Mathematics 556 Fall 1994

HOMEWORK ASSIGNMENTS

Assignment # 1. Due 16 September.

1. Folland page 67, problem 1. Denoted 67/1 in future. Do this is a general scalar product space in place of $\mathbf{C}^k$.

2. Folland 67/3.

3. Folland 68/9. Again in a general scalar product space.

4. If u and w are unit vectors in $\mathbf{R}^3$ with $u \perp w$, describe in words the curve given parametrically by $u \cos t + w \sin t$.

5. (**i.**) If A is a real $k \times k$ matrix prove that the real vectors in the kernel of A span the kernel. (**ii.**) If A is a real $k \times k$ symmetric matrix, prove that A has an orthonormal basis of real eigenvectors. **Hint.** Do not start from scratch. Use part **i**.

6. i. Suppose that A_1 and A_2 are symmetric linear transformations on a finite dimensional vector space V. Denote by $\lambda_! < \lambda_2, < \ldots, \lambda_m$ the eigenvalues of A_1. The eigenspace is denoted E_{λ_j}. Prove that A_2 maps E_{λ_j} to itself.

ii. Prove that V has an orthonormal basis each of whose elements is an eigenvector of both A_1 and A_2. **Hint.** Apply the spectral theorem on the vector space E_{λ_j}. **Beware.** There is a trap. It is not true that an eigenvector of A_1 is automatically an eigenvector of A_2 as you see by considering the 2×2 case with $A_1 = I$ and $A_2 = \operatorname{diag}\{1, -1\}$.

iii. Suppose that $\mathbf{A}$ is a finite commuting family of symmetric linear transformations on a scalar product space V, that is any pair A_1, A_2 of elements of $\mathbf{A}$ satisfies $A_1 A_2 = A_2 A_1$. Prove that there is an orthonormal basis of V consisting of vectors each of which is an eigenvector of all elements of $\mathbf{A}$. **Extra.** Prove the result for infinite families without using the Axiom of Choice.

Discussion. Extra means not required. These are often the most intriguing parts of the problems.

7. Suppose that $u \neq 0$ and v are elements of a finite dimensional vector space V and that any vector w which is ortogonal to u is orthogonal to v. Prove that v is equal to a scalar multiple of u. **Hint.** Show that v is equal to its projection on the subspace of scalar multiples of u.

8. Let $\mathcal{P}_2$ be the vector space of polynomials of degree less than or equal to 2. Define a scalar product on this space by

$$\langle p, q \rangle := \int_{-\infty}^{\infty} p(x)\, q(x)\, e^{-x^2/2}\, dx\,.$$

Perform the Gramm-Schmidt process on the polynomials, $1, x, x^2$.
Discussion. Continuing in this way generates the *Hermite Polynomials*.

Assignment # 2. Due 23 September.

1. Show that a linear transformation on a finite dimensional scalar product space which is both symmetric and antisymmetric must vanish. Prove that a linear transformation on a finite dimensional scalar product space has at most one expression as a sum of a symmetric plus an antisymmetric transformation.

2. Prove that a transformation of a finite dimensional scalar product space is normal if and only if its symmetric and antisymmetric parts commute.

2.3 If B is a normal transformation on a finite dimensional scalar product space, prove that there is a polynomial p of degree at most equal to the dimension of the space such that $p(B) = B^*$. **Hint.** Compute the matrix of $p(B) - B^*$ when the space is given an orthonormal basis consisting of eigenvectors of for B.

2.5 Prove that two normal transformations, B_1 and B_2 commute if and only if the family of symmetric transformations $\{B_1 + B_1^*,\ i(B_1 - B_1^*),\ B_2 + B_2^*,\ i(B_2 - B_2^*)\}$ is a commuting family as in problem 6 of assignment # 1.

3. Prove the **Riesz Representation Theorem.** *If $L : V \to \mathbf{C}$ is a linear functional on a finite dimensional scalar product space, then there is a unique vector w such that*

$$L(v) = \langle v, w \rangle \tag{1}$$

for all $w \in V$. Furthermore, if $\{e_j\}$ is an orthonormal basis for V then $w = \sum \overline{L(e_j)} e_j$.

The modeling of diffusion had some implicit simplifications. For example we have not at all considered the diffusion of the solute from one container to the other. Thus we have supposed that each container holds a fixed number M_j of molecules of solute.

Next, molecules per unit volume is not the usual way to measure concentration. It ignores the solute! Here is a more careful analysis. Suppose that a container has N molecules of solute and L molecules of solvent. Denote by m_{solute} the mass of a molecule of solute and similarly for solvent. Then, the concentration by mass of solute is given by

$$C = \frac{m_{solute} N}{m_{solute} N + m_{solvent} L} . \tag{2}$$

The model discussed in this section is appropriate for dilute solutions where N is very small compared to L so the terms involving N in the denominators of 2, can be safely neglected. In this situation the diffusion will also not much affect the temperature and pressure in the containers. Then the concentration by mass is propotional to N and one justifies the equations of this section.

4. Consider the diffusion between two containers of solute when it need not be dilute. Write the equations for the evolution of the mass concentrations $C_j(t)$. Note that the equations are not linear. Show that $N_1 + N_2$ is independent of time. Use this relation to eliminate the unknown C_2 leaving a single autonomous nonlinear differential equation for C_1. Analyse this equation by whatever method you know to show that there is exactly one stationary solution and that every solution tends to that solution as t tends to infinity. This proves approach to the equilibrium even for the nonlinear model. **Hint.** The easiest method uses the phase "plane" which in this case is the real line. The vector field on the line points either to the left or the right at each point. The extreme opposite approach is to solve explicitely by separation of variables and to analyse the resulting formulas.

