The Error Formula in Taylor Approximation

Begin with the Fundamental Theorem of Calculus,

$$f(x) - f(a) \;=\; \int_0^x f'(s)\; ds, \tag{1}$$

which is an error formula for approximation of $f(x)$ by the zeroth order Taylor polynomial

$$P_0(x;a) \;:=\; f(a)\,.$$

The inspired idea is to write the integrand in (1) as

$$f'(s)\;\frac{d}{ds}\Big(s-x\Big)\,.$$

Integrating by parts then yields

$$\begin{aligned} f(x) - f(a) \;&=\; -\int_0^x \big(s-x\big)f''(s)\; ds \;+\; \big(s-x\big)\,f'(s)\Big|_{s=a}^{s=x} \\ &=\; \int_0^x \big(x-s\big)f''(s)\; ds \;-\; \big(x-a\big)f'(a)\;. \end{aligned}$$

Moving the second term on the right hand side to the left yields

$$f(x) - \big[f(a) + (x-a)f'(a)\big] \;:=\; f(x) - P_1(x;a) \;=\; \int_0^x \big(x-s\big)\,f''(s)\; ds\,. \tag{2}$$

This is an *error formula* for approximation by the first order Taylor polynomial.
Continue in this fashion, writing the integrand in (2) as

$$f''(s)\;\frac{d}{ds}\,\frac{-(s-x)^2}{2}\,.$$

An integration by parts as above (exercise) yields the error formula

$$f(x) - P_2(x;a) \;=\; \int_0^x \frac{\big(x-s\big)^2}{2}\, f'''(s)\; ds\,. \tag{3}$$

Inductively one finds (exercise),

$$f(x) - P_n(x;a) \;=\; \int_0^x \frac{\big(x-s\big)^n}{n!}\, f^{(n+1)}(s)\; ds\,. \tag{3}$$