

Lab 5: Nonlinear Systems

Goals

In this lab you will use the `pplane6` program to study two nonlinear systems by direct numerical simulation. The first model, from population biology, displays interesting nonlinear oscillations (so-called **limit cycles**). The second is a system whose solutions depend on a parameter. Neither of these systems is described by exactly solvable systems of differential equations. Although much may be learned from strictly theoretical analyses, we must ultimately rely on computational methods to extract their quantitative predictions.

Application 1: Predator-Prey Species Interactions

In class we considered a model of predator-prey species interactions known as the Lotka-Volterra model (referred to in Section 6.3 as the predator-prey system). If x describes the size of a population of rabbits and y describes a population of foxes (which like to eat said rabbits) then the Lotka-Volterra model of their interactions says that there are positive constants a, b, c, d so that

$$\frac{dx}{dt} = x(a - by) \quad (1)$$

$$\frac{dy}{dt} = y(-c + dx) \quad (2)$$

That is, the exponential growth rate of rabbits is decreased by the presence of foxes and the exponential death rate of foxes is decreased by the presence of rabbits. This model predicts some unlikely behavior. In the absence of foxes ($y = 0$), equation (1) becomes $\frac{dx}{dt} = ax$. In other words, without any foxes the rabbits will always grow exponentially without bound. And even if the predator population is small, they will always eat the prey at a rate proportional to their product. In other words, 10 foxes surrounded by 100,000 rabbits would each have to eat ten times more than 10 foxes surrounded by 10,000 rabbits. If the rabbit population could be held at a fixed level $x_0 > c/d$, equation (2) becomes $\frac{dy}{dt} = Cy$ where $C = -c + dx_0 > 0$. In other words, if the rabbit population is maintained at a given level, above some threshold, the fox population will always grow exponentially without bound. None of these predictions are ecologically reasonable. The following model addresses these problems.

For positive values of r , the two populations are modelled by:

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{x + \frac{1}{5}} \quad (3)$$

$$\frac{dy}{dt} = ry \left(1 - \frac{y}{x}\right). \quad (4)$$

In the absence of predators, the prey satisfies the logistic equation with equilibrium population $x = 1$. In the presence of predators, prey is consumed at a rate $\frac{xy}{x + \frac{1}{5}}$. That is, if $x \gg \frac{1}{5}$, then the predators consume prey at a rate proportional to the predator population. Only if $x \ll \frac{1}{5}$ do the predators consume prey at a rate proportional to xy as in the Lotka-Volterra model. And for a fixed prey population, the predator population satisfies logistic growth with equilibrium population x . The parameter r is the inverse relaxation time for the predator population, i.e. $\frac{1}{r}$ is proportional to the time it takes the predator population to equilibrate. (Note: We have already scaled the variables and chosen some parameter values in equations (3) and (4). The general version of the model would have many more parameters.)

Application 2: Bifurcation

When one tries to understand the behavior of a nonlinear system one of the first things one looks at is the set of equilibrium solutions. The number and type of equilibrium solutions may well depend on some parameter(s) of the system: the mass of a component, the stiffness of a spring, the length of a lever, the resistance of an electronic component, etc. In this section of the lab you will observe in a very simple case how the structure of the equilibrium points of a system of equations changes as a parameter varies. Such a qualitative change is called a bifurcation and the associated parameter value is called a bifurcation point. The system in question is:

$$\frac{dx}{dt} = ax - y \quad (5)$$

$$\frac{dy}{dt} = x + ay + x^2 \quad (6)$$

In this system a is the parameter.

Prelab Assignment

Before arriving in lab, answer the following questions. Your answers should be neatly presented and handed in at the beginning of lab.

1. Find the fixed points (critical points) of the Predator-Prey system, equations (3) and (4). Calculate the numerical value of the coexistence point corresponding to positive values of x and y .
2. Find the curve in the phase plane where the trajectories of (3) and (4) are vertical (the x -nullcline) and the curve in the phase plane where the trajectories are horizontal (the y -nullcline). Use the information from Prelab Problem 1 along with these curves to sketch possible phase portraits.
3. The system exhibits very different behavior depending on whether $r > r_c$ or $r < r_c$, where $r_c = .053576\dots$. In one regime, the coexistence point is stable and all solutions are attracted to it. In the other region, the coexistence point is unstable and population levels starting near the point spiral outward. Which do you think happens for which values of r ? That is, do you think that a large or a small value of r ought to correspond to the stable coexistence or to the oscillatory behavior? (Either provide a coherent logical argument or do a stability analysis of the coexistence fixed point to justify your prediction.)
4. Find the equilibrium points for the second system; that is, for equations (5) and (6).

Phase Portraits

To study the evolution of the fox and rabbit populations over time, you will want to generate a phase portrait plotting x against y . The following describes how you are to use **pplane6** to generate these phase portraits:

After you log on and open Matlab, type **pplane6**. As happened with the **dfield6** program you used in Lab 1 a dialog box will open with lots of little boxes all filled in; ignore them and click on the **Proceed** button. You will see a graph with a direction field corresponding to a system of equations. Put the cursor on any point and click. You have just chosen initial conditions for the system. Now you know what the solution to the system of ODEs with your choice of initial conditions looks like.

Plot a few solutions (your graph should suggest an insect) and then go to the **Graph** menu and select **y vs t**. Your cursor will become cross hairs; center the cross hairs on a solution curve and click. You will see a plot of - surprise - y vs t . You can change your mind and click on **x vs t** or both or **3D**, etc.

If your graph is getting too cluttered go to the **Edit** menu and select **Erase all solutions**.

Finally, go to the **Solutions** menu and choose **Find an equilibrium point**; your cursor will again become a cross hair and if you position the cross hairs near an equilibrium point and click you will get a red dot at the equilibrium point and some info in a dialog box. You can repeat the command and find another equilibrium point for this system.

When you use **pplane6** to do this lab you will, of course, change the system in the **pplane6** dialog box to the system you want to study. In the first case, the predator-prey system, there is a parameter r in the system. You can enter the equations with the r in them and then, underneath the equations box, enter $r=0.5$ or whatever in the parameters or expressions box. (A parameter is a constant of the problem that changes from one problem to the next.)

Some important points: when you set up your equation you also enter the minimum and maximum values for x and y as you did with **dfield6** but it is often more convenient to **zoom in** or **zoom out**. You will find those commands under the **Edit** menu. In addition you should have the solver evolve the solution *forward* in time. This can be done by changing the solution direction in the **options** menu for the **pplane6 Display** window. (Looking at the solution only in the forward direction tells you whether

solutions are moving towards or away from an equilibrium point.) Now you have the tools to do the lab.

Lab Problems

1. Use **pplane6** to solve equations (3) and (4) and print phase portraits. Start from various initial conditions, and use $r = .07, .05$ and $.03$. **Zoom in** on important features.
2. Check your prediction from Prelab Problem 3. What, if anything, surprised you about the behavior of the system?
3. What is different about the oscillatory state here compared to that of the Lotka-Volterra predator-prey model? Discuss. (Hint: Consider the dependence of the steady state oscillation amplitude on the initial condition. How many different closed orbits do you see for each value of r ?)
4. Classify the equilibrium points of the second system (equations (5) and (6)) when $a = -0.5$, $a = 0$ and $a = 0.5$.
5. Provide a sketch or printout of the behavior of solutions near the equilibrium points in each of the three cases.