐

3.2. **Singularities and spaces of arcs.** In birational geometry singularities are measured via divisorial valuations. We will be interested in singularities of pairs $(X,Y)$, where $Y$ is a proper subscheme of $X$, and for simplicity we assume that $X$ is a nonsingular variety. In order to define invariants, we will consider various *divisors over* $X$: these are prime divisors $E\subset X'$, where $f\colon X'\to X$ is a birational morphism and $X'$ is nonsingular. Every such divisor $E$ gives a discrete valuation $\mathrm{ord}_E$ of the function field $K(X')=K(X)$, corresponding to the DVR $\mathcal{O}_{X',E}$. We will identify two divisors over $X$ if they give the same valuation of $K(X)$.

Let $E$ be a divisor over $X$. If $Y$ is a closed subscheme of $X$, then we define $\mathrm{ord}_E(Y)$ as follows: we may assume that $E$ is a divisor on $X'$ and that the scheme-theoretic inverse image $\pi^{-1}(Z)$ is a divisor. Then $\mathrm{ord}_E(Z)$ is the coefficient of $E$ in $f^{-1}(Z)$. We also define

$\mathrm{ord}_E(K_{-/X})$ as the coefficient of $E$ in $K_{X'/X}$. Note that both $\mathrm{ord}_E(Y)$ and $\mathrm{ord}_E(K_{-/X})$ do not depend on the particular $X'$ we have chosen.

The way to obtain invariants is to look at every divisor $E$ over $X$ and compare $\mathrm{ord}_E(Y)$ with $\mathrm{ord}_E(K_{-/X})$. We will restrict in what follows to one invariant of the pair, its *log canonical threshold*. This is defined by

$$\mathrm{lc}(X,Y) := \inf_E \frac{1+\mathrm{ord}_E(K_{-/X})}{\mathrm{ord}_E(Y)}.$$

It is a general principle that such invariants of singularities can be computed on certain kind of resolutions. In the case of the log canonical threshold this means the following. let $f\colon X' \to X$ be a *log resolution* of the pair $(X,Y)$. This means that $f$ is proper and birational, $X'$ is nonsingular, and $f^{-1}(Y)$ is a divisor such that $f^{-1}(Y)+K_{X'/X}$ is a divisor with simple normal crossings (i.e. we can find algebraic local coordinates $x_1,\ldots,x_n$ on $X'$ such that the corresponding divisor is defined by an equation of the form $x_1^{a_1}\cdots x_r^{a_r}$). One can show that in the definition of the log canonical threshold it is enough to consider only divisors that lie on $X'$. In other words, if we write

$$f^{-1}(Y)=\sum_{i=1}^{s} a_iE_i,\ \text{and}\ K_{X'/X}=\sum_{i=1}^{s}k_iE_i, \tag{3}$$

then $\mathrm{lc}(X,Y)=\min_i \frac{k_i+1}{a_i}$.

The importance of the log canonical threshold in birational geometry comes from the fact that it gives the largest $q>0$ such that the pair $(X,q\cdot Y)$ is log canonical (log canonical pairs–when $Y$ is a divisor–form the largest class of pairs for which the Minimal Model Program is expected to work). For other points of view on log canonical thresholds see [Kol].

We will describe now an interpretation of the log canonical threshold of $(X,Y)$ in terms of the codimensions of the contact loci of $Y$. In fact, we will first use the Birational Transformation Theorem to give a formula for the codimensions of the contact loci of $Y$ in terms of a log resolution of singularities.

Suppose that $f$ is a log resolution of $(X,Y)$ and suppose that the $a_i$ and the $k_i$ are as in (3). There is such a log resolution that is an isomorphism over $X\smallsetminus Y$, i.e. $a_i>0$ for every $i$. For simplicity, we assume that $f$ has this property.