5. i. Find the trigonometric polynomial $p(\theta)$ of degree 1 such that $p(0) = 0$ and $p(\pi) = 1$. **ii.** Find the trigonometric polynomial $q(\theta)$ of degree 2 such that $q(0) = 0$ and $q(2\pi/3) = 1$ and $p(4\pi/3) = -1$. **Hint**. If you draw a graph you should be able to see the answer on sight. This is a short answer question!

6. Folland 31/6.

7. Folland 31/16.

8. Consider diffusion among m containers all of which have equal volume and all of which are connected to all others with all permeabilities equal. That is $V_j = V$ for all j and $k_{ij} = k > 0$ for all $i \neq j$. For the resulting matrix A find all the eigenvalues and eigenspaces. Verify that all solutions tend to an equilibrium of equal concentrations, and that the rate of approach increases if k increases or V decreases.

Discussion. Another highly symmetric case is if there are $m = 2^K$ containers of equal volumes which are arranged at the vertices of K dimensional cube with equal nonzero permeabilities along the edges of the cube. The other permeabilities are zero. I have not yet solved this one. If anyone sees an elegant computation, please tell me.

Assignment # 3. Due 30 September.

1. Folland 37/3.

Discussion. This and the next question involve functions which are not continuously differentiable, but which are Lipshitz continuous, that is there is a constant L such that

$$|f(\theta) - f(\phi)| \le L\,|\theta - \phi|$$

for all θ, ϕ. Our proof of convergence is valid for such functions with the only change being that wherever $\max|f'|$ appears in the proof it is replaced by L. The conclusion is that we have the right to sum these series even without doing §2.2 of Folland.

2. Folland 37/4.

3. Folland 42/2.

Discussion. Theorem 2.4 of Folland is not in our handouts and is important in the problems that follow. You will need to read and digest it.

4,5,6,7. Folland 42/3,5,6,7.

8. Exhibit explicitely the orthonormal basis for the three dimensional space of 2π periodic functions on the discrete line $\mathbf{R}_{2\pi/3}$ which is given in the theorem on the Discrete Fourier Transform.

Assignment # 4. Due 7 October.

Folland's Theorem 2.1 shows that for piecewise smooth functions, the symmetric partial sums $\sum_{-N\le n\le N} c_n(f)\, e^{in\theta}$ converge to $(f(\theta+)+f(\theta-))/2$. It is important to notice that this does not show that the Fourier series converges to this value. Recall that for a two sided series, $\sum_{n=-\infty}^{\infty} a_n \;= s$, means that

$$\lim_{m,l\to\infty} \sum_{-m}^{l} a_n = s\,.$$

Equivalently, one must show that BOTH sums $\sum_0^\infty a_n$ and $\sum_{-\infty}^{-1} a_n$ converge say, to s_1 and s_2, and that $s_1+s_2=s$.

1. Prove that a symmetric linear transformation A on a finite dimensional real or complex scalar product space V has an eigenvalue without using the Fundamental Theorem of Algebra as follows.

i. Define $f : V \to \mathbf{R}$ by $f(v) := \langle Av, v\rangle$. Prove that f is continuous. **Hint.** Use the continuity of the scalar product and of A to show that if $v_n \to v$ then $f(v_n)\to f(v)$.

ii. Denote by S the unit sphere, $S := \{v\in V \,:\, ||v|| = 1\}$. Show that there is a vector $u\in S$ where $f(v)$ attains its maximal value, that is $f(u)\ge f(v)$ for all $v\in S$. **Hint.** Apply a theorem from advanced calculus.

iii. Show that u is an eigenvector, that is Au is parallel to u by showing that Au is orthogonal to any vector w which is orthogonal to u. **Hint.** Construct a nice curve $\gamma(t)$ in S with $\gamma(0)=u$ and $\gamma'(0)=w$. Show that $w\perp Au$ by considering $f(\gamma(t))$ for $t\approx 0$.

Discussion. This variational approach is very important and we will return to it many times in the course. It gives one of the best methods for constructing approximate eigenvectors and for deriving estimates for eigenvalues.

The matrices A in equation (5) of the notes on discrete diffusion are diagonal dominant according to the following definition.

Definition. A $k \times k$ matrix is (strictly) diagonal dominant if the diagonal element in each row is larger in modulus that the sum of the moduli of the off diagonal elements in the row, that is, for $1 \leq i \leq k$

$$|A_{ii}| \leq \sum_{j \neq i} |A_{ij}| .$$

2. Prove that a strictly diagonal dominant matrix is invertible. **Hint.** Show that $Av = 0$ has only the trivial solution.

Discussion. In the next homework set you will use this to prove a very useful Theorem of Schur which approximately locates eigenvalues of diagonally dominant matrices.

3. Folland 47/3.

4. Folland 47/4.

5. Folland 47/5.

6. Folland 61/1.

Discussion. Be sure to read the discussion of Gibb's phenomenon on page 60. This problem analyses the phenomenon for the particular function in Figure 2.8 of Folland. In the next assignment you will show that knowing this case is enough to treat any piecewise smooth function

7. Consider diffusion among three containers. Suppose that $V_j = 1$ for all j, $k_{1,3} = k_{3,1} = 0$, and, $k_{1,2} = 1 \leq k_{2,3}$. The eigenvalue equation is a real cubic equation such that $\lambda = 0$ is a root. The equation for the other two roots is therefore quadratic so explicitely solvable. **i.** Show that increasing any of the k_{ij} decreases both negative eigenvalues. **ii.** Compare the sensitivity of the eigenvalues to changes in $k_{1,2}$ and $k_{2,3}$

Discussion. This is the first of many questions you will have in *perturbation theory.* Since the eigenvalues are explicit there is no need for the theory. One can compute sensitivities by differentiating directly. An alternate strategy is to write the quadratic equation for the other eigenvalues and differentiate implicitely.

