Math 256 — Applied Honors Calculus IV, Differential Equations
Prof. J. Rauch — Fall 2007

Homework Set 7 Due Friday, October 26, 2007

Reading. i. For Linear Algebra read Chapter 1 of Keith Miller's Linear Algebra Notes. A link to the URL is on the course web site.

ii. For perturbation theory, my lecture notes are available on my office door for consultation and/or copying. I'd like to make a clearer version available so if someone volunteers their better notes, that would be good. If someone finds a good online source for perturbation theory that too would be good.

iii. The proof of the error formula for Taylor Polynomials is posted on the web site. The error estimate

$$\big|f(x) \; - \; P_n(x;a)\big| \;\le\; \frac{M_{n+1}|x-a|^{n+1}}{(n+1)!}\,, \qquad |x-a|\le\mu, \quad M_{n+1} \;:=\; \max_{|x-a|\le\mu}\big|f^{n+1}(x)\big|,$$

for approximation by $P_n(x;a)$ on the interval $|x-a|\le\mu$ is a consequence. To use it you need an estimate on how large M_n is, as well as the distance $|x-a|$ to the base point (it *must* be less than μ).

Advice. Rather than working many routine examples it is better to work a few and to think carefully about what you have done to make sure that you master the fundamentals. Ask yourself what variations are possible and how that might change what needs to be done.

Problems to Study.

- Routine Gaussian elimination. K. Miller Chapter 1 problems 1-8. Boyce 383/1-5
- Taylor polynomials. See your Calculus text. Typical problems are given next.
- Routine Taylor polynomial error estimates. **a.** If you approximate $\ln x$ by $P_2(x;1)$ on $|x-1|\le\mu$, how small must μ be taken so that the error is no larger than .001?

  b. Same question for $\sin x$ and $P_2(x;0)$ on $|x|\le\mu$.

  Remark. This is the key idea about Taylor approximation. It gets better as μ gets smaller. The higher is n the faster it gets better.
- Routine Taylor polynomial estimate. **a.** If you approximate $\ln x$ on $|x-1|\le 1/2$ by $P_n(x;1)$ how large must n be taken so that the error is no larger than .001?

  b. Same question for $\sin x$ on $|x|\le\pi/4$ and $P_n(x;0)$.
- Routine perturbation theory. Solve the variant of the slightly nonlinear spring equation treated in class,

  $$\frac{d^2y}{dt^2} \;+\; \omega^2 y \;+\; \epsilon\, y^2 \;=\; 0, \qquad y(0)=0, \quad y'(0)=1.$$

  The unperturbed solution is $\sin\omega t$.

Problems to Hand In.

- Routine Gaussian elimination. K. Miller Chapter 1 Problem 3a. You should work either by hand showing the row reduction to upper triangular form or you should use MATLAB following the model in gauss.m. On balance the latter is quicker and a more reliable way to do the arithmetic. Remember `>>format rat` to get answers in fraction rather than decimal form. You will need to print a part of your MATLAB session to document the computation.

- Gaussian elimination. K. Miller Chapter 1 Problem 5. This is a hand computation. It can be done using symbolic computations in MATLAB, MAPLE, or MATHEMATICA. The point is to learn the nature of row operations. So, it is BY HAND.

- Perturbation theory. Compute the first order perturbation theory approximation $y_0(t) + \epsilon y_1(t)$ to the solution of

$$\frac{dy}{dt} - y + \epsilon t y^2 \ = \ 0, \qquad y(0) = 1\,.$$

  Here is a simple problem which cannot be solved exactly, but which *can* be solved in perturbation theory.

- Perturbation theory. **a.** Compute the first order perturbation theory approximation to the solution of the slightly damped oscillator,

$$y'' \ + \ \epsilon y' \ + \ \omega^2 y \ = \ 0, \qquad y(0) = 1, \quad y'(0) = 0\,.$$

  b. The goal is to reduce the friction ϵ to a level that for $0 \le t \le 2\pi/\omega$ the friction modifies the solution by less than 2%. *Using the result in part* **a.** estimate how large a value of ϵ can be tolerated.

  Discussion. This problem is sufficiently simple that it can be easily analysed using exact solutions. But, it is good practice in perturbation theory in a familiar context.

- Perturbation Theory. Boyce/DiPrima 206/32. This problem asks you to experiment with a problem in perturbation theory. The program gravityspring.m is a model for how to use Matlab's ode45 for such a problem. That program offers three graphing options. There is a fourth, `plot(Y(:,1),Y(:,2))`, which plots $\big(y(t), y'(t)\big)$ in the y, y' plane. It is particularly useful for part **c**, since in this graph periodic solutions close on themselves.

  We have the skills to go further.

  g. Compute the first order perturbation theory approximation to u, that is

$$u(t,\epsilon) \ \approx \ u_0(t) \ + \ \epsilon\, u_1(t)\,, \qquad u_1(t) \ = \ \frac{\partial u}{\partial \epsilon}(t,0)\,.$$

  h. Explain why this matches or not the observations about the amplitude A in part **e.**

  Discussion. 1. In contrast to the small nonlinearity computation in class, this one yields a perturbation of size ϵ. The in class problem yielded a perturbation of (relative) size ϵ^2. **2.** The first order perturbation calculation can be used to answer part **c.** The method is due to Poincaré. We will *not* present it. **3.** An alternate method which definitively answers part **c** will be given near the end of the course.