**Theorem 3.7.** ([ELM]) *With the above notation, for every nonnegative integer $m$ we have*

$$\mathrm{codim}(\mathrm{Cont}^m(Y))=\min_{\nu}\sum_{i=1}^{s}(k_i+1)\nu_i,$$

*where the minimum is over all $\nu=(\nu_i)\in\mathbb{N}^s$ such that $\sum_{i=1}^{s}a_i\nu_i=m$ and $\cap_{\nu_i\geq 1}E_i\neq\emptyset$.*

*Proof.* After possibly extending the ground field, we may assume that it is uncountable. Note that

$$f^{-1}(\mathrm{Cont}^m(Y))=\mathrm{Cont}^m(f^{-1}(Y))=\coprod_{\nu}\mathrm{Cont}^{\nu}(E), \tag{4}$$

where $\mathrm{Cont}^{\nu}(E)=\cap_{i=1}^{s}\mathrm{Cont}^{\nu_i}(E_i)$. Here $\nu\in\mathbb{N}^s$ is such that $\sum_{i=1}^{s}a_i\nu_i=m$. Note that since every $a_i>0$, this is a finite set.

The divisor $\sum_i E_i$ has simple normal crossings, hence in order to compute the codimension of $\mathrm{Cont}^{\nu}(E)$ we may take an étale morphism to $\mathbb{A}^n$ to reduce to the case when the $E_i$ are coordinate hyperplanes in an affine space. Using this one sees that $\mathrm{Cont}^{\nu}(E)$ is nonempty if and only if $\cap_{\nu_i\geq 1}E_i\neq\emptyset$, and in this case $\mathrm{codim}(\mathrm{Cont}^{\nu}(E))=\sum_{i=1}^{s}\nu_i$.

On the other hand, note that $\mathrm{Cont}^{\nu}(E)\subseteq\mathrm{Cont}^{e}(K_{X'/X})$, where $e=\sum_{i=1}^{s}k_i\nu_i$. If $p\gg 0$, it follows from Theorem 2.9 i) that $\psi_p^X(\mathrm{Cont}^{\nu}(E))$ is a union of fibers of $f_p$. Moreover, it follows from part ii) of the same result that

$$\mathrm{codim}(f_{\infty}\mathrm{Cont}^{\nu}(E))=\sum_{i=1}^{s}(k_i+1)\nu_i.$$

The decomposition (4) gives a decomposition

$$\mathrm{Cont}^{m}(Y)=f_{\infty}(\mathrm{Cont}^{m}(f^{-1}(Y))=\coprod_{\nu}f_{\infty}(\mathrm{Cont}^{\nu}(E))$$

(recall that by Proposition 2.1, $f_{\infty}$ is bijective over $\mathrm{Cont}^m(Y)$). Therefore

$$\mathrm{codim}(\mathrm{Cont}^{m}(Y))=\min_{\nu}f_{\infty}\left(\mathrm{Cont}^{\nu}(E)\right),$$

and we get the formula in the theorem. □

**Corollary 3.8.** *If $Y$ is a proper closed subscheme of the nonsingular variety $X$, then*

$$\mathrm{lc}(X,Y):=\dim(X)-\max_{m}\frac{\dim Y_m}{m+1}. \tag{5}$$

*Proof.* It is easy to see that the formula in the theorem implies

$$\mathrm{codim}\ \mathrm{Cont}^{\geq m}(Y)=\min_{\nu}\sum_{i}(k_i+1)\nu_i,$$

where the minimum is over those $\nu=(\nu_i)\in\mathbb{N}^s$ such that $\sum_i a_i\nu_i\geq m$ and $\cap_{\nu_i\geq 1}E_i\neq\emptyset$. For every $i$ we have $k_i+1\geq\mathrm{lc}(X,Y)\cdot a_i$, hence

$$m\cdot\mathrm{lc}(X,Y)\leq\mathrm{codim}\ \mathrm{Cont}^{\geq m}(Y)=\mathrm{codim}(Y_{m-1},X_{m-1})=m\cdot\dim(X)-\dim(Y_{m-1}).$$

Suppose now that $i$ is such that $\frac{k_i+1}{a_i}=\mathrm{lc}(X,Y)$. If we take $\nu_j=0$ for $j\neq i$ and $\nu_i=\ell\geq 1$, we see that

$$\mathrm{codim}\ \mathrm{Cont}^{\geq a_i\ell}(Y)\leq a_i\ell\cdot\mathrm{lc}(X,Y),$$

and therefore $\dim(Y_{m-1})\geq m(\dim(X)-\mathrm{lc}(X,Y))$ whenever $m$ is divisible by $a_i$. This completes the proof. □

3.3. **Complements.**

**Exercise 3.9.** Show that if $X$ and $X'$ are $K$-equivalent, then for *every* nonsingular variety $Z$ and proper birational morphisms $Z \to X$ and $Z \to X'$ we have $K_{Z/X} = K_{Z/X'}$.