Assignment # 5. Due 14 October.

1. Prove. **Schur's Theorem.** . Suppose that A is a $k \times k$ matrix. For each $1 \leq i \leq k$ let B_i be the closed disk of complex numbers with center at A_{ii} and radius equal to $r_i = \sum_{j \neq i} |A_{ij}|$. If z does not belong to the union of the B_i, then $A - zI$ is invertible. **Hint.** Use problem 1 of homework 3.

Discussion. This shows that the eigenvalues of A belong to the union of the disks B_i. In other words, spectrum of A is contained in the union of the B_i.

2. Apply Shur's criterion to the matrix (5) from *Notes on Discrete Diffusion.* You should be able to conclude at least that the eigenvalues lie in the left half plane, which is one of the questions raised in the *Notes.*

3. Consider heat flow in a ring with an external supply of heat which is equal to h calories per unit length per unit time. Here h is a positive real number. Write the appropriate heat equation and solve the initial value problem using Fourier series. Describe the behavior of the average temperature in the ring. Describe the size of the difference between u and its average value.

4. Consider heat flow in a ring with an external supply of heat which at t, x is proportional to the the difference $\underline{u} - u(t,x)$ between the temperature and the temperature $\underline{u}$ of the external source. Write the appropriate heat equation and solve the initial value problem using Fourier series. How does the average temperature $u_{av}(t)$ behave in this case? And the difference $u(t) - u_{av}(t)$?

5. Folland 56/1.

6. Folland 43/4.

7. Folland 48/7. Also give the cosine series on $[0, 6\pi]$.

8. Folland 48/12.

Assignment # 6. Due 21 October.

1. Folland 79/1.

2. Folland 80/9.

3. Folland 80/10a,b..

4. Folland 80/11.

5. Let V be the space of continuous 2π periodic functions with the usual scalar product.
a. Is $f_n(\theta) := (\sin\theta)^n$ a Cauchy sequence? If so, toward which equivalence class of 2π periodic square integrable functions does it converge? Is there a continuous function in this equivalence class? Explain clearly your answers.
b. Same question for the periodic functions with $g_n(\theta) = (\theta^2 + 1/n)^{1/2}$ for $|\theta| \le \pi$.
Discussion. Part b. yields is a good trick for smoothing corners.

6. Folland 86/6.

7. Folland 86/7b.

8. Folland 56/2.

9. Folland 56/4.

The next result will be used to show that the eigenvalues of the Discrete Diffusion matrices are real. It is useful to recall that if M is a matrix and B is an invertible matrix of the same size then the eigenvalues of M and BMB^{-1} are the same. In fact λ, v is an eigenvalue eigenvector pair for M if and only if λ, Bv is such a pair for BMB^{-1}.

10. Suppose that A and K are symmetric $k \times k$ matrices and that A is invertible. Prove that the eigenvalues of A^2K are all real.

Assignment #7. Due 31 October.

1. Use problem 10 of Homework 6, to show that all eigenvalues of the all the discrete diffusion matrices from *Notes on Diffusion* are real.

The next three problems almost appeared on the midterm exam.

2. Find a formula for the Fourier coefficient $c_n\big(\sin(\theta)\, f(\theta)\big)$ in terms of the Fourier coefficients of f.

3. Prove the following estimate on Fourier coefficients,

$$|c_{-1}(f) + c_1(f)|^2 \le \frac{1}{\pi} \int_0^{2\pi} |f(\theta)|^2 \, d\theta \,.$$

4. Solve the initial value problem for the differential equation in problem 6 of the midterm exam. Show that $\lim_{t\to\infty} u(t,\theta)$ exists if and only if $c_0(f) = 0$ and compute the limit in that case.

5. Folland 57/5.

6. Folland 57/7.

7. Folland 80/4. **Hint.** Don't forget completeness!

8. Folland 107/1. **Hint.** Either follow Folland pp 101-103, or use a suitable method of images to reduce to a periodic problem. The method of images is a blend of the two used for the $u = 0$ and $u_x = 0$ boundary conditions.

Discussion. This is about the last of the explicitly solvable boundary conditions.

9. Folland 107/2. **Hint.** Subtract a steady state.

Assignment #8. Due 7 November

1 . 234/1.

2 . 235/2.

3 . 235/3. In addition answer

c. Show that the solution contains only left moving waves, that is has the form $F(t+x)$, if and only if $u_t(0,x) = u_x(0,x)$.

4 . 235/4. 235/2,3,4,6, 240/1,3

5 . 235/6.

6 . 240/1.

7 . 240/3.

8. Sometimes physical or mathematical intuition helps you guess an approximate eigenvalue and eigenvector of a matrix, that is a vector v and scalar μ such that $||v|| = 1$ and $Av - \mu v$ is small. For symmetric (or even normal) matrices this is sufficient to show that there is an eigenvalue close to μ. There need not be an eigenvector close to v!

Prove the following Theorem.

