Problem session 1

Problem 1. Describe the closed algebraic subsets of $\mathbb{A}^1$.

Problem 2. Let k be an infinite field (in particular, this applies if k is algebraically closed). Show that if $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, then there is $x = (x_1, \ldots, x_n) \in k^n$ such that all $f_i(x_1, \ldots, x_n)$ are nonzero.

In the following problems, we review the definition and some basic facts about Noetherian rings. We assume that all rings are commutative, and have an identity element.

Problem 3. Given a ring R, show that the follwing are equivalent:

i) Every ideal in R is finitely generated.
ii) For every sequence of ideals in R
$$I_1 \subseteq I_2 \subseteq \cdots ,$$
there is m such that $I_i = I_{i+1}$ for all $i \geq m$.
iii) Every family $\mathcal{I}$ of ideals in R has a maximal element with respect to inclusion.

A ring that satisfies these properties is called *Noetherian*.

Problem 4. Prove that if R is a Noetherian ring, then every quotient R/I is again Noetherian.

Problem 5 Prove that if R is a Noetherian ring, and if S is a multiplicative system in R, then the localization $S^{-1}R$ is again Noetherian.