Problem session 9

We will describe the Grassmannian as an algebraic variety. By definition, the Grassmannian $G(r,n)$ is the set of r-dimensional linear subspaces in k^n. When we don't want to make reference to a fixed basis, and we talk about the set of r-dimensional linear subspaces of a vector space V, we write $G(r,V)$. Note that we can alternatively think of $G(r,n)$ as the set of $(r-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$.

If W is a k-vector space, then we use the notation $\mathbb{P}(W)$ for the projective space of lines through the origin in W. Of course, after the choice of a basis $e_1,\ldots,e_n$ in W, this becomes isomorphic to $\mathbb{P}^{n-1}$.

Problem 1. We first describe the Grassmannian as an algebraic variety via its *Plücker embedding* in a projective space, as follows.

i) Show that there is an injective map $G(r,V) \to \mathbb{P}(\wedge^r V)$ that takes the subspace U of V to $\wedge^r U \subseteq \wedge^r V$.

ii) Show that this identifies $G(r,V)$ with the set of *decomposable* vectors in $\mathbb{P}(\wedge^r V)$, that is with (classes of) nonzero vectors $v \in \wedge^r V$ that can be written as $v_1 \wedge \cdots \wedge v_r$ for some $v_1,\ldots,v_r \in V$.

iii) Show that $v \in \wedge^r V$ is *divisible* by $v_1 \in V$ (that is, one can write $v = v_1 \wedge w$ for some $w \in \wedge^{r-1} V$) if and only if $v_1 \wedge v = 0$ in $\wedge^{r+1} V$. Deduce that a nonzero vector $v \in \wedge^r V$ lies over the image of $G(r,V)$ if and only if there is a vector subspace $W \subseteq V$ of dimension $\geq r$ such that $w \wedge v = 0$ in $\wedge^{r+1} V$ for every $w \in W$.

iv) Consider the linear map

$$\wedge^r V \to \mathrm{Hom}_k(V, \wedge^{r+1} V),$$

that takes v to the linear map $w \to w \wedge v$. Deduce from iii) that the cone over $G(r,V)$ is the inverse image of the locus of maps $V \to \wedge^{r+1} V$ of rank $\leq d-r$. Deduce that $G(r,V) \subseteq \mathbb{P}(\wedge^r V)$ is a closed subset, cut out by degree $(d-r+1)$ homogeneous polynomials, where $d = \dim(V)$. (Note: these polynomials do not generate the ideal of $G(r,V)$. In fact, this ideal can be generated by quadrics).

v) Note that if $r = 1$, then the Plücker embedding is simply the isomorphism $G(1,V) \simeq \mathbb{P}(V)$, as expected. Similarly, if $r = d-1$, then the Plücker embedding is again an isomorphism $G(d-1,V) \simeq \mathbb{P}(\wedge^{d-1} V)$. Note that using the isomorphism $\wedge^{d-1} V \simeq V^*$ induced by $\wedge^{d-1} V \otimes V \to \wedge^d V \simeq k$, this simply recovers the parametrization of the hyperplanes in V by $\mathbb{P}(V^*)$.

Problem 2. Show that as a set, $G(r,d)$ can be identified to the set of $r \times d$ matrices of maximal rank with entries in k, modulo the left action by multiplication of the group $GL_r(k)$. Moreover, the Plücker embedding $G(r,d) \hookrightarrow \mathbb{P}^N$, where $N = \binom{d}{r} - 1$, is induced by the map that takes the matrix A to the tuple with entries given by the r-minors of the matrix A.

Problem 3. For every $1 \leq i_1 < \ldots < i_r \leq n$, consider the subset $U_{i_1,\ldots,i_n}$ of $G(r,d)$ corresponding to matrices whose r-minor on the columns $i_1,\ldots,i_r$ is nonzero.

i) Show that this is an open subset of $G(r, d)$, isomorphic to $\mathbf{A}^{r(d-r)}$ (recall that the algebraic variety structure on $G(r, d)$ has been defined via the Plücker embedding).
ii) Show that such open subsets cover $G(r, d)$, and every two of them intersect non-trivially. Deduce that $G(r, d)$ is irreducible.