Theorem. *If A is normal linear transformation on a finite dimensional scalar product space V, v is a unit vector in V, and, μ is a scalar satisfying*

$$||Av - \mu v|| \le \epsilon\,, \tag{1}$$

then A has an eigenvalue λ such that $|\lambda - \mu| \le \epsilon$.

Hint. Choose an orthonormal basis of eigenvectors and then compute $||Av - \mu v||^2$.

9. Consider diffusion among three containers at the vertices of a triangle. Suppose that all the $k_{ij} = 1$. Suppose that the volumes $V_1 = V_3 = 1$ and that $V_2 = 1/M^2 << 1$. The basic diffusion matrix is equal to the product

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & M^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \tag{2}$$

Zero is an eigenvalue with eigenvector $(1, 1, 1)$.

i. If the initial temperature of the large containers are T and $-T$ what do you expect to happen? Get a second exact eigenvalue-eigenvector pair from this insight.

ii. Use the idea of assignment 6 #10 to show that the eigenvalues are equal to the eigenvalues of the symmetric matrix

$$\begin{pmatrix} -2 & M & 1 \\ M & -2M^2 & M \\ 1 & M & -2 \end{pmatrix} . \tag{3}$$

iii. Use Shur's Theorem to estimate the location of the eigenvalues. **Discussion.** Shur's Theorem applied to the original form, (2), yields a much weaker conclusion.

iv. Show that there is an eigenvalue λ with $|\lambda - (-2M^2)| \leq 2M$. Thus, the third eigenvalue is equal to $-2M^2(1 + O(1/M))$ as $M \to \infty$. **Hint.** To do this you need to show that there is an eigenvalue in the disk from **iii** which is far from the origin. Here are two strategies. If the eigenvalues were all in the other disks, the trace of the matrix (which is equal to the sum of the diagonal elements and also to the sum of the eigenvalues) would not be large enough. Alternatively, you can reason that Schur's theorem is about diagonal dominance. If the off diagonal terms were zero, $(0, 1, 0)$ would be an eigenvector with eigenvalue equal to $-2M^2$. Use this as an approximate eigenvalue eigenvector pair in the Theorem of problem 8.

Extra. Explain physically why you expect a large negative eigenvalue and the form of the eigenvector of the original diffusion matrix. **Ans.** $(0, 1, 0) + o(1)$.

Assignment # 9 . Due 14 November.

1. Consider diffusion among three unit volume containers at the vertices of a triangle. Suppose that $k_{ij} = k_{ji}$, $k_{12} = k_{13} = 1$, and, that the third $k_{23} = M >> 1$. Physical intuition suggests that what will happen is that the second and third containers will equilibrate in $t = O(1/M)$. Then the system will essentially behave as if there were two containers one of which is twice as large as the other and with conductivity equal to 2.

i. Find the diffusion matrix.

As always, $\lambda_1 = 0$ is an eigenvalue corresponding to the equilibrium eigenvector $(1, 1, 1)$.

ii. Prove that as $M \to \infty$ there is an eigenvalue $\lambda_3 = -bM(1 + O(1/M))$ with $b > 0$ and find b. **Hint.** Problem 8 of Assignment 8. **Discusssion.** This confirms the next to last sentence of the intuition.

Extra. Show that the third eigenvalue is of the form $\lambda_2 = -a + O(1/M)$ with strictly positive $a > 0$. This confirms the last sentence of the physical intuition.

2. The small vibrations of a semiinfinite string fixed at one end are solutions of the of the boundary value problem

$$u_{tt} = u_{xx} \quad \text{for} \quad x \geq 0, \quad -\infty < t < \infty , \qquad u\big|_{x=0} = 0 .$$

i. Suppose that $\phi(x)$ is an infinitely differentiable function which vanishes outside the interval $[1,2]$. Find the solutions $u(t,x)$ whose initial state is given by

$$u(0,x) = \phi(x), \qquad u_t(0,x) = \frac{d\phi(x)}{dx} \,.$$

Hints. Assignment 8 problem 3c explains the peculiar choice of initial data. Use the method of images.

ii. Sketch $u(t,1)$, the wave viewed by an observer at $x = 1$, assuming that ϕ has graph which is a single positive bump. Sketch $u(4,x)$.

3. Show that if g is an integrable function with $\int g(x)\, dx = 1$ then the sequence $g_\epsilon := \epsilon^{-1} g(x/\epsilon)$ satisfies the three hypotheses for a summability kernel. **Hint.** You will need the Legesgue Dominated Convergence Theorem.

4. In class we solved the the two point boundary value problem

$$w_{xx} = f(x) \quad \text{for} \quad 0 \le x \le L \qquad \text{and,} \quad w(0) = w(L) = 0 \,. \tag{1}$$

The solution, which represents a steady state or equilibrium temperature, involved the eigenfunctions $\phi_n(x)$ and eigenvalues λ_n of the operatore $\frac{d^2}{dx^2}$ with domain restricted to functions which vanish at $x = 0, L$,

$$\phi_n(x) := \frac{\sin n\pi x/L}{\sqrt{L/2}} \,, \qquad \lambda_n := -(n\pi/L)^2 \,. \tag{2}$$

Then

$$w(x) = \int_0^L K(x,y)\, f(y)\, dy \tag{3}$$

where the Green's function K is given by the explicit formula

$$K(x,y) = \sum \frac{\phi_n(x)\, \phi_n(y)}{\lambda_n} \,. \tag{4}$$

We will later see how to compute an explicit formula for K relying on insights from the theory of distributions. That approach has wide utility. For the simple problem (1), K can be evaluated using only Taylor's Theorem and elementary results from first year calculus.

