Problem session 11

The following is a useful criterion for proving that a variety is irreducible.

Problem 1. Let $f\colon X \to Y$ be a surjective morphism of projective varieties. Suppose that for every $y \in Y$, the fiber $f^{-1}(y)$ is irreducible, and that all fibers have the same dimension. Show that if Y is irreducible, then X is irreducible, too.

Recall that in the previous problem set we defined the variety of complete flags. In the following problem we generalize this to arbitrary flags.

Problem 2. Let V be a vector space over k, and $1 \leq \ell_1 < \ldots < \ell_r < n$. A *flag of type* $(\ell_1, \ldots, \ell_r)$ in V is a sequence of linear subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_r \subseteq V$, where $\dim_k(V_i) = \ell_i$.

i) Show that the set

$$\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) := \{(V_1, \ldots, V_r) \in G(\ell_1, V) \times \cdots \times G(\ell_r, V) \mid V_1 \subseteq \cdots \subseteq V_r\}$$

is a closed subset in the product of Grassmanians. In particular, this is a projective variety that parametrizes flags in V of type $(\ell_1, \ldots, \ell_r)$.

ii) Show that the projection on the last component gives a surjective morphism $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) \to G(\ell_r, V)$, such that each fiber is isomorphic to $\mathrm{Fl}_{\ell_1,\ldots,\ell_{r-1}}(k^{\ell_r})$.

iii) Use induction on r to prove that each flag variety $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ is irreducible, of dimension

$$\sum_{i=1}^{r} \ell_i(\ell_{i+1} - \ell_i)$$

(where we put $\ell_{r+1} = \dim(V)$). In particular, the dimension of the complete flag variety $\mathrm{Fl}(V)$ on V is $\frac{n(n-1)}{2}$, where $n = \dim(V)$.

The following problem discusses the nilpotent cone of (the Lie algebra of) GL_n.

Problem 3. Let $\mathcal{N}(n)$ denote the *nilpotent cone* of $\mathfrak{gl}_n$

$$\{A \in M_n(k) \mid A \,\text{is nilpotent}\}.$$

i) Show that the coefficients of the characteristic polynomial of a matrix A (namely $\mathrm{trace}(\wedge^i A)$, for $1 \leq i \leq n$) give n equations that define $\mathcal{N}(n)$ in $M_n(k)$.

ii) Consider the set

$$Z = \{(A, (V_1, \ldots, V_{n-1})) \in \mathcal{N}(n) \times \mathrm{Fl}(k^n) \mid A(V_i) \subseteq V_{i-1} \,\text{for every}\, i\}$$

(where in the above formula we make the convention that $V_0 = \{0\}$). Show that the above set is a closed subset of the product $\mathcal{N}(n) \times \mathrm{Fl}(k^n)$.

iii) Let $p\colon Z \to \mathcal{N}(n)$ and $q\colon Z \to \mathrm{Fl}(k^n)$ denote the maps induced by the two projections. Show that each fiber $q^{-1}(y)$ is isomorphic to the set of strictly upper-triangular matrices in $M_n(k)$, hence to $\mathbf{A}^{n(n-1)/2}$.

iv) Deduce that Z is irreducible of dimension $n^2 - n$. Since p is surjective, conclude that $\mathcal{N}(n)$ is irreducible.

v) Show that there is an open subset of $\mathcal{N}(n)$ consisting of those A with $\mathrm{rk}(A) = n-1$. Prove that the fiber $p^{-1}(A)$ consists of one point for every $A \in U$. In particular, $\dim \mathcal{N}(n) = n^2 - n$.

Remark. It is not hard to see that in fact, p is an isomorphism over U, but we will not discuss this. On the other hand, we have seen that $\mathcal{N}(n)$ can be defined by n equations in $M_n(k)$, and that its codimension is $n^2 - n$. The following problem shows that, in fact, these n elements generate the ideal of $\mathcal{N}(n)$. In other words, $\mathcal{N}(n)$ is a complete intersection. This problem requires some background in commutative algebra.

Problem 4. We use the notation in the previous problem. Let R be the quotient of the coordinate ring $\mathcal{O}(M_n(k))$ by the equations $f_1, \ldots, f_n$ defining $\mathcal{N}(n)$.

i) Show that the ring R is *generically reduced*, that is, there is a non-nilpotent $f \in R$ such that the localization R_f is reduced (Hint: consider how the equations $f_1, \ldots, f_n$ look around the point given by $A = (a_{i,j})$, where $a_{i,i+1} = 1$ for $1 \leq i \leq n-1$, and $a_{i,j} = 0$ otherwise).
ii) Show that the results in the previous problem imply that R is a Cohen-Macaulay ring, having only one minimal prime. Deduce from this and i) that R is reduced.