Math 412 - Intorduction to Abstract Algebra

Homework 4 solutions

1. (4.1.13) If $f(x)$ is a zero divisor, then there exists a non zero polynomial $g(x) = b_0 + b_1 x + \cdots + b_m x^m$ (with $b_m \neq 0_R$) such that $f(x)g(x) = 0_{R[x]}$. The product expansion of $f(x)g(x)$ has highest order term $a_n b_m x^{n+m}$. Since $a_n b_m x^{n+m} = 0_{R[x]}$, this implies that $a_n b_m = 0_R$. Therefore, a_n is a zero divisor.

2. (4.1.15a) We see that

$$(1_R + ax)(1_R - ax + a^2x^2) = (1_R - ax + a^2x^2)(1_R + ax) = 1_R + a^3x^3,$$

which equals 1_R since $a^3 = 0_R$. Therefore, $(1_R + ax)$ is a unit.

3. (4.1.18) To show that ϕ is surjective, we consider the images of constant polynomials. That is, for $a_0 \in R$, the image $\phi(a_0) = a_0$. Since every element of R is mapped to, ϕ is surjective. To show that ϕ is a homomorphism of rings, we must show for any $f(x), g(x) \in R[x]$ that $\phi(f(x) + g(x)) = \phi(f(x)) + \phi(g(x))$ and $\phi(f(x)g(x)) = \phi(f(x))\phi(g(x))$. Without loss of generality, assume that $n \geq m$. Using the notation from problem (4.1.13) above,

$$\phi(f(x) + g(x)) = \phi((a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_mx^m))$$

which equals

$$\phi((a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n) = a_0 + b_0.$$

This equals $\phi(f(x)) + \phi(g(x))$. The constant term of the expression $f(x)g(x)$ equals a_0b_0, therefore, $\phi(f(x)g(x)) = \phi(f(x))\phi(g(x))$. (Note: we could write out the general formula for this step, but it gets a little messy.)

4. (4.1.20) This is not a ring homomorphism because of the product rule. That is, taking derivatives is not multiplicative. For example, $D(2x) = 2 \neq D(2)D(x) = 0 \cdot 1 = 0$. This is not a ring isomophism. For example, it is not injective since any constant maps to 0.

5. (4.2.8) By definition, the greatest common divisor $d(x)$ is the monic polynomial of greatest degree that divides both $f(x)$ and $g(x)$. (Note: The unicity was proved in Theorem 4.8 on page 97 of the text.) If we assume the second part of Corollary 4.9 on page 99, then this implies that $h(x)|d(x)$. This means that there exists a polynomial $c(x)$ where $c(x)h(x) = d(x)$. By hypothesis, $\deg h(x) = \deg d(x)$, which implies $c(x) = c$, a constant. Therefore, we have $h(x) = cd(x)$.

   Note: There was a minor typo on the original assignment 4 page in the numbering of the problems below.

6. (4.2.14) If $f(x)$ and $g(x)$ are relatively prime, then (by Theorem 4.8 on page 97) there exist polynomials $u(x)$ and $v(x)$ such that

$$f(x)u(x) + g(x)v(x) = 1_F.$$

If $f(x)|h(x)$ and $g(x)|h(x)$, then we know there exist polynomials $a(x)$ and $b(x)$ such that

$$f(x)a(x) = g(x)b(x) = h(x).$$

If we multiply the first equation by $h(x)$ we obtain

$$f(x)u(x)g(x)b(x) + g(x)v(x)f(x)a(x) = h(x).$$

Therefore, we have

$$f(x)g(x)[u(x)b(x) + v(x)a(x)] = h(x).$$

This implies that $f(x)g(x)|h(x)$.

7. (4.2.16) The idea is to show that the two greatest common divisors divide each other, and since they are both monic, this implies they are equal.

The greatest common divisor $d_1(x)$ of $h(x)$ and $g(x)$ divides both $h(x)$ and $g(x)$. Therefore, $d_1(x)$ divides both $f(x)h(x)$ and $g(x)$. If we let $d_2(x)$ denote the greatest common divisor of $f(x)h(x)$ and $g(x)$, then, by part two of Corollary 4.9, we have $d_1(x)|d_2(x)$.

If $a(x)|g(x)$, then by part two of Corollary 4.9, $a(x)$ and $f(x)$ are relatively prime. Since (in particular) $d_2(x)|g(x)$, we know, by Theorem 4.10, that $d_2(x)|h(x)$. Therefore, we know, by Corollary 4.9, that $d_2(x)|d_1(x)$.