i. Use Taylor's Theorem with remainder to show that the solution of the initial value problem $u_{xx} = f$, $u(0) = u'(0) = 0$ is given by

$$u(x) = \int_0^x (x-s)\, f(s)\, ds \,.$$

ii. Then, $v := w - u$ must satisfy $v'' = 0$, $v(0) = 0$, $v(L) = -\int_0^L (L-s) f(s)\, ds$. Find the unique solution v of this two point boundary value problem for v.

iii. Write $w = u + v$ to derive the following formula for K

$$K(x,s) = \frac{x-L}{L}\,s \quad \text{for} \quad 0 \le s \le x\,,$$
$$K(x,s) = \frac{x}{L}\,(s-L) \quad \text{for} \quad x \le s \le L\,. \tag{5}$$

iv. For x between 0 and $L/2$ sketch the graph of $K(x,s)$ as a function of s. Verify that the slope jumps by one at the point $s = x$.

v. Does K ever change sign? If f is everywhere nonnegative, what can you say about the sign of w? Is this result reasonable given the interpretation as a steady state heat distribution? **Hint.** Beware of a minus sign in front of the source when you write the equation for steady heat flow.

Discussion. The results about the sign and the formula (5) would be difficult to verify using the formula (4). This shows that one often needs more than one way to view a problem.

5. 313/1

6. 313/3

7. 313/4

Assignment # 10. Due 21 November.

1. 240/8

2. Perform the same computation as that in problem 2 of Assignment 9, with the boundary condition replaced by

$$u_x\Big|_{x=0} = 0\,.$$

What would you say was the refection coefficient at the boundary $x = 0$ for problems 1 and 2? **Ans.** One of them has reflection coefficient equal to 1, and the other has coefficient equal to -1.

3. In problems 2 and 1 of Assignments 5 and 7 respectively we obtained a good deal of information about the spectrum of general diffusion matrices. We also know that equal temperatures always is an equilibrium. There can be otherequilibria. For example, if the objects can be split into two groups so that heat can never flow from a container of one group to a container of the other, then if one group is a a temperature T_1 and the other at T_2 then this is an equilibrium. If such a division in two separate parts is not possible then equal temperatures is the only equilibrium. This is the content of Theorem below which you are asked to prove. The diffusion matrix is given by equation (5) in *Discrete Diffusion.*

For the present problem, the conductivities $k_{ij} \geq 0$ are not necessarily symmetric, that is, we do not suppose that $k_{ij} = k_{ji}$.

Defintion. A system of N heat containers is **connected** if whenever they are divided into two disjoint subsets $\{1, 2, \ldots, N-1, N\} = I \cup J$, there is an $i \in I$ and a $j \in J$ so that $k_{ij} \neq 0$.

This says that heat can flow from any subset of containers to the complementary subset.

Theorem. *For a connected set of containers, all eigenvectors with eigenvalue 0 are proportional to* $(1, 1, \ldots, 1)$

Hints. From the solution to problem 5 on assignment 1, you know that the real and imaginary parts of any eigenvector with real eigenvalue must themselves be eigenvectors. Use this to reduce to the case of real eigenvectors. Replace v by $-v$ if necessary so that there is a component equal to $M = \max_j |v_j|$. The matrix is diagonal dominant but not strictly diagonal dominant. Use the proof that strictly dominant matrices are invertible to show that all nonzero components of v must be equal to M. Let I be the set of indices of nonzero coordinates, and let J be the complementary set. Use connectivity together with $Av = 0$ to show that $I = \{1, 2, \ldots, N-1, N\}$.

4. Compute $(\sin x)^j\, \delta'(x)$ for $j = 1, 2$.

5. 213/3 Warning. Yes this is 213.

6. 313/7

7. 320/2

8. 320/6

9. 320/9

Assignment #11 . Due 30 November.

In problem 1 of Assignment 7, you showed that the eigenvalues of diffusion matrices were real provided that the conductivities were symmetric, that is $k_{ij} = k_{ji}$. That is, heat travels as easily from container i to j as from j to i. This reality does not necessarily hold when the conductivities are not symmetric. In the next problem this is illustrated by finding the exact eigenvalues for a ring of containers for which heat can flow only in the clockwise direction.

1. Suppose that $n \geq 3$ containers of unit volume have conductivities given by

$$k_{1,2} = k_{2,3} = \ldots = k_{n-1,n} = k_{n,1} = 1$$

and all other conductivities vanish.

i. Find the associated diffusion matrix. **Partial Ans.** For $n = 3$ it is

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

ii. Find the eigenvalues and a basis of eigenvectors for all n. **Hint.** If τ denotes translation in the clockwise direction by one container, then τ is a symmetry of the problem. This allows one to construct a matrix corresponding to τ which commutes with A. The matrix τ has distinct eigenvalues and easy eigenvectors. The commutativity implies that the eigenvectors of τ are eigenvectors of A allowing you to simply compute the eigenvalues. **Partial Ans.** For $n = 3$ the eigenvalues are 0 and $(-3 \pm i\sqrt{3})/2$.

2. If F_k converges to F in $\mathcal{D}'(\Omega)$, that is $F_k \rightharpoonup F$, and $a(x) \in C^\infty(\Omega)$, prove that $aF_k \rightharpoonup aF$.

3. 320/5.

4. 320/7.

5. 320/10.

6. 358/1. Do not use Laplace transform, solve for the Green's Function with your bare hands.