**Exercise 3.10.** Recall that the Negativity Lemma says that if $f\colon X \to Y$ is a proper birational morphism of nonsingular varieties and if $-D$ is a nef divisor on $X$, then $D$ is effective if and only if $f_*D$ is effective. Use this to show that if $Y \to X$ and $Y \to X'$ are proper birational morphisms of nonsingular varieties such that $K_{Y/X}$ and $K_{Y/X'}$ are numerically equivalent, then $K_{Y/X} = K_{Y/X'}$.

We sketch now the basics of motivic integration. For simplicity, we will discuss only the Hodge realization of motivic integration, that was introduced in [Ba1], following [Kon]. Suppose that $X$ is a nonsingular variety and let $F\colon \mathcal{L}(X) \to \mathbb{Z} \cup \{\infty\}$. We assume that $F^{-1}(m)$ is a cylinder for every integer $m$, and that $F^{-1}(\infty)$ is thin. Consider the sum

$$\sum_{m\in\mathbb{Z}} E(F^{-1}(m)) \cdot (uv)^{-m}. \tag{6}$$

As in §1, we work in the Laurent power series ring in $u^{-1}$ and $v^{-1}$, hence convergence means that the degree with respect to both $u$ and $v$ goes to $-\infty$. If the sum (6) is convergent, then we say that $f$ is integrable. The sum is denoted by $\int_{\mathcal{L}(X)} (uv)^{-f}$ and it is called the (Hodge realization of the) motivic integral of $f$. In fact, one can make the definition for slightly more general functions $F$ (see [Ba1]), but the present one covers essentially all applications.

The fundamental result of the theory is the following consequence of the Birational Transformation Theorem.

**Exercise 3.11** (Change of Variable Formula)**.** Let $f\colon X' \to X$ be a proper birational morphism between nonsingular varieties. Suppose that $F\colon \mathcal{L}(X) \to \mathbb{Z} \cup \{\infty\}$ is a function such that $F^{-1}(m)$ is a cylinder for every integer $m$ and $F^{-1}(\infty)$ is thin. Show that $F$ is integrable if and only if $F \circ f_\infty + \mathrm{ord}_{K_{X'/X}}$ is integrable, and in this case

$$\int_{\mathcal{L}(X)} (uv)^{-F} = \int_{\mathcal{L}(X')} (uv^{-(F\circ f_\infty + \mathrm{ord}_{K_{X'/X}})}.$$

The basic examples of functions to integrate are obtained from functions of the form $\mathrm{ord}_Y$ for a proper closed subscheme $Y$ of $X$ (note that this function satisfies the condition on the level sets for the definition of motivic integrals). Suppose that $f\colon X' \to X$ is a log resolution of $(X, Y)$. Note that we have

$$\mathrm{ord}_Y \circ f_\infty = \mathrm{ord}_{f^{-1}(Y)}.$$

Using the Change of Variable Formula, we get

$$\int_{\mathcal{L}(X)} (uv)^{-\mathrm{ord}_Y} = \int_{\mathcal{L}(X')} (uv)^{-(\mathrm{ord}_{f^{-1}(Y)} + \mathrm{ord}_{K_{X'/X}})}.$$

On the other hand, functions as the one on the right-hand side of the above formula can be explicitly integrated, as the following exercise shows.

**Exercise 3.12.** Let $X$ be a nonsingular variety and $D = \sum_{i=1}^{s} a_i D_i$ an effective divisor on $X$ with simple normal crossings. Show that $\mathrm{ord}_D$ is integrable, and

$$\int_{\mathcal{L}(X)} (uv)^{-\mathrm{ord}_D} = \sum_{J\subseteq\{1,\ldots,s\}} E(D_J^\circ; u, v) \cdot \prod_{i\in J} \frac{uv-1}{(uv)^{a_i}-1},$$

hence it is a rational function. Here we put $D_J^\circ = \cap_{i\in J} D_i \smallsetminus \cup_{j\not\in J} D_j$.

