Problem session 3

Problem 1. Let $X \subseteq \mathbb{A}^n$ be a closed algebraic subset, and let $f \in \mathcal{O}(X)$. When is the principal affine open subset $D_X(f)$ equal to X ?

Problem 2. Suppose that $\mathrm{char}(k) = p > 0$, and consider the map $f\colon \mathbb{A}^n \to \mathbb{A}^n$ given by $f(x_1, \ldots, x_n) = (x_1^p, \ldots, x_n^p)$.

i) Show that f is a morphism of affine algebraic varieties, and that it is a homeomorphism, but it is not an isomorphism.
ii) Show that if Y is a closed subset of $\mathbb{A}^n$ defined by equations with coefficients in $\mathbb{F}_p$, then f induces a morphism from Y to Y.

Problem 3. Let $Y \subseteq \mathbb{A}^2$ be the cuspidal curve defined by the equation $x^2 - y^3 = 0$. Construct a bijective morphism $f\colon \mathbb{A}^1 \to Y$. Is it an isomorphism ?

Problem 4.

i) Show that $\mathbb{A}^1 \smallsetminus \{0\}$ is an affine variety.
ii) Let $U = \mathbb{A}^2 \smallsetminus \{(0,0)\}$. What is $\mathcal{O}(U)$?
iii) Deduce that U is not an affine variety.