7. Compute the Green's kernel $G(t, s)$ for the initial value problem

$$\frac{du}{dt} + t\,u = f(t)\,, \qquad u = f = 0 \quad \text{form} \quad t < 0\,.$$

The Green's kernel then has two properties. First, the solution of the initial value problem is given by

$$u(t) = \int G(t, s)\, f(s)\ ds\,.$$

Second, for each s, the Green's function is determined as the solution of

$$\frac{dG(t, s)}{dt} + t\, G(t, s) = \delta(t - s), \qquad G(t, s) = 0 \quad \text{for} \quad t < s\,.$$

It is the response to a delta function source at s.

8. i. 358/6. **Note.** The number μ is assumed real.

ii. Invent a heat conduction problem for which this boundary value problem determines the steady state response.

Assignment # 12. Due 7 December.

1. i. In the halfspace H defined to be the subset $x_1 > 0$ in $\mathbf{R}^3$, find the Green's function for the boundary value problem

$$\begin{aligned}
&\Delta\, u = f \quad \text{for} \quad x \in H\,,\\
&\partial u/\partial \mathbf{n} = 0 \quad \text{for} \quad x \in \partial H\,,\\
&u \to 0 \quad \text{as} \quad |x| \to \infty\,.
\end{aligned}$$

ii. Express the solution u in terms of the data f using this Green's function.

iii. Observe that nonnegative f yield nonpositive solutions u. Use heat conduction to explain in 60 words or less why this result is physically reasonable.

2. Find a distribution F on $\mathbf{R}^3$ which satisfies

$$\Delta\, F = \frac{\partial}{\partial x_1}\, \delta\,.$$

Hint. This is easy.

3. i. Find all rotationally symmetric solutions of the differential equation $(1-\Delta)u = 0$ on $\mathbf{R}^3 \setminus 0$.

ii. Use scalings $v = u(ax)$ to find the general rotationally symmetric solution of $(k^2-\Delta)u = 0$ on $\mathbf{R}^3 \setminus 0$, for $k \in \mathbf{R} \setminus 0$. **Ans.** $Ae^{-kr}/r + Be^{kr}/r$.

Discussion. These formulas are correct for complex values of k. The cases $k \in i\mathbf{R}$ are particularly important in scattering theory.

4. Compute in the sense of distributions on $\mathbf{R}^3$,

$$(k^2 - \Delta)\ \frac{Ae^{-kr} + Be^{kr}}{r}\,.$$

Hints. The function to be differentiated is locally integrable so unambiguously defines a distribution. Do it for one term. The other is similar. You can repeat the classroom proof for the case $k = 0$, or, you can use Leibniz' rule for distributions.

5. Find the Green's kernel for the Neumann problem for $k^2 - \Delta$ in the half space $x_1 \geq 0$. That is, find G tending to zero at infinity and satisfying

$$(1-\Delta_x)G(x,y) = \delta_y \quad \text{for} \quad x_1 \geq 0, \qquad \text{and} \qquad \mathbf{n}.\nabla_x\, G(x,y) = 0 \quad \text{for} \quad x_1 = 0\,.$$

Hints. Use images as in Problem (1) above. Be sure that the expression that you find unambiguously defines a distribution.

6. 368/6a. **Discussion.** This is the geometric fact used in the Courant-Hilbert handout to construct the Green's function for a ball. Part **b.** of this problem is that construction and you need not repeat it.

7. Explain in 100 words or less why the construction of the Green's function for Δ in the unit ball of $\mathbf{R}^3$ (Courant-Hilbert page 377 handout) does not work to construct a Green's kernel for $k^2 - \Delta$ in the ball.

8. Here are two application of a Green's kernel. **i.** 377/1.

ii. Show that the solution of the boundary value problem in 358/6 satisfies the integral equation

$$u + \mu^2 \int_0^{\pi/2} K(x,y)\, u(y)\ dy = g(x)$$

where $K(x,y)$ is the Green's kernel for $\mu = 0$, and,

$$g(x) := \int_0^{\pi/2} K(x,y)\, f(y)\ dy$$

is the solution for the case $\mu = 0$.

Discussion. This can be used to give a series expansion for u in powers of μ. The rigorous justification of such expansions uses Neumann series which we will shortly discuss.

9. 358/5. **Expanded Hint.** An Euler equation in this context is an ordinary differential equation of the form

$$\sum_{j=0}^{n} a_j\, x^j\, \frac{d^j}{dx^j}\, u = 0$$

with constants $a_j \in \mathbf{C}$ and $a_n \neq 0$. The key point is that the j^{th} derivative terms have an x^j in front. Euler's observation is that the general solution of such linear homogeneous equations are usually linear combination of functions t^r for appropriate powers r which can be found by plugging into the equation and solving the resulting n^{th} order polynomial equation for r. When the polynomial has no multiple roots this is the end of the story. For second order equations the polynomial is quadratic, hence always explicitly solvable.

Assignment #13 . Due 14 December.

1. Denote by $K_D(x,y)$ the Green's functions in a halfspace $\{x_1 > 0\} := H \subset \mathbf{R}^3$ for the Laplacian Δ with Dirichlet boundary condition at $x_1 = 0$, computed in class. Similary denote by $K_N(x,y)$ the Green's function with Neumann boundary condition at $x_1 = 0$, computed in Problem 1 of Assignment 12.

i. Which is larger, K_N or K_D?

ii. For $f \le 0$, use this result of compare the solutions of $\Delta\, u_D = f$ and $\Delta\, u_N = f$ in H, with Dirichlet and Neumann boundary conditions respectively.

iii. Explain in 60 words or less why this last result is physically reasonable.

