Partial Solutions for First Midterm Review Problems
Math 412, Winter 2014

3.3.15) If $f : R \to S$ be a homomorphism of rings and r is a zero divisor of R, then $f(r)$ need not be a zero divisor of S.

Examples: If $f : \mathbb{Z}_6 \to \mathbb{Z}_8$ is the zero map, then 3 is a zero divisor in $\mathbb{Z}_6$, but $f(3) = 0$ is not a zero divisor in $\mathbb{Z}_8$ (since it is zero). A less trivial example is given by the homomorphism $f : \mathbb{Z}_6 \to \mathbb{Z}_2$ given by taking $[a]_6$ to $[a]_2$. Then 3 is a zero divisor in $\mathbb{Z}_6$, since $3 \cdot 2 = 0$ in $\mathbb{Z}_6$, but $f(3) = 1$ is not a zero-divisor in $\mathbb{Z}_2$. Recall that $\mathbb{Z}_2$ is an integral domain, so has no zero divisors.

3.3.28) a) There exists an example of a homomorphism of rings $f : R \to S$ such that R has an identity but S does not. This does not contradict part 4 of Theorem 3.10.

Examples: If R has an identity and S does not have an identity, one can still define the zero map $f : R \to S$. Less trivially, let $f : \mathbb{Z}_2 \to \mathbb{Z}_2 \times E$ be given by $f(0) = (0, 0)$ and $f(1) = (1, 0)$. Neither of these examples contradicts Theorem 3.10, since the homomorphisms are not surjective.

b) There exists an example of a homomorphism of rings $f : R \to S$ so that S has an identity but R does not.

Examples: The zero map can again be used to construct trivial examples. Less trivially, consider the homomorphism $f : E \to \mathbb{Z}$ given by $f(n) = n$ for all $n \in E$.

3.3.30) Claim: If $f : R \to S$ is a homomorphism of rings and

$$K = \{r \in R \mid f(r) = 0_S\},$$

then K is a subring of R.

Proof: We first notice that $f(0_R) = 0_S$, by Theorem 3.10, so $0_R \in K$, which implies that K is non-empty.

Suppose that $a, b \in K$, then $f(a) = 0_S$ and $f(b) = 0_S$. Then, by part (3) of Theorem 3.10,

$$f(a - b) = f(a) - f(b) = 0_S - 0_S = 0_S$$

(where the first equality follows from Theorem 3.10). So, $a - b \in K$. Therefore, K is closed under subtraction. Similarly, $f(ab) = f(a)f(b) = 0_S 0_S = 0_S$ (where the last equality follows from Theorem 3.5), so $ab \in K$. Therefore, K is closed under multiplication. Theorem 3.6 then implies that K is a subring of R.

3.3.35) a) Claim: There does not exist an isomorphism $f : \mathbb{Z} \to E$.

Proof: If such an isomorphism existed, then, by Theorem 3.10, $f(1)$ would be an identity for E and we previously observed that E does not have an identity.

b) Claim: There does not exist an isomorphism $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to M(\mathbb{R})$.

Proof: This follows from the fact that $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is commutative, by Theorem 3.1, while we know that $M(\mathbb{R})$ is non-commutative. More explicitly, suppose that f is an isomorphism and $A, B \in M(\mathbb{R})$ are chosen so that $AB \neq BA$. Then, there would exist $a, b \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ so that $f(a) = A$ and $f(b) = B$. Since $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is a commutative ring, by Theorem 3.1,

$$AB = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = BA$$

which gives a contradiction.

c) Claim: There does not exist an isomorphism $f : \mathbb{Z}_4 \times \mathbb{Z}_{14} \to \mathbb{Z}_{16}$

Proof: $\mathbb{Z}_4 \times \mathbb{Z}_{14}$ has 56 elements and $\mathbb{Z}_{16}$ has 16 elements, so there is no bijection between $\mathbb{Z}_4 \times \mathbb{Z}_{14}$ and $\mathbb{Z}_{16}$, hence no isomorphism.

e) Claim: There does not exist an isomorphism $f : \mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z}$.

Proof: If such an isomorphism existed, then $f(0, 1)$ would be a non-zero element of $\mathbb{Z}$ so that $f(0, 1) + f(0, 1) = f((0, 1) + (0, 1)) = f((0, 0)) = 0$. However, $\mathbb{Z}$ does not contain any such element.

f) Claim: There does not exist an isomorphism $f : \mathbb{Z}_4 \times \mathbb{Z}_4 \to \mathbb{Z}_{16}$.

Proof: If such an isomorphism existed, then since $\mathbb{Z}_4 \times \mathbb{Z}_4$ has identity $(1, 1)$ and $\mathbb{Z}_{16}$ has identity 1, Theorem 3.10 implies that $f(1, 1) = 1$. But then

$$f(2, 2) = f((1, 1) + (1, 1)) = f(1, 1) + f(1, 1) = 1 + 1 = 2$$

and

$$f(0, 0) = f((2, 2) + (2, 2)) = f(2, 2) + f(2, 2) = 2 + 2 = 4 \neq 0.$$

However, this contradicts part (1) of Theorem 3.10, which guarantees that $f(0, 0) = 0$.