Problem session 7

Problem 1. Let X be a prevariety, and $f \in \mathcal{O}(X)$ a regular function on X. Put $X_f := \{x \in X \mid f(x) \neq 0\}$.

1) Show that the restriction map $\mathcal{O}(X) \to \mathcal{O}(X_f)$ induces a ring homomorphism $\rho\colon \mathcal{O}(X)_f \to \mathcal{O}(X_f)$, where $\mathcal{O}(X)_f$ is the ring of fractions of $\mathcal{O}(X)$ with denominators powers of f.
2) Show that ρ is an isomorphism. (Hint: you can use the case when X is affine which we proved in class; cover X by affine open subsets to reduce to this case; prove first injectivity of the map, then you can use this to prove surjectivity).

Use the above problem to prove the following criterion for a prevariety to be affine.

Problem 2. Suppose that X is a prevariety, and $f_1, \ldots, f_r \in \mathcal{O}(X)$ are such that

1) Each X_{f_i} is an affine variety.
2) The ideal generated by $f_1, \ldots, f_r$ in $\mathcal{O}(X)$ is equal to $\mathcal{O}(X)$.

Show that X is an affine variety.

The following problem considers the *Segre embedding* to show that the product of two projective varieties is again projective.

Problem 3. Consider two projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$. Let $N = (m+1)(n+1)-1$, and we denote the coordinates on $\mathbb{A}^{N+1}$ by $z_{i,j}$, with $0 \leq i \leq m$ and $0 \leq j \leq n$.

1) Show that the map $\mathbf{A}^{m+1} \times \mathbf{A}^{n+1} \to \mathbf{A}^{N+1}$ given by
$$((x_i)_i, (y_j)_j) \to (x_i y_j)_{i,j}$$
induces a morphism
$$\phi_{m,n}\colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N.$$
2) Consider the ring homomorphism $f_{m,n}\colon k[z_{i,j} \mid 0 \leq i \leq m, 0 \leq j \leq n] \to k[x_1, \ldots, x_m, y_1, \ldots, y_n]$, given by $f_{m,n}(z_{i,j}) = x_i y_j$. Show that $\ker(f_{m,n})$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\phi_{m,n}$ (in particular, this image is closed).
3) Show that $\phi_{m,n}$ is a closed immersion.
4) Deduce that if X and Y are (quasi)projective varieties, then $X \times Y$ is a (quasi)projective variety.