**Remark 3.13.** In the above construction, the important thing is that the Hodge-Deligne polynomial is additive. A similar construction can be done starting from any other additive invariant. In fact, *the motivic integral* is constructed with respect to the universal such invariant, taking value in the Grothendieck group of varieties over $k$. However, in order to make sense of sums like the one in (6) we need to complete the corresponding value ring (see, for example, [Ve] for details).

One can use the above formalism in order to define *stringy invariants* for singular varieties. We sketch here the definition of stringy Hodge numbers, following [Ba1].

Suppose that $X$ is a normal Gorenstein variety of dimension $n$. Since $X$ is normal, if $X_{\mathrm{reg}}$ is the smooth part of $X$, then its complement in $X$ has codimension at least two. Therefore there is on $X$ a Weil divisor $K_X$, unique up to linear equivalence such that $\mathcal{O}(K_X)|_{X^{\mathrm{reg}}} \simeq \Omega^n_{X_{\mathrm{reg}}}$. One can show that since $X$ is Cohen-Macaulay, the sheaf $\mathcal{O}(K_X)$ is isomorphic to the dualizing sheaf $\omega_X$ of $X$. Since $X$ is, in fact, Gorenstein, it follows that $K_X$ is a Cartier divisor.

Suppose now that $\pi\colon X' \to X$ is a resolution of singularities of $X$ such that the exceptional locus of $\pi$ is a divisor with simple normal crossings. One can show that there is a unique divisor $K_{X'/X}$ supported on the exceptional locus of $\pi$ such that $K_{X'/X}$ is linearly equivalent with $K_{X'} - \pi^* K_X$. As in Lecture 2, this is called the discrepancy of $f$. Unlike the case when $X$ is nonsingular, this is not necessarily an effective divisor. The condition that $K_{X'/X}$ is effective means that $X$ has canonical singularities (one can show that this condition does not depend on the choice of the resolution). For more on canonical singularities, and for singularities of pairs, in general, see [Kol].

Suppose now that $X$ is a Gorenstein variety with canonical singularities. We define the *stringy E-function* of $X$, as follows. If $\pi\colon X' \to X$ is a resolution of $X$, let $E_{\mathrm{st}}(X) := \int_{\mathcal{L}(X')} (uv)^{-\mathrm{ord}_{K_{X'/X}}}$.

**Exercise 3.14.** i) Show that if $\pi'\colon X'' \to X'$ is proper and birational such that $\pi\circ\pi'$ is again a resolution of $X$, then

$$K_{X''/X} = K_{X''/X'} + (\pi')^*(K_{X'/X}).$$

ii) Deduce that the stringy $E$-function of $X$ does not depend on the choice of resolution.

iii) One says that a resolution of singularities $\pi\colon X' \to X$ is crepant if $K_{X'/X} = 0$. Show that if $X$ has a crepant resolution $X'$, then $E_{\mathrm{st}}(X; u, v) = E(X; u, v)$.

**Remark 3.15.** If one works with rational powers of $u$ and $v$, then one can defined the stringy $E$-function in the case when $X$ is only $\mathbb{Q}$-Gorenstein and having log terminal singularities.

**Remark 3.16.** Note that $E_{\rm st}(X;u,v)$ is not necessarily a polynomial, but only a rational function in $u$ and $v$.

**Exercise 3.17.** Let $X\subseteq\mathbb{A}^4$ be defined by the cone over a smooth quadric in $\mathbb{P}^3$. Compute $E_{\rm st}(X;u,v)$.

