Math 256
Applied Honors Calculus IV: Differential Equations, Fall 2007

Homework Set 13
Due Monday, December 10 , 2007

Road map for §9.1,9.2,9.3. §9.1 reviews 2×2 constant coefficient first order systems which are characterized by the eigenvalues and eigenvectors of matrix. The determination of the asymptotic stability and/or stablility of the equilibrium 0 is important.

The ellipticity of orbits for a center is Problem 493/19. The proof is different from that in class.

The damped oscillator is discussed on pages 497-499. §9.2 presents in pages 499-501 the method of finding the integral curve in phase space by eliminating the time variable. In class we applied this to the undamped oscillator. Our second derivation of the same formulas (problem 512/20 and pages 537-538) is of much wider applicability.

§9.2 pages 495-499 presents the careful definition of **stability** and **asymptotic stability**. It discusses the damped nonlinear pendulum which was explored more thoroughly in class using `pplane`. The separatrices appear in problems §9.3/21,22,23.

§9.3 begins with a glimpse at structuraly stability, then presents the extremely important process of linearization at an equiilbrium leading to the linearized equation 508/(13). This leads the important first two steps in the analysis of nonlinear systems.

Step I. *Find the equilibria.*

Step II. *Find the linearized equation at the equilibrium and determine the qualitative behavior of the linear system. In most cases (exceptions are roots that lie on the imaginary axis) this determines the behavior of the nonlinear system near the equilibrium.*

The example of the damped pendulum is presented in the remainder of the section.

Problems to Study.

- Phase portraits. §9.1/1-12. In addition to the questions posed, find the directions of the asymptotes in the case of saddles and of the two invariant lines in the case of proper nodes.
- Equilibria other than the origin. §9.1/13-16. The interest of these is that they are an especially simple case of the linearization algorithm from §9.3.
- Finding the phase plane trajectory without finding $(x_1(t)\,,\,x_2(t))$. §9.2/1-4. These problems are similar. Number 4 is a variant of the non-hyperbola problem from the last assignment.
- Linearization. Use the two step algorithm on examples 5-14 of §9.2. Determine the qualitative behavior near the equilibrium (this is part (c) of the question). Do part (b) using `pplane` to verify your conclusion.
- Linearization. §9.3/5-17. For saddles be sure to identify the "in and out directions". For proper node sinks determine the slope at which most solutions approach the equilibrium, and the slope at which the two exceptional solutions approach. For proper node sources determine the analogous directions valid as $t \to -\infty$.

Problems to Hand In.

- Orbits don't reach equilibria in finite time. §9.2/25. **Discussion.** This shows that trajectories in the phase plane can approach and equilibrium but cannot reach an equilibrium in finite time. We proved the phase line version in class. The proof here is the same and works for $n \times n$ autonomous systems.
- Linearization.§9.3/7. Answer the questions proposed in the "to study" part. Double problem.

- Linearization.§9.3/12. For the indeterminate stability case, try to determine the stability using `pplane`. Attach printout. Double problem.

  Discussion. In case of complex conjugate purely imaginary eigenvalues, $\pm i\omega$, the back of the book says "center or spiral, indeterminate". The correct answer is "indeterminate". An infinity of behaviors which are neither spirals nor centers are possible. Imagine concentric rings each of which can be either a center, spiral in, or spiral out. They can be infinite in number, for example, in the rings $2^{-n-1} < r < 2^{-n}$, with $n = 1, 2, \ldots$. If there are infinitely many like this, the equilibrium is stable. If there are only a finite number then the innermost rules and it is a center or spiral.

  Things get really interesting when you think that in a single ring, for example $1 < r < 2$ you could nest an infinity of shrinking rings which cluster at the $r = 1$ side. Then in the original example of rings you can insert these infinitely complex rings in place of the centers or spirals. And the complexity can continue to augment without limit. This can all be done with infinitely differentiable functions.

- Linearization. §9.3/18. Double problem.

- A `pplane` problem. §9.3/21 **Discussion.** If the term were longer, the follow-up question 27 would be assigned.