Problem session 6

Problem 1. Show that if X is a closed subset in $\mathbb{P}^n$, then X is irreducible if and only if the ideal $I(X) \subseteq k[x_0, \ldots, x_n]$ is prime.

Problem 2. Let $S = k[x_0, \ldots, x_n]$ and $R = k[x_1, \ldots, x_n]$. Recall that if J is an ideal in R, then

$$J^{\mathrm{hom}} := \left(f^{\mathrm{hom}} \mid f \in J\right),$$

where $f^{\mathrm{hom}} = x_0^{\deg(f)} \cdot f(x_1/x_0, \ldots, x_n/x_0) \in S$. On the other hand, if $\mathfrak{a}$ is a homogeneous ideal in S, then $\overline{\mathfrak{a}} := \{h(1, x_1, \ldots, x_n) \mid h \in \mathfrak{a}\} \subseteq R$.

An ideal $\mathfrak{a}$ in S is called x_0-saturated if $(\mathfrak{a} \colon x_0) = \mathfrak{a}$ (recall that $(\mathfrak{a} \colon x_0) := \{u \in S \mid x_0 u \in \mathfrak{a}\}$.

i) Show that the above maps give inverse bijections between the ideals in R and the x_0-saturated homogeneous ideals in S.

ii) Show that we get induced bijections between the radical ideals in R and the homogeneous x_0-saturated radical ideals in S. Moreover, a homogeneous radical ideal $\mathfrak{a}$ is x_0-saturated if and only if either no irreducible component of $V(\mathfrak{a})$ is contained in the hyperplane $(x_0 = 0)$, or if $\mathfrak{a} = S$.

iii) The above correspondence induces a bijection between the prime ideals in R and the prime ideals in S that do not contain x_0.

iv) Recall that we have an open immersion

$$\mathbb{A}^n \hookrightarrow \mathbb{P}^n, \ \ (u_1, \ldots, u_n) \to (1 \colon u_1 \colon \ \ldots \colon u_n),$$

which allows us to identify $\mathbb{A}^n$ with the complement of the hyperplane $(x_0 = 0)$ in $\mathbb{P}^n$. Show that for every ideal J in R we have $\overline{V_{\mathbb{A}^n}(J)} = V_{\mathbb{P}^n}(J^{\mathrm{hom}})$.

v) Show that for every homogeneous ideal $\mathfrak{a}$ in S, we have $V_{\mathbb{P}^n}(\mathfrak{a}) \cap \mathbb{A}^n = V_{\mathbb{A}^n}(\overline{\mathfrak{a}})$.

vi) Deduce in particular that the maps given by $Z \subseteq \mathbb{A}^n \to \overline{Z}$ and $W \subseteq \mathbb{P}^n \to W \cap \mathbb{A}^n$ give inverse bijections (preserving the irreducible decompositions) between the nonempty closed subsets of $\mathbb{A}^n$ and the nonempty closed subsets of $\mathbb{P}^n$ that have no irreducible component contained in the hyperplane $(x_0 = 0)$.