Math 256
Applied Honors Calculus IV: Differential Equations, Fall 2007

Homework Set 12
Due Monday, December 3 , 2007

Reading. Boyce and DiPrima 7.8 is devoted to multiple eigenvalues. We have and alternate approach which has the advantage of working in all cases. It is summarized in an online handout.

The only method for inhomogeneous systems presented in class is Variation of Parameters which relies on forming a Fundamental Matrix from n linearly independent solutions of the homogeneous equation. In Boyce and DiPrima this material is covered in §7.7 and 7.9. Our derivation of the identity (31) was somewhat different, and I hope clearer in its motivation.

For numerical methods we will treat the important topics of *order of accuracy, local truncation error, global or accumulated truncation error, roundoff error, and variation of time step.* In Boyce and DiPrima this material is discussed in sections 8.1,8.2, 8.3, 8.5. We will not cover multiple step methods.

Problems to Study.

- Multiple roots from B&D. §7.8/1-12.
- Variation of Parameters from Boyce and DiPrima. §7.9/1-12.

Problems to Hand In.

- Multiple Eigenvalues from Boyce and DiPrima. §7.8/16. Add.

  c. Use `pplane` to sketch the phase diagram and sketch a few integral curves. Hand in the printed results.

  d. Compute by hand the direction vector at the point $(1,0)$ on the x-axis and $(1,0)$ on the y-axis.

- Multiple Eigenvalues from Boyce and DiPrima. §7.8/18. You may ignore all the parts (a.-d.) of this problem, simply apply the algorithm for multiple roots to compute the general solution of the homogeneous equation which is the subject of this problem.

- Variation of Parameters from Boyce and DiPrima. §7.9/13

- Behavior of Euler's method as $h \to 0$. Consider the initial value problem which is the subject of Table 8.1.1.

  a. Write an m-file which allows you to solve this initial value problem using Euler's method. You can use the file eulertest.m as a starting point. For $n = 1, 2, 3, \ldots, 10$ use Euler's method to compute approximate values of $y(0.5)$ using $h = 1/10n$.

  b. The in class theorem asserts that in exact arithmetic if you plot the approximate values computed against h then as $h \to 0$, the values very nearly lie on a straight line which hits $h = 0$ at the exact value. Verify this by plotting.

  c. From table 8.1.1 you have an exact value,. Verify the prediction in **b.** by plotting

$$\frac{y_{\text{approx}}(0.5) - y_{\text{exact}}(0,5)}{h}$$

  against h. The theorem asserts that as $h \to 0$ the values should converge to a constant which is the slope of the line in **b.**.

- Behavior of the improved Euler method as $h \to 0$. Return to the example above but with a better method as displayed in Table 8.2.2.

  a. Write an m-file which implements the improved Euler method. Use $h = 1/10n$ as in the preceding example. If you are weak at programing work in a group. Write up your own homework.

  b. Plot the approximate values for $y(0.5)$ against h. The in class theorem asserts that for h small, the values very nearly lie on a parabola $y_{\text{exact}}(0.5) + C\,h^2$. Plot the computed values against h with the goal of displaying this parabolic behavior.

  c. Verify the prediction in a second way by plotting

  $$\frac{y_{\text{approx}}(0.5) - y_{\text{exact}}(0,5)}{h^2}$$

  against h. The theorem asserts that the quotient should approach a constant as $h \to 0$.

  Discussion. If you computed this ratio for Euler's method the answers would be $\approx C/h$ so would diverge to infinity. The error for improved Euler is infinitely small compared to that of Euler. Thie same conclusion holds if one compares any method of higher order with one of lower order. To see the enormous gain MAY require one to look at very small h. Otherwise it would always pay to use very high order.

- Local truncation error.§8.1/22 a, d. Parts b and c can be skipped.

  Discussion. This is a theory question. The point of the tests outlined in b. and c. is to show you that those values of h give bad approximations. You should not be surprised to find a much smaller value of h predicted in d. If you want to try b. and c. use Matlab. Then you can go further and test if the value of h predicted in d. is overly conservative. The truncation error estimate is an estimate from above. The actual error can be smaller. For one step, it is sharp for small h as can be proved using Taylor's Theorem (see 447/(21)).

Chapter 8 review.

Euler method. Derivations. Backward Euler **not** covered.
Local and global truncation error. 446-449.
Estimate for local truncation error. §8.1/(22).
Guestimate for the global truncation error. §8.1/(24). (A rigorous treatment is given in §8.1 problem 23 for those interested.)
First order accuracy, after 448/(24).
Improved Euler. Derivation.
Local and global truncation error. pg. 453.
Variable stepsize algorithm using Euler and Improved Euler. pg 455.
Roundoff. pg 468. The rest of section 8.5 has good additional insight into error causing aspects of ode.
§8.3 and §8.4 **not** covered.

Denote by $y^h_{\text{Euler}}(T)$ the approximate value for a solution $y(t)$ at time T computed with the Euler algorithm with $h = T/N$ for a positive integer N. Euler reaches time T from time 0 in N steps. Analogously, $y^h_{\text{Improved}}(T)$ is the value from the improved Euler method.

The error in Euler (resp. improved Euler) is bounded by a constant time h (resp. h^2). In the limit as $h \to 0$ it is very nearly equal to Ch (resp. $\tilde{C}h^2$).

In class Theorem. There are constants $C, \tilde{C}$, and M depending on T and the solution $y(t)$ so that as $N \to \infty$ (so $h \to 0$),

$$\left| y^h_{\text{Euler}}(T) \;-\; y(T) \;-\; C\,h \right| \;\le\; M\,h^2,$$

$$\left| y^h_{\text{improved}}(T) \;-\; y(T) \;-\; \tilde{C}\,h^2 \right| \;\le\; M\,h^3.$$