2. Use the method of images to construct the Green's function in the quarter space $Q := \{x_1 > 0 \ \text{ and } \ x_2 > 0\} \subset \mathbf{R}^3$ with Dirichlet boundary conditions. That is find the distribution G tending to zero at infinity and satisfying

$$\Delta_x G(x,y) = \delta_y\,, \quad \text{and} \quad G = 0 \ \text{ when } \ x_1 = 0 \text{ or } x_2 = 0\,.$$

Discusion. In the same way, one can solve with Neumann boundary conditions or with the mixed conditions, Dirichlet on $x_1 = 0$ and Neumann on $x_2 = 0$

3. Compute the norm of each of the following matrices.

$$\textbf{i.}\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \qquad \textbf{ii.}\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

4. Consider diffusion among a nearly triangular array of unit volume containers with symmetric conductivities satisfying $k_{1,2} = 1+\epsilon$, $k_{1,3} = k_{2,3} = 1$ with small ϵ. An oscillating heat source $(\cos\omega t)h$ is applied, so that the the temperatures are given by the real parts $u = \Re v$ with

$$v' = Av + e^{i\omega t} h\,. \tag{1}$$

Here $A = A(\epsilon)$ is the diffusion matrix. A particular solution, the so called steady state, is given by $v(t) = e^{i\omega t} w$ where the vector $w \in \mathbf{C}^3$ is determined by

$$(i\omega I - A)w = h\,. \tag{2}$$

For the unperturbed problem, $\epsilon = 0$, one knows explicitly the eigenvectors e_j and eigenvalues λ_j of A.

i. Show that for $\epsilon = 0$, equation (2) has the unique solution

$$w = (i\omega I - A)^{-1} h = \sum (i\omega - \lambda_j)^{-1} \langle h, e_j \rangle\, e_j\,.$$

ii. Denote by $C := i\omega - A(0)$ the unperturbed matrix and by $P := A(0) - A(\epsilon)$, the perturbation matrix. Then $C^{-1}h$ is the unperturbed steady state. Compute the correction to first order in ϵ to this steady state using the standard scientist idea of writing the equation in the form $(C+P)(w+\delta w) = h$ and neglecting quadratically small terms. **Ans.** $\delta w = -C^{-1} P\, w = -C^{-1} P C^{-1}\, h$.

Discussion. The first formula gives the perturbation in the solution in terms of the unperturbed solution, w, which one often has in hand. The second formula is an expression in terms of the data, h.

iii. Compute the first order correction by *assuming* that the solution is given as a power series in ϵ,

$$w = w(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j \, w_j = w_0 + \epsilon \, w_1 + \epsilon^2 \, w_2 + \dots \tag{3}$$

with vectors w_j. Expand $C(\epsilon)$ in a similar series, then insist that the coefficients of the successive powers of ϵ in the equation

$$C(\epsilon) w(\epsilon) - h = 0 \tag{4}$$

vanish. The term in ϵ^0 gives the unperturbed solution and the term in ϵ^1 gives the first order correction.

iv. Compute the first order correction by *assuming* that the solution $w(\epsilon)$ is differentiable with respect to ϵ with $w(0)$ equal to the unperturbed solution. Differentiate equation (4) with respect to ϵ then set $\epsilon = 0$ to find an equation which determines $dw/d\epsilon|_{\epsilon=0}$. Then use

$$w(\epsilon) = w(0) + \epsilon \, dw/d\epsilon|_{\epsilon=0} + O(\epsilon^2) \, .$$

Discussion. These are the three standard methods in perturbation theory. The first is purely formal. To justify the second and third, it suffices to show that $w(\epsilon)$ is an infinitely differentiable function of ϵ. Then, (3) is a Taylor expansion. The differentiability, and, the convergence of (3) for small ϵ, follow from the implicit function theorem from advanced calculus. Then, the computations in parts **iii.** and **iv.** are rigorous approaches to computing the Taylor expansion of w at $\epsilon = 0$. The use of the implicit function theorem even works for nonlinear problems.

5. This problem continues the study from the last. The goal is to justify the approximations and to assess the size of the error.

i. Show that $||P|| = 2\epsilon$ and $||(i\omega I - A(0))^{-1}|| = 1/\omega$. **Hint.** Each of these can be computed by finding a simple orthonormal basis of eigenvectors.

ii. The error from first order perturbation theory is equal to the sum of the series $\sum_{n=2}^{\infty} (-C^{-1}P)^n \, C^{-1}h$. Use this to show if $|\epsilon/\omega| < 1$ then

$$||\,\text{error}\,|| \le \frac{(2\epsilon/\omega)^2}{1 - 2\epsilon/\omega} \, ||C^{-1}h|| = \frac{4\epsilon^2}{\omega^2 - 2\epsilon\omega} \, ||C^{-1}h|| \, . \tag{5}$$

Discussion. If 2ϵ is small compared to ω, then the prefactor is close to $4\epsilon^2/\omega^2$ so the percent error is estimated from above by a number close to $400 \, \epsilon^2/\omega^2$.

This Neumann series idea is the fourth approach to our linear perturbation problem. It has the advantages of being rigorous, giving the most rapid computation of the high order terms, and giving a rigorous error bound. It has the disadvantage of applying only to linear problems.

