# Problem session 12

Let $X = \mathrm{Specm}(k[S])$ be the n.n.n. affine toric variety corresponding to the (finitely generated, integral) monoid $S$. Let $M$ be the finitely generated free abelian group such that $S$ is a submonoid of $M$ and $S$ generates $M$ as a group.

We have seen in class that we have a bijection between $T$-invariant irreducible closed subsets of $X$ and faces of $S$ (these are submonoids $F \subseteq S$ such that if for $u, v \in S$ we have $u + v \in F$, then $u, v \in F$), such that $Y$ corresponds to $F$ if $I(Y) = \bigoplus_{u \in S \smallsetminus F} k\chi^u$.

**Problem 1**. Show that if $Y \subseteq X$ is as above, then $Y$ itself has a natural structure of n.n.n. affine toric variety, as follows (moreover, by taking the torus in each such $Y$, we recover precisely all the orbits of the action of $T$ on $X$).

i) Show that if $F$ corresponds to $Y$ as above, then we have a canonical isomorphism $\mathcal{O}(Y) \simeq k[F]$.
ii) Let $T_Y := \mathrm{Specm}(M_Y)$, where $M_Y \subseteq M$ is the subgroup generated by $F$. Show that we have an open immersion $T_Y \hookrightarrow Y$ that makes $Y$ a n.n.n. affine toric variety.
iii) Denote by $O_Y$ the image of the torus $T_Y$ in $Y$. Show that $O_Y$ is an orbit of the action of $T$ on $X$.
iv) Show that the map that takes $Y$ to $O_Y$ induces a bijection between the $T$-invariant irreducible closed subsets of $X$ and the orbits of the $T$-action on $X$.

**Problem 2**. Let $S$ be a finitely generated, integral monoid.

i) Show that if $S$ is generated (as a monoid) by $u_1, \ldots, u_r$, then every face $F$ of $S$ is generated by those $u_i$ that lie in $F$.
ii) Deduce that every n.n.n. affine toric variety has only finitely many orbits.

**Problem 3**. Let $M$ and $M'$ be two finitely generated free abelian groups, and $T = \mathrm{Specm}(k[M])$ and $T' = \mathrm{Specm}(k[M'])$ the corresponding tori. Show that there is a natural bijection

$$\mathrm{Hom}_{\mathrm{gp}}(M, M') \simeq \mathrm{Hom}_{\mathrm{alg-gp}}(T', T).$$

**Problem 4**. Let $X$ and $Y$ be n.n.n. affine toric varieties, and $T_X \subseteq X$ and $T_Y \subseteq Y$ the corresponding tori. A *morphism of toric varieties* $f \colon X \to Y$ is a morphism of algebraic varieties such that it induces a morphism of algebraic groups $g \colon T_X \to T_Y$.

i) Show that in this case $f$ is compatible with the group actions, i.e.

$$f(\lambda \cdot u) = g(\lambda) \cdot f(u)$$

for every $\lambda \in T_X$ and every $u \in X$.
ii) Show that the correspondence $S \to \mathrm{Specm}(k[S])$ defines an anti-equivalence of categories between the category of finitely generated integral monoids and that of n.n.n. affine toric varieties.