Problem session 4

Problem 1. (Gluing morphisms). Let X and Y be algebraic prevarieties. Suppose that we have an open cover $X = U_1 \cup \ldots \cup U_n$ and morphisms $f_i \colon U_i \to Y$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every i and j. Show that in this case there is a unique morphism $f \colon X \to Y$ such that $f|_{U_i} = f_i$ for every i.

Problem 2. Let $f \colon X \to Y$ be a morphism of prevarieties and let $Y = V_1 \cup \ldots \cup V_r$ be an open cover.

i) Show that f is a closed immersion if and only if the induced morphism $f^{-1}(V_i) \to V_i$ is a closed immersion for every i.

ii) The same assertion holds for open or locally closed immersions. In fact, in this case it is enough to assume that $f(X) \subseteq V_1 \cup \ldots \cup V_r$.

iii) If Y is affine, then f is a closed immersion if and only if X is affine, too, and the induced homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective.

Problem 3. Show that a disjoint union of finitely many (affine) prevarieties is again an (affine) prevarieties.

Problem 4. Let $R = \oplus_{m \in \mathbb{N}} R_m$ be a graded ring, and I a homogeneous ideal in R.

i) Show that I is a radical ideal if and only if for every homogeneous $f \in R$ such that $f^r \in I$ for some $r \geq 1$, we have $f \in I$.

ii) Show that I is a prime ideal if and only if $I \neq R$, and for every two homogeneous elements f, $g \in I$ with $fg \in I$, either $f \in I$, or $g \in I$.

Problem 5. Show that $\mathbb{A}^1$ is *not* isomorphic to any proper open subset of itself.