We give now some applications of the description of the log canonical threshold in terms of the dimensions of jet schemes, Recall that a monomial ideal $\mathfrak{a}$ in the polynomial ring $k[x_1,\ldots,x_n]$ is an ideal generated by monomials. If $u=(u_1,\ldots,u_n)\in\mathbb{N}^n$, we write $x^u$ for the monomial $x_1^{u_1}\ldots x_n^{u_n}$. The *Newton polyhedron* of the monomial ideal $\mathfrak{a}$ is the convex hull $P_{\mathfrak{a}}$ of the set

$$\{u\in\mathbb{N}^n\mid x^u\in\mathfrak{a}\}.$$

The polar polyhedron of $P_{\mathfrak{a}}$ is

$$P_{\mathfrak{a}}^{\circ}=\{v\in\mathbb{R}^n\mid\sum_i u_iv_i\geq 1\,\text{for every}\,u\in P_{\mathfrak{a}}\}.$$

It is a standard fact of convex geometry that $(P_{\mathfrak{a}}^{\circ})^{\circ}=P_{\mathfrak{a}}$.

**Exercise 3.18.** Let $\mathfrak{a}$ be a nonzero monomial ideal in $k[x_1,\ldots,x_n]$.

i) For every $a=(a_i)\in\mathbb{N}^n$, put ${\rm Cont}^{\geq a}(x)=\cap_i{\rm Cont}^{\geq a_i}(x_i)$. Show that for every $m\in\mathbb{N}$ we have
$$\mathrm{Cont}^{\geq m}(\mathfrak{a})=\bigcup_{a\in\mathbb{N}^n}\mathrm{Cont}^{\geq a}(x),$$
where the union is over those $a\in\mathbb{N}^n\cap(m+1)P_{\mathfrak{a}}$.

ii) Deduce that ${\rm lc}(\mathbb{A}^n,V(\mathfrak{a}))$ is that positive number $c$ such that $\left(\frac{1}{c},\ldots,\frac{1}{c}\right)$ lies on the boundary of $P_{\mathfrak{a}}$

**Exercise 3.19.** Let $X\subseteq\mathbb{P}^n$ be a smooth hypersurface of degree $d$ and let $Y\subseteq\mathbb{A}^{n+1}$ be the cone over $X$.

i) Show that if $X$ is nonsingular, then
$${\rm lc}(\mathbb{A}^{n+1},Y)=\min\{1,(n+1)/d\}.$$

ii) Show that if dim $X_{\rm sing}=r$, then
$${\rm lc}(\mathbb{A}^{n+1},Y)\geq\min\{1,(n-r)/d\}.$$

**Exercise 3.20.** Show that if $X\subseteq\mathbb{A}^2$ is defined by $x^2-y^3=0$, then ${\rm lc}(\mathbb{A}^2,X)=\frac{5}{6}$.

**Exercise 3.21.** Let $X$ be a scheme of finite type over $k$.

i) Show that for every $m$, there is a section $s_m\colon X\to X_m$ of the canonical projection $\pi_m\colon X_m\to X$, that takes every point $x$ to the *constant $m$-jet* at $x$.

ii) Show that we have an action of the torus $k^*$ on $X_m$ such that the action of $\lambda\in k^*$ is induced by the ring morphism $k[t]/(t^{m+1})\to k[t]/(t^{m+1})$ that takes $t$ to $\lambda t$.

iii) Show that the fixed points of this action are given by $s_m(X)$.

iv) Deduce that for every irreducible component $W$ of $X_m$, if $x\in\pi(W)$, then $s_m(x)\in W$.

There is a local version of the log canonical threshold. Suppose that $X$ is a smooth variety and $x \in X$. If $Y$ is a closed subscheme of $X$ containing $x$, then

$$\mathrm{lc}_x(X, Y) := \max_U \{\mathrm{lc}(U, Y|_U)\}$$

where the maximum is over all open neighborhoods $U$ of $x$ in $X$ (show that this is indeed a maximum).

**Exercise 3.22.** Prove the following version of *Inversion of adjunction*: if $H$ is a smooth divisor in the smooth variety $X$, and if $Y$ is a closed subscheme not containing $H$, then for every $x \in Y \cap H$ we have

$$\mathrm{lc}_x(X, Y) \geq \mathrm{lc}_x(H, Y|_H).$$

**Exercise 3.23.** Let $X$ be a smooth variety, $Y$ a closed subscheme of $X$ and $x$ a point on $Y$. Show that if $\pi_m \colon X_m \to X$ is the canonical projection, then

$$\mathrm{lc}_x(X, Y) = \dim(X) - \sup_m \frac{\dim (\pi_m)^{-1}(x)}{m + 1}.$$

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