Problem session 10

Problem 1. Let $G(r, n+1)$ be the Grassmannian of $(r-1)$-dimensional linear spaces in $\mathbb{P}^n$.

1) Show that the *incidence correspondence*
$$\Gamma := \{(p, L) \in \mathbb{P}^n \times G(r, n+1) \mid p \in L\}$$
is a closed subset of $\mathbb{P}^n \times G(r, n+1)$.
2) Use this to show that if X is a closed subset of $\mathbb{P}^n$, then the set of $(r-1)$-dimensional linear subspaces of $\mathbb{P}^n$ that intersect X non-trivially is a closed subset of $G(r, n+1)$.
3) Let $p\colon \Gamma \to \mathbb{P}^n$ and $q\colon \Gamma \to G(r, n+1)$ be the morphisms induced by the two projections. Show that $\mathbb{P}^n$ can be covered by open subsets U_i, such that $p^{-1}(U_i) \simeq U_i \times G(r-1, n)$ (over U_i). Similarly, $G(r, n+1)$ can be covered by open subsets V_i, such that $q^{-1}(V_i) \simeq \mathbb{P}^{r-1} \times V_i$ (over V_i).
4) In particular, deduce that the two maps p and q are open.

Problem 2. Let X and Y be two disjoint closed subsets of $\mathbb{P}^n$. The *join* of X and Y is the union $J(X, Y)$ of all lines $\overline{pq}$ in $\mathbb{P}^n$, where $p \in X$ and $q \in Y$. Show that $J(X, Y)$ is a closed subset of $\mathbb{P}^n$.

Problem 3. Let X be a closed subset of $\mathbb{P}^n$. The *Fano variety of lines* on X consists of the lines $\ell \in G(2, n+1)$ such that $\ell \subseteq X$. Show that this is a closed subset of $G(2, n+1)$. Can you describe the Fano variety of lines for the quadric $xy - zw = 0$ in $\mathbb{P}^3$?

Problem 4. Let V be an n-dimensional vector space. A *complete flag* in V is a sequence of vector subspaces of V
$$V_1 \subset \cdots \subset V_{n-1} \subset V_n = V,$$
with $\dim_k(V_i) = i$. Show that there is a closed subset of $\prod_{i=1}^n G(i, V)$ that parametrizes the complete flags in V. This is the (complete) flag variety $\mathrm{Fl}(V)$ of V.