Math 256 Fall 2007
Professor J.Rauch Lesson #1

Playing with Direction Fields

Reading. Text Chapter 1. You have likely met this material in your calculus course. There are several tricky traps to avoid which will be discussed in class. This homework achieves the computer use goal set out in the second paragraph on page 5.

Remark on exactly solvable equations. The phrase "may be quite difficult to solve" in that second paragraph on page 5 is too weak. More accurate is "can almost never be solved by formulas using the elementary functions".

Exactly solvable equations, are very important. There are an amazing variety of important applications which are adequately modeled by such equations. To read the scientific literature one must know them. In fact, much modeling involves simplifications until one reaches an exactly solvable model. Understanding the simplified model is an important step towards understanding the full, and usually not exactly solvable models.

This series of exercises is aimed at introducing you to the plug and play aspects of direction fields for single scalar fist order ordinary differential equations of the general form

$$x' \;=\; f(t,x)\,.$$

The unknown is here a real valued function $x(t)$ of the real variable t which you are encouraged to think of as time. The prime denotes differentiation with respect to t.

We use of software written by John POLKING and available on his web page,

http://math.rice.edu/˜polking/

Links to this and to the page below are included in the online Math 256 materials.

The MATLAB m-file dfield7.m can be downloaded the Math 256 page or from Polking's site. If you are at a university machine, it is almost surely loaded already. Versions of this file for earlier versions of MATLAB are available on Polking's site. This is the preferred method, but there is a simpler one which is to use the internet version which can be reached from the "Java Versions" link on Polking's page or from the Math 256 web page. The URL is,

http://math.rice.edu/˜dfield/dfpp.html

Experiment number 1. Use the dfield method to solve the trivial equations

$$x' = 0,$$

$$x' = 1,$$

and

$$x' = t.$$

Play with the size of the display window to get used the software. Make sure that you understand how the geometric output corresponds to these equations.

Experiment number 2. See if you can figure out ahead of time what the phase plane will look like for

$$x' = t^2$$

Run the software to see if you figured out right. Choose a window that extends into negative values as far as it extends into positive values.

Experiment number 3. Next run dfield on the exactly solvalbe equation

$$x' = x^2.$$

Make sure that you understand why the phase plane looks as it does. For example all arrows point up. Why? Make sure that you understand why this equation is VERY different from $x' = t^2$.

From the exact solution of this equation, which you should find by hand, show that if $x(0) > 0$ then the solution tends to infinity at the positive time $t = 1/x(0)$.

From the phase plane you can easily see that solutions grow very rapidly, but the fact that there is finite time explosion for this equation and NOT for the equation $x' = x$ is *not* easily seen from the geometry of the direction field.

For the example

$$x' = x^2 \sin^2(x + t) \tag{1}$$

there is finite time blow up but you cannot solve exactly to find out. It is important to have methods for finding out properties of solutions for equations which cannot be exactly solved, as these form the vast majority of equations. Just as the vast majority of indefinite integrals

$$\int f(t)\, dt$$

can *not* be found explicitly. The indefinite integral is the family of solutions of

$$x' \;=\; f(t)\,,$$

where the right hand side does not depend on x.

Exercise. Though one cannot find formulas for the solutions of equation (1), it is easy to tell if a given function is or is not a solution. For example, check whether $x(t) = 2t$ is a solution. It is important in understanding what an ordinary differential equation is to see clearly this distinction. It also makes an easy exam question.

Speaking of easy exam questions, the classic about direction fields is the following. Find the direction field of the differential equation (1) at the point $t = 0,\ x = -\pi/2$. Is it pointing up or down? What does this tell you about the solution $x(t)$ with $x(0) = -\pi/2$? **Hint.** For t slightly larger (respectively smaller) than 0.

A brain teaser. For the unknown function $t(x)$ find the general solutions of the four simple equations

$$t' = t, \qquad t' = x, \qquad t' = t^2, \qquad t' = x^2\,.$$

Here prime denotes differentiation with respect to x.