6. This question presents computations in infinite dimensions which are analogous to those in problems (4). You will assess the change in the steady state heat distribution in a uniform rod, produced by small nonuniformities, for example a clean weld. The steady state temperature when the ends are kept at temperature zero is determined as the solution of the boundary value problem

$$\frac{d}{dx}\big(k(x)\frac{d}{dx}\,u(x)\big) = -h(x) \quad 0 \le x \le l\,, \qquad u(0) = u(l) = 0\,, \tag{6}$$

where $k(x)$ is the conductivity. The unperturbed conductivity is equal to a constant k and the perturbed conductivity is equal to $k(x) = k + \epsilon q(x)$ where q is of order 1, and, ϵ is small.

The change of independent variable, $x' = lx$ converts these problem to problems of the same form with $l = 1$, so we may, without loss of generality, suppose that $l = 1$. This simplifies some of the formulas. In class we computed the Green's function for the unperturbed problem

$$K(x, y) = -\min\{x, y\}\,(\max\{x, y\} - 1)/k\,. \tag{7}$$

This also follows from formula (10.15) of Folland with

$$v_a(x) := x\,, \quad \text{and} \quad v_b(x) := x - 1\,.$$

The associated integral operator is denoted by $\mathbf{K}$. Then, $-\mathbf{K}h$ is the solution of the unperturbed problem. Denote the unperturbed operator and the perturbation by

$$L := \frac{d}{dx}\,k\,\frac{d}{dx} \quad \text{and} \quad P := \epsilon Q := \epsilon\,\frac{d}{dx}\,q(x)\,\frac{d}{dx}\,. \tag{8}$$

The perturbed steady state is the solution of

$$(L + \epsilon Q)\,u = -h\,, \qquad u(0) = u(1) = 0\,. \tag{9}$$

i. Compute the first order perturbation theory approximation to the solution using a method analogous to that in part **ii.** of problem (4).

Ans. $u \approx -\mathbf{K}h + \mathbf{K}P\mathbf{K}h = u_0 - \mathbf{K}Pu_0$ where $u_0 := -\mathbf{K}h$ is the solution of the unperturbed problem. Note that $\mathbf{K}P\mathbf{K}h$ is a concretely specified function once h is given. It will be simplified in part **v.** below.

ii. Compute the first order perturbation theory approximation to the solution using a method analogous to that in part **iii.** of problem (4). That is, *assume* that the answer is given by a formal power series

$$u = u(x, \epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j\, u_j(x) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \dots\,. \tag{10}$$

Plug this into equation (9) and equate the coefficients of the powers in ϵ to zero. The power ϵ^0 determines u_0 and the next power determines the first order correction.

iii. Compute the first order perturbation theory approximation to the solution using the method analogous to that in part **iv.** of problem (4). That is, differentiate with respect to ϵ in

$$\frac{d}{dx}\left((k + \epsilon q(x))\, \frac{d}{dx}\, u(x,\epsilon) \right) = -h(x) \quad 0 \le x \le l\,, \tag{11}$$

$$u(0,\epsilon) = u(1,\epsilon) = 0\,, \tag{12}$$

to find a boundary value problem for $\partial u/\partial\epsilon|_{\epsilon=0}$ which can be solved using the operator $\mathbf{K}$. Then use $u(x,\epsilon) \approx u(x,0) + \epsilon\, \partial u/\partial\epsilon|_{\epsilon=0}$.

iv. Explain why

$$k\, \frac{d^2}{dx^2}\, \mathbf{K}\, g = g\,, \quad \text{and,} \quad \frac{d}{dx}\, \mathbf{K}\, g = \int_0^1 \frac{\partial K}{\partial x}(x,y)\, g(y)\, dy$$

for all g.

v. Use the results of part **iv.** and one integration by parts to show that the correction from first order perturbation theory is equal to the explicit expressions

$$\epsilon \int_0^1 \frac{\partial K}{\partial y}(x,y)\, q(y)\, u_0'(y)\ dy = -\epsilon \int_0^1 \int_0^1\ \partial_2 K(x,y)\ \partial_1 K(y,s)\ q(y)\, h(s)\ dy\, ds\,. \tag{13}$$

Here ∂_j denotes partial derivative with respect to the j^{th} slot and the the partial derivatives of K are given by explicit computation as

$$\frac{\partial K}{\partial x}(x,y) = \begin{cases} -(1+y)/k & \text{for} \quad x < y \\ -y/k & \text{for} \quad x > y \end{cases}, \qquad \frac{\partial K}{\partial y}(x,y) = \begin{cases} -(1+x)/k & \text{for} \quad y < x \\ -x/k & \text{for} \quad y > x \end{cases}.$$

Discussion. The computation in part **i.** is purely formal. To justify the computation in parts **ii.** and **iii.** it suffices to show that $u(x,\epsilon)$ is an infinitely differentiable function of ϵ. This can be done using an infinite dimensional version of the implicit function theorem, which also proves convergence of (10) for small ϵ. The methods of parts **i.,ii., iii.,** work for nonlinear problems.

7. Find a convergent power series in ϵ for the steady state u in problem (6) using Neumann's series. Show that the series converges provided $||\epsilon \mathbf{K} Q|| < 1$. **Partial Answer.** The second order correction is equal to $\epsilon^2 \mathbf{K} Q \mathbf{K} Q \mathbf{K} h$.

Discussion. The convergence criterion is satisfied provided ϵ is sufficiently small. When it is violated, the use of perturbation theory is dubious. The Neumann series approach has the disadvantage of working only for linear equations, and the advantages of rapid access to higher order corrections as well as rigorous error bounds.