Math 256
Applied Honors Calculus IV: Differential Equations, Fall 2007

Homework Set 6
Due Friday, October 19, 2007

Omissions/replacements. We will not be covering, using or examining Abel's formula from section 4.1, nor the method of variation of parameters from section 4.4. The method of undetermined coefficients of section 4.3 is replaced by the annihilator method.

A word to the wise. Though the routine problems are exactly that, exams will test you on those skills. To earn a good grade in 256 you must at least prove that you would have earned such a grade in 216.

Problems to Study.

- Interval of existence. 222/5.
- Linear independence. 222/7-10, your pick.
- Wronskian with variable coefficients. 222/16.
- Amplitude phase representation. 230/1-6, your pick.
- Routine homogeneous Euler. 230/1-24.
- Routine inhomogeneous Euler. 235/1-12 (no plot on the last four).
- Routine inhomogeneous. 235/13-18. Instead of undetermined coefficients, find an annihilator M of low order, and the general solution of the homogeneous equation $MLy = 0$, noting which terms comprise the general solution of $Ly = 0$.

Problems to Hand In.

- Identification of parameters. In many situations one tries to find the coefficients of the governing equations by making observations on solutions. Here is a model case.

  An oscillatory system (a tuning fork or tapped wine glass) emits a sound of frequency 523 oscillations per second (a.k.a. 523 Hertz). This is the C above middle C on a piano. The signal decays with half life equal to 3 seconds.

  Assume that the oscillation is described by a spring mass equation,

  $$y'' + by' + cy = 0.$$

  Find the coefficients b and c.

  Discussion. Oscillating with frequency 523 Hertz suggests periodicity. Half life describes decay. The verbal description is imprecise or even contradictory. What it means is that on intervals of time long compared to 1/523 and short compared to 3 the solutions look periodic with frequency 523 Hz. This requires that the two time scales 1/523 and 3 be well separated, as they are.

- Periodic harvesting. This problem cannot be solved explicitly and cannot be understood by the Phase Line Method as it is nonautonomous. You will need to compute approximate solutions using dfield, or ode45, or some such method. Model `m-files` are on the course home page.

In the absence of harvesting, the population $Q(t)$ of algae follows a logistic equation,

$$\frac{dQ}{dt} = k\,Q\,(L - Q),$$

with positive constants k, L. It is discovered that the algae can be profitably harvested for use in fertilizer. The harvest is easier in summer than other seasons and the harvesting rate is therefore peaked in the summer. In fact, the rate of harvesting is equal to

$$H\big(\sin(\pi t)\big)^2, \qquad H \text{ is constant.}$$

which is periodic with period equal to 1 year.

a. Write a differential equation modelling this situation.

b. Find the smallest value $q > 0$ (which depends on k, L) so that the direction field of this equation points downward for all points (t, Q) with $Q \geq q$. Conclude that if the population starts in $Q \leq q$ it remains there for all $t \geq 0$.

c. Take $k = 0.01$ and $L = 1$. By finding approximate solutions with some computing package (support your conclusions with printout), estimate the maximal sustainable harvesting constant H given that $Q(0) = 0.5$. **Hint.** This question resembles an earlier HW question concerning a linear population growth model. This time the problem can NOT be solved in terms of definite or indefinite integrals.

- An independence argument. 232/40.
- One routine, but messy, problem to hand in. 235/